Dagmar Medková

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Boundary Value Problems on Bounded and Unbounded Lipschitz Domains

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Introduction

This book is devoted to the Laplace equation as one of the most important equations in mathematical physics. Results of the theory of boundary value problems for the Poisson equation are useful in many fields of physics, and also in other mathematical theories—for example, in the theory of boundary value problems for elliptic equations and in hydrodynamics. So, we aimed to write this text to be as accessible as possible to experts in other fields and to research students. In the preliminary chapter, we gather all fundamental definitions and propositions from various fields of mathematics that we use in this book.

Boundary value problems for the Laplace equation are studied independently in potential theory, in the theory of partial differential equations, and in harmonic analysis by distinct methods. Here, we attempt to gather the most important results of these theories concerning boundary value problems for the Laplace equation or, more generally, for the Poisson equation. Containing only classical results from potential theory, this book is intended as an introduction to the modern theory of partial differential equations. We believe that every expert in this field should know classical results such as the regularity properties of solutions of the Poisson equation. Moreover, some objects from potential theory correspond to objects in other theories. Thus, we see that superharmonic functions are nothing other than supersolutions of the Laplace equations, subharmonic functions are subsolutions of the Laplace equations, and solutions of the Dirichlet problem in different senses (in Sobolev spaces, in Besov spaces, or in the sense of non-tangential limits) are special classes of the Perron-Wiener-Brelot solutions studied in potential theory. We also include several classical results that are not in contemporary books on potential theory but that are important for applications of boundary value problems on unbounded domains. Take, for example, the behavior of volume potentials: Every expert in the theory of partial differential equations knows what holds true, but it is almost impossible to find a reference. As a consequence of the properties of volume potentials, we can study the Laplace operator as a mapping between two homogeneous Sobolev spaces. Thus, we can prove some results above homogeneous Sobolev spaces that are very important for problems on unbounded domains and that are also very difficult to find in the literature.

After gathering properties of solutions of the Laplace equation and of the Poisson equation, we start to study boundary value problems on bounded and unbounded domains with compact boundary. First, we construct a Perron-Wiener-Brelot solution of the Dirichlet problem on general bounded open sets using the explained potential theory. Then, we study by the integral equation method several boundary value problems on bounded and unbounded Lipschitz domains with the boundary condition fulfilled in the sense of non-tangential limits (so-called L^q-solutions of the problems). Because a solution of the Dirichlet problem in this sense is a Perron-Wiener-Brelot solution of the problem, we deduce that the harmonic measure and the surface measure on the boundary are mutually absolutely continuous. It would be extremely difficult to prove this result solely by the methods of potential theory. The integral equation method for L^q-solutions is a starting point for the study of classical solutions, solutions in Besov spaces, and solutions in Sobolev spaces or more generally in homogeneous Sobolev spaces. It enables us to prove results for these solutions of boundary value problems by an easier and shorter way than appears in literature where only standard methods are used.

Let us now summarize the content of the book. The first chapter is preliminary, where we explain the mathematical notions and theorems used in the book: the fundamentals of the theory of metric spaces, the theory of Banach spaces and linear operators on them, the theory of interpolation, the theory of measures and integrals, the theory of distributions, theories of Sobolev, Besov, and Lizorkin-Triebel spaces, the theory of homogeneous Sobolev spaces, and the theory of integral operators.

Chapter 2 studies harmonic functions, i.e., classical solutions of the Laplace equation. We prove classical properties of harmonic functions such as the mean value theorem, the maximum principle, the Harnack inequality, and the stability of harmonic functions under the Kelvin transformation. We show that separately harmonic functions are harmonic, and we study the convergence of sequences of harmonic functions. Some results in this chapter are quite modern. We demonstrate that solutions of the Laplace equation in the sense of distributions are harmonic functions, i.e., solutions of the Laplace equation in classical sense. Additionally, we show that a tempered distribution satisfying the Laplace equation in the whole space is a polynomial, and we study the growth of derivatives of harmonic functions at infinity. These results are important for boundary value problems on exterior domains.

Chapter 3 presents properties of solutions of the Poisson equation. We define superharmonic and subharmonic functions and study their fundamental properties. We show that superharmonic functions are distributions for which $-\Delta u \geq 0$. Also, we define Green's function and study its properties. Then, we explore Green's potentials and volume potentials. As a consequence, we obtain regularity results for solutions of the Poisson equation. The behavior of the Laplace operator on homogeneous Sobolev spaces is also explored. These results are important for boundary value problems.

Chapter 4 is devoted to the Perron-Wiener-Brelot solution of the Dirichlet problem on bounded open sets. We define an upper solution, a lower solution, and for resolutive boundary conditions also Perron-Wiener-Brelot solutions of the Dirichlet problem for the Laplace equation. We show that continuous functions are resolutive. Then, we define harmonic measures, study regular sets, and show that domains with Lipschitz boundary are regular. We shall see later in Chap. 7 that solutions of the Dirichlet problem in Sobolev and Besov spaces are special classes of Perron-Wiener-Brelot solutions of the Dirichlet problem.

Chapter 5 studies so-called L^p-solutions of boundary value problems for the Laplace equation, i.e., that boundary conditions are p-integrable, they are fulfilled in the sense of a non-tangential limit, and the maximal function of a solution is p-integrable on the boundary. A single layer potential and a double layer potential with density from $L^p(\partial\Omega)$ are defined. We study properties of layer potentials and search for solutions of boundary value problems in the form of an appropriate linear combination of a single layer potential and a double layer potential. By this method we explore L^p-solutions of the Dirichlet problem, the Neumann problem, the Robin problem, the oblique derivative problem, the transmission problem, the jump problem, and mixed problems for bounded and unbounded open sets with compact Lipschitz boundary. Moreover, we show that an L^p-solution of the Dirichlet problem is a Perron-Wiener-Brelot solution of the Dirichlet problem. As a consequence, we obtain that the harmonic measure and the surface measure on the boundary are mutually absolutely continuous.

Chapter 6 examines classical solutions of boundary value problems on bounded and unbounded domains, utilizing results of Chaps. 4 and 5. We show that there exists a solution of the Dirichlet problem for the Poisson equation in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for continuous data. Solutions in $\mathcal{C}^{0,\lambda}(\overline{\Omega})$ and in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ are also studied. In general, there exists no solution in $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ of the Neumann problem or the Robin problem for a continuous boundary condition. So, classical solutions of these problems are a bit weaker, and they are functions from $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for which there exists a normal derivative on the boundary. We find a necessary and sufficient condition for the existence of a classical solution of the Neumann problem and the Robin problem. Also, we show that solutions of these problems are from $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ for boundary conditions from $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Moreover, we study classical solutions of the oblique derivative problem, the transmission problem, the jump problem, and mixed problems. At the end of this chapter, we present the symmetry of solutions of the Dirichlet problem and the Robin problem for symmetric domains and symmetric boundary conditions.

Chapter 7 treats boundary value problems for the Poisson equation in Sobolev and Besov spaces on bounded domains and in homogeneous Sobolev spaces on unbounded domains. Some results are given by the variational method, and others we achieve using results of Chap. 5 and interpolation. For unbounded domains, we utilize results of Chap. 3. We study the Dirichlet problem, the Neumann problem, the Robin problem, the oblique derivative problem, the transmission problem, the obstacle problem, and mixed problems. We compare the Perron-Wiener-Brelot solution of the Dirichlet problem and solutions of this problem in Sobolev and Besov spaces. At the end of this chapter, we examine the eigenvalue problem for the Dirichlet problem, for the Neumann problem, for the Robin problem, and for the mixed problem.

The book was inspired by results in [3], [6–15], [18], [20–23], [25–37], [39], [44–48], [50–54], [56], [58, 59], [61], [63–66], [68–71], [72–76], [77–80], [82–85], [88–93], [95–97], [99–111], [113–120], [122], [124], [126], [129], [131–134], [135], [138, 139], [148, 149].

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Chapter 1
Preliminaries

Abstract Chapter 1 gathers fundamental facts from functional analysis.

When we study boundary value problems for the Poisson equation we need basic knowledge of the functional analysis: metric spaces, Banach spaces and theory of continuous linear operators on Banach spaces, interpolation theory, theory of measures and integrals, theory of function spaces and theory of integral operators. For a convenience of a reader we remember in this chapter all we need in the text. A reader can skip this chapter.

1.1 Notation

First of all, we introduce some notation we use in the whole book.

N denotes the set of all positive natural numbers; $N_0 = N \cup \{0\}$.

If $j, k \in N$ denote

$$\delta_{jk} = \begin{cases} 1, j = k, \\ \\ 0, j \neq k \end{cases}$$

the Kronecker delta.

C denotes the complex plane. If $z = x + iy \in C$, then $\bar{z} = x - iy$ denotes its conjugate.

R^m denotes the m-dimensional Euclidean space. If $x = [x_1, \ldots, x_m] \in R^m$, $y = [y_1, \ldots, y_m] \in R^m$ denote

$$x \cdot y = \sum_{j=1}^{m} x_j y_j, \qquad |x| = \sqrt{x \cdot x}.$$

$R^m_+ = \{[x_1, \ldots, x_m] \in R^m; x_m > 0\}$.

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If $x = (x_1, \dots, x_m)$, $\alpha = (\alpha_1, \dots, \alpha_m)$ then

$$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}.$$

If $M, K \subset R^m$ denote $M \setminus K = \{x \in M; x \notin K\}$.

$\partial\Omega$ denotes the boundary of Ω.

$\overline{\Omega} = \Omega \cup \partial\Omega$ denotes the closure of Ω.

$\operatorname{int}\Omega$ denotes the interior of Ω,

$\operatorname{diam}\Omega = \sup\{|x - y|; x, y \in \Omega\}$ denotes the diameter of Ω.

$\operatorname{dist}(x, \Omega) = \inf\{|x - y|; y \in \Omega\}$ denotes the distance of the point x from the set Ω.

If $M \subset R^m$, denote

$$\chi_M(x) = \begin{cases} 1, \, x \in M, \\ \\ 0, \, x \notin M. \end{cases}$$

the characteristic function of M.

If $x \in R^m$, $r > 0$ denote $B(x; r) = \{y \in R^m; |x - y| < r\}$ the open ball with the center x and the diameter r.

Denote by $P_k(R^m)$ the space of homogeneous polynomials of degree k.

If f is a function, denote $\operatorname{spt} f = \overline{\{x; f(x) \neq 0\}}$ the support of f.

If f is a function on G and $M \subset G$, denote by $f|M$ the restriction of f onto M.

If f is a function, denote $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$.

If f is a function, denote

$$\partial_j f = \frac{\partial f}{\partial x_j}$$

If f is differentiable at $x \in R^m$, denote $\nabla f(x) = (\partial_1 f(x), \dots, \partial_m f(x))$ the gradient of f at x. If $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multiindex, then $|\alpha| = \alpha_1 + \dots \alpha_m$ denotes its length, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_m!$,

$$\partial^{\alpha} f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_m^{\alpha_m} f.$$

Remark that

$$f = \partial^{(0,\dots,0)} f.$$

If $k \in N$ denote

$$\nabla^k f = \{\partial^{\alpha} f; |\alpha| = k\}.$$

If

$$p(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha$$

is a polynomial, denote

$$p(D) = \sum_{|\alpha|=k} a_\alpha \partial^\alpha.$$

Let f and g be functions defined on $B(z;r) \setminus \{z\}$, $r > 0$, and $g > 0$. We say that $f(x) = O(g(x))$ as $x \to z$ if there are positive constants ρ, K such that $|f(x)| \le Kg(x)$ for $x \in [B(z;\rho) \setminus \{x\}]$. We say that $f(x) = o(g(x))$ as $x \to z$ if

$$\lim_{x\to z} \frac{f(x)}{g(x)} = 0.$$

Let f and g be functions defined on $R^m \setminus B(z;r)$ and $g > 0$. We say that $f(x) = O(g(x))$ as $|x| \to \infty$ if there are positive constants ρ, K such that $|f(x)| \le Kg(x)$ for $x \in R^m \setminus B(z;\rho)$. We say that $f(x) = o(g(x))$ as $|x| \to \infty$ if

$$\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = 0.$$

If $K \subset R^m$, then $\mathcal{C}^0(K)$ denotes the set of all real continuous functions on K. By $\mathcal{C}^{0,0}(K)$, denote the set of all bounded continuous functions on K equipped with the norm

$$\|f\|_{\mathcal{C}^{0,0}(K)} = \sup_{x\in K} |f(x)|.$$

If $0 < \beta \le 1$, denote by $\mathcal{C}^{0,\beta}(K)$ the set of all functions $f \in \mathcal{C}^{0,0}(K)$ for which

$$\|f\|_{\mathcal{C}^{0,\beta}(K)} \equiv \|f\|_{\mathcal{C}^{0,0}(K)} + \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

Let $\Omega \subset R^m$ be an open set, $k \in N$. Denote by $\mathcal{C}^k(\Omega)$ the set of all functions f defined on Ω such that for each multiindex α with $|\alpha| \le k$, we have $\partial^\alpha f \in \mathcal{C}^0(\Omega)$. If $0 \le \beta \le 1$, denote by $\mathcal{C}^{k,\beta}(\Omega)$ the set of all functions f such that for each multiindex α with $|\alpha| \le k$, we have $\partial^\alpha f \in \mathcal{C}^\beta(\Omega)$. Denote by $\mathcal{C}^{k,\beta}(\overline{\Omega})$ the set of all functions $f \in \mathcal{C}^{k,\beta}(\Omega)$ such for each multiindex α with $|\alpha| \le k$ there is an extension of $\partial^\alpha f$ onto $\overline{\Omega}$ which is from $\mathcal{C}^\beta(\overline{\Omega})$. Denote

$$\mathcal{C}^\infty(\Omega) = \bigcap_{k=0}^{\infty} \mathcal{C}^k(\Omega)$$

and by $\mathcal{C}_c^\infty(\Omega)$ the set of all functions from $\mathcal{C}^\infty(\Omega)$ with compact support in Ω.

If μ is a nonnegative measure and f is μ-measurable function denote

$$\|f\|_{L^\infty(\mu)} = \inf\{c; |f(x)| \leq c \text{ for } \mu - \text{almost all } x\},$$

$$\|f\|_{L^p(\mu)} = \Big(\int\limits_\Omega |f|^p \, d\mu\Big)^{1/p} \qquad \text{for } 1 \leq p < \infty.$$

If $1 \leq p \leq \infty$, then $L^p(\mu)$ denotes the Banach space of all μ-measurable functions f such that

$$\|f\|_{L^p(\mu)} < \infty.$$

The elements of $L^p(\mu)$ are classes of equivalent functions. The space $L^p(\mu)$ with the norm $\| \dots \|_{L^p(\mu)}$ is a Banach space.

If μ is a nonnegative measure, M is a μ-measurable set, $\mu(M) > 0, f$ is μ-measurable function, and there exists finite

$$\int\limits_M f \, d\mu$$

denote

$$⨍_M f \, d\mu = \frac{1}{\mu(M)} \int\limits_M f \, d\mu.$$

$\mathcal{H}_k$ denotes the k-dimensional Hausdorff measure.

If μ is a measure and B is a μ-measurable set, denote by $\mu|B$ the restriction of μ onto B, i.e., for $\nu = \mu|B$ we have $\nu(A) = \mu(B \cap A)$.

Let $\Omega \subset R^m$ be open, $1 \leq p \leq \infty$. Denote $L^p(\Omega) = L^p(\mathcal{H}_n|\Omega)$. By $L^p_{loc}(\Omega)$, we denote the set of all Lebesgue measurable functions f such that $|f|^p$ is Lebesgue integrable over all compact subsets of Ω. If $\mathcal{H}_{m-1}(\partial\Omega) < \infty$, denote $L^p(\partial\Omega) = L^p(\mathcal{H}_{m-1}|\partial\Omega)$.

We say that a sequence of functions f_j converges to f uniformly on a set K if

$$\lim_{j\to\infty} \sup_{x\in K} |f_j(x) - f(x)| = 0.$$

We say that a sequence of functions f_j converges to f locally uniformly on an open set Ω if f_j converges to f uniformly on each compact subset of Ω.

We say that a family $\mathcal{F}$ of functions on a set M is equibounded from above on M if

$$\sup_{f\in\mathcal{F}} \sup_{x\in M} f(x) < \infty.$$

We say that $\mathcal{F}$ is equibounded from below on M if

$$\inf_{f\in\mathcal{F}}\inf_{x\in M} f(x) > -\infty.$$

We say that $\mathcal{F}$ is equibounded on M if it is equibounded from below and from above on M.

If $K \subset R^m$ is a closed set, denote by $\mathcal{C}'(K)$ the Banach space of all finite real Borel measures with support in K with the total variation as a norm.

If f is a mapping from X to Y, then for $B \subset Y$ denote $f_{-1}(B) = \{x \in X; f(x) \in B\}$.

1.2 Metric Spaces

In this section, we remember the notion of metric space and its basic properties.

Definition 1.2.1 Let X be a nonempty set. Let $\rho(x, y)$ be a nonnegative function defined on the set $X \times X$ of all ordered pairs (x, y) of members of X, and satisfy the following conditions:

- $\rho(x, y) = \rho(y, x)$ for every pair (x, y) of $X \times X$;
- $\rho(x, y) = 0$ if and only if $x = y$;
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, where x, y, and z are any three elements of X.

Such a function $\rho(x, y)$ is called a metric on X. A set X with metric ρ is called a metric space.

Definition 1.2.2 Let X be a metric space with metric ρ. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if for every $\epsilon > 0$ there exists an integer n_0, such that $\rho(x_n, x_m) < \epsilon$ whenever $m \geq n \geq n_0$. A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists an integer n_0, such that $\rho(x, x_m) < \epsilon$ whenever $n \geq n_0$. A metric space X is said to be complete if every Cauchy sequence of points of X converges to a point of X.

Example 1.2.3 The Euclidean space R^m with the metric $\rho(x, y) = |x - y|$ is a complete metric space.

Definition 1.2.4 Let X be a metric space with metric ρ. A subset M of X is called open if for each $x \in M$ there is $r > 0$ such that the ball $B(x; r) = \{y \in X; \rho(x, y) < r\}$ is a subset of M. A set M is called a neighborhood of $x \in X$ if there exists $r > 0$ such that $B(x; r) \subset M$. The closure of M, denoted by $\overline{M}$, is the set of all limits of sequences $\{x_n\}$, where $x_n \in M$. A set M is said to be closed if $\overline{M} = M$. A subset M of X is called dense if $\overline{M} = X$.

Theorem 1.2.5 (Baire Category Theorem) *If a complete metric space is a union of a countable number of closed sets, then at least one of these closed sets contains a nonempty open set.*

(See [43], Theorem I.6.9.)

Definition 1.2.6 Let X be a metric space. A subset K of X is called compact if for every system $\{V_\alpha; \alpha \in A\}$ of open sets with $K \subset \cup\{V_\alpha; \alpha \in A\}$ there is a finite subsystem $\{V_1, \ldots, V_n\}$ such that $X = V_1 \cup \cdots \cup V_n$. A subset K of X is called relatively compact if its closure is compact. A metric space X is said to be locally compact if for each $x \in X$ there exists a relatively compact open set $V \subset X$ such that $x \in V$.

Lemma 1.2.7 *A subset K of a metric space X is compact if and only if every sequence $\{x_n\}$ of members of K has a subsequence that converges to a member of K. If K is compact, then it is closed.*

(See [142], Theorem 8.2.4.)

Example 1.2.8 $M \subset R^m$ is relatively compact if and only if it is bounded.

(See [149], Introduction, §2.)

Definition 1.2.9 A metric space is said to be σ-compact if X is locally compact and there is a sequence $\{X_j\}$ of compact subsets of X such that $X = \cup X_j$.

Remark 1.2.10 Let Ω be an open subset of R^m and K be a compact subset of Ω. Then there exists a positive constant r such that $B(x; r) \subset \Omega$ for each $x \in K$.

Proof If $x \in K$, then there exists a positive constant $r(x)$ such that $B(x; 2r(x)) \subset \Omega$ because Ω is open. Since K is compact, there exist $x_1, \ldots, x_k \in K$ with $K \subset B(x_1; r(x_1)) \cup \cdots \cup B(x_k; r(x_k))$. Put $r = \min(r(x_1), \ldots, r(x_k))$. Fix $x \in K$. Then there is $j \in \{1, \ldots, k\}$ such that $x \in B(x_j; r(x_j))$. Since $r \le r(x_j)$, we have $B(x; r) \subset B(x_j; 2r(x_j)) \subset \Omega$.

Lemma 1.2.11 *Let Ω_j be an increasing sequence of open subsets of a metric space X. If $K \subset X$ is compact and $K \subset \cup\Omega_j$, then there is an index l such that $K \subset \Omega_l$.*

Proof Since K is compact, there are $i_1, \ldots, i_k$ such that

$$K \subset \Omega_{i_1} \cup \cdots \cup \Omega_{i_k}.$$

Put $l = \max(i_1, \ldots, i_k)$. Then $K \subset \Omega_l$.

Definition 1.2.12 Let X and Y be metric spaces. A mapping T from X to Y is said to be continuous if for each $x \in X$ and $\epsilon > 0$ there is $\delta > 0$ such that $T(B(x; \delta)) \subset B(T(x); \epsilon)$.

Lemma 1.2.13 *Let f be a continuous real function on a compact metric space X. Then f achieves its maximum and minimum values.*

(See [55], Chapter 6, Theorem 6.14.)

Definition 1.2.14 Let X and Y be metric spaces. A mapping T from X to Y is said to be uniformly continuous if for each $\epsilon > 0$ there is $\delta > 0$ such that $T(B(x; \delta)) \subset B(T(x); \epsilon)$ for each $x \in X$.

Lemma 1.2.15 *Let X and Y be metric spaces and T be a continuous mapping from X to Y. If X is compact, then T is uniformly continuous.*

(See [142], Theorem 8.3.8.)

Definition 1.2.16 Let X be a metric space, f_n, and f be functions on X. We say that f_n converges uniformly to f on X if for each $\epsilon > 0$ there is $n_0 \in N$ such that $|f_n(x) - f(x)| \leq \epsilon$ for each $x \in X$ and $n \geq n_0$.

Theorem 1.2.17 (Dini Theorem) *Let X be a compact metric space and f_n be a monotonic sequence of continuous functions on X, $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in X$. If f is a continuous function on X, then f_n converges uniformly to f on X.*

(See [55], Theorem 9.36.)

Definition 1.2.18 A metric space X is connected if the only nonempty subset of X which is both open and closed is X itself. A connected open subset of X is called a domain. A nonempty set M is a component of a metric space X if it is a closed open connected subset of X.

Remark 1.2.19 Let $\Omega \subset R^m$ be open. If $f : \langle 0; 1\rangle \to \Omega$ is a continuous mapping, then $\gamma = \{f(t); t \in \langle 0; 1\rangle\}$ is a continuous curve within Ω that connects $f(0)$ and $f(1)$. Remark that Ω is connected if and only if for any $x, y \in \Omega$ there is a continuous curve within Ω that connects x and y.

(See [55], Theorem 6.63.)

Definition 1.2.20 Let X be a metric space with a metric ρ. If A is a nonempty subset of X, then A is a metric space with the induced metric ρ. We say that B is a relatively open (closed) subset of A if $B \subset A$ and B is open (closed) in the metric space A with the metric ρ.

1.3 Banach Spaces

A special case of a metric space is a normed linear space.

Definition 1.3.1 Let X be a vector space over the field of real or complex numbers. A norm on X, denoted by $\| \;\|$, is a nonnegative function on X with the following properties:

- $\|x\| = 0$ if and only if $x = 0$.
- $\|\alpha x\| = |\alpha| \|x\|$ for $x \in X$ and a scalar α,
- $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$.

The vector space X, together with a norm X, is called a normed linear space.

Definition 1.3.2 Let X be a normed linear space. The metric d induced by the norm is defined by $d(x, y) = \|x - y\|$. If X is a complete metric space with respect to d, X is called a Banach space.

Example 1.3.3 R^m is a Banach space. (See [43], Chapter IV, §3, Lemma 1.)

Definition 1.3.4 Let X be a linear space, and suppose that it has two norms $\|\cdot\|_1$, $\|\cdot\|_2$. We call them equivalent if there is a positive number a such that

$$a^{-1}\|x\|_1 \le \|x\|_2 \le a\|x\|_1, \qquad \forall x \in X.$$

If X is a linear space with equivalent norms $\|\cdot\|_1$, $\|\cdot\|_2$, then $\{x_n\}$ converges in one norm if and only if it converges in the other.

Proposition 1.3.5 *Let X be a finite-dimensional vector space; $\|\cdot\|_1$, $\|\cdot\|_2$ are two norms on X. Then these norms are equivalent.*

(See [128], Theorem 4.2.)

Proposition 1.3.6 *Let X be a vector space endowed with norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If X is a Banach space with respect to both norms $\|\cdot\|_1$, $\|\cdot\|_2$, then the norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent.*

(See [149], Chapter II, §5, Corollary.)

Proposition 1.3.7 *Let X be a closed subspace of a normed vector space Y. If Z is a finite-dimensional subspace of Y, then $X + Z = \{x + z; x \in X, z \in Z\}$ is a closed subspace of Y.*

(See [127], Chapter V, Lemma 1.2.)

Definition 1.3.8 If X and Y are subspaces of a vector space Z such that $X \cap Y = \{0\}$, then $X + Y$ is called the direct sum of X and Y and is denoted by $X \oplus Y$.

Proposition 1.3.9 *If X is a finite-dimensional subspace of a normed vector space Z, then there is a closed subspace Y of Z such that $Z = X \oplus Y$.*

(See [128], Lemma 5.1.)

Notation 1.3.10 *Let X be a subspace of a normed vector space Z. If there exists a finite-dimensional space Y such that $Z = X \oplus Y$, we denote* $\operatorname{codim} X = \dim Y$ *the codimension of X. If such Y does not exist, we write* $\operatorname{codim} X = \infty$.

Lemma 1.3.11 *Let X be a normed vector space. Then $\{x \in X; \|x\| = 1\}$ is a compact set if and only if X is finite-dimensional.*

(See [149], Chapter III, §2, Corollary 2 or [43], Chapter IV, §3, Theorem 5.)

Definition 1.3.12 Let X be a normed vector space and M be a nonempty closed subspace of X. For $x \in X$ denote $[x] = \{y \in X; x - y \in M\}$. If $x, y \in X$ and α is a scalar, denote $[x] + [y] = [x + y]$, $\alpha[x] = [\alpha x]$. The space $X/M = \{[x]; x \in X\}$ is called a factor space. The space X/M is equipped with the norm

$$\|[x]\| = \inf_{y \in M} \|x - y\|.$$

Proposition 1.3.13 *If M is a closed subspace of a Banach space X, then X/M is a Banach space.*

(See [128], Chapter 3, Theorem 3.13.)

1.4 Linear Operators

In this section, we gather fundamental properties of continuous linear operators and closed linear operators. We formulate the closed-graph theorem, the Banach–Steinhaus theorem, and the fixed-point theorem.

Definition 1.4.1 Let X and Y be vector spaces over the same space of scalars. An operator T from X into Y is called linear if for all x and y in X and all scalars α and β,

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

If $X \subset Y$, define a linear operator I from X to Y by $I(x) = x$. Then I is called the identity operator.

Definition 1.4.2 Let X and Y be normed vector spaces. If T is an injective (i.e., one to one) continuous linear map X onto Y such that T^{-1} is continuous, then T is called an isomorphism X onto Y. An isomorphism T is called isometric if $\|Tx\|_Y = \|x\|_X$ for each $x \in X$. We say that the spaces X and Y are isomorph if there exists an isomorphism X onto Y. We say that the space X is continuously imbedded into Y (and write $X \hookrightarrow Y$) provided that $X \subset Y$ and the inclusion map I are continuous.

Theorem 1.4.3 *Let X and Y be normed linear spaces, and let T be a linear operator from X into Y. The following statements are equivalent:*

1. *T is continuous at one point $x \in X$.*
2. *T is continuous at all points $x \in X$.*
3. *T is bounded, i.e., there exists a number M such that for all $x \in X$*

$$\|Tx\| \le M\|x\|.$$

(See [57], Theorem 1.3.2 or [149], Chapter I, §6.)

Definition 1.4.4 Let X and Y be normed linear spaces. Denote by $\mathcal{L}(X, Y)$ the vector space of all bounded linear operators which map X into Y. (If $Y = X$, we write $\mathcal{L}(X)$.) Define the norm on $\mathcal{L}(X, Y)$ by

$$\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Remark 1.4.5 Let X, Y, and Z be normed linear spaces, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Then $ST \in \mathcal{L}(X, Z)$ and $\|ST\| \le \|S\| \cdot \|T\|$.

(See [149], Chapter I, §6, Proposition 2.)

Theorem 1.4.6 *If X is a normed linear space and Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.*

(See [57], Theorem 1.3.5.)

Lemma 1.4.7 *Let X and Y be normed linear spaces and T be a linear operator from X into Y. If X is finite-dimensional, then $T \in \mathcal{L}(X, Y)$.*

(See [43], Chapter IV, §3, Corollary 4.)

Lemma 1.4.8 *Let X and Y be subspaces of a vector space Z such that $Z = X \oplus Y$. Then for each $z \in Z$, there is unique $Pz \in X$ such that $(I - P)z \in Y$. The operators P and $I - P$ are linear operators on Z. (P is called the projection of Z onto X along Y, and $I - P$ is called the projection of Z onto Y along X.)*

Proof If $z \in Z$, then there are $x \in X$ and $y \in Y$ such that $z = x + y$. If $\tilde{x} \in X$, $\tilde{y} \in Y$, and $z = \tilde{x} + \tilde{y}$, then $0 = (x - \tilde{x}) + (y - \tilde{y})$. Since $x - \tilde{x} = \tilde{y} - y \in X \cap Y = \{0\}$, we deduce that $x = \tilde{x}$ and $y = \tilde{y}$. Putting $Pz = x$, we get $(I - P)z = y$. Clearly, the operators P and $I - P$ are linear.

Corollary 1.4.9 *Let X and Y be closed subspaces of a Banach space Z such that $Z = X \oplus Y$. Denote by P the projection of Z onto X along Y. Then $P \in \mathcal{L}(Z)$. Denote*

$$\|z\|_1 = \|Pz\| + \|(I - P)z\|, \quad z \in Z$$

Then $\|\cdot\|_1$ is a norm on Z which is equivalent to the original norm $\|\cdot\|$.

Proof $\|\alpha z\|_1 = |\alpha| \cdot \|z\|_1$, $\|x + y\|_1 = \|P(x + y)\| + \|(I - P)(x + y)\| \le \|Px\| + \|Py\| + \|(I - P)x\| + \|(I - P)y\| = \|x\|_1 + \|y\|_1$, $\|z\| \le \|z\|_1$. Therefore $\|\cdot\|_1$ is a norm.

If z_n is a Cauchy sequence with respect to the norm $\|\cdot\|_1$, then Pz_n and $(I-P)z_n$ are Cauchy sequences with respect to the norm $\|\cdot\|$. Since X and Y are closed subspaces of the Banach space Z, there are $x \in X$ and $y \in Y$ such that $\|Pz_n - x\| \to 0$, $\|(I - P)z_n - y\| \to 0$ as $n \to \infty$. Put $z = x + y$. Then $Pz = x$, $(I - P)z = y$ and $\|z_n - z\|_1 \to 0$ as $n \to \infty$. Therefore X equipped with the norm $\|\cdot\|_1$ is a Banach space. Proposition 1.3.6 gives that the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. This forces that $P \in \mathcal{L}(Z)$.

Definition 1.4.10 Let X and Y be normed vector spaces and $T \in \mathcal{L}(X, Y)$. Then $\operatorname{Ker} T = \{x \in X; Tx = 0\}$ is the kernel of T.

Lemma 1.4.11 *Let X and Y be normed vector spaces and $T \in \mathcal{L}(X, Y)$. Then* Ker T *is a closed subspace of X.*

Proof Let $x_k \in \operatorname{Ker} T$, $x_k \to x$. Since T is continuous by Theorem 1.4.3, we have $Tx = 0$.

Definition 1.4.12 Let X and Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is bounded bellow if there exists $\epsilon > 0$ such that $\|Tx\|_Y \geq \epsilon \|x\|_X$ for each $x \in X$.

Proposition 1.4.13 *Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. If T is bounded below, then $T(X)$ is a closed subspace of Y.*

(See [128], Theorem 3.12.)

Definition 1.4.14 Let X and Y be normed linear spaces over the same scalar. Let $\mathcal{D}(T)$ be a subspace of X and T be a linear operator from $\mathcal{D}(T)$ into Y. The operator T is called closed if $x \in X$, $y \in Y$, $\{x_n\} \subset \mathcal{D}(T)$, $x_n \to x$, $Tx_n \to y$ imply $x \in \mathcal{D}(T)$ and $Tx = y$.

Theorem 1.4.15 (Closed-Graph Theorem) *Let X and Y be Banach spaces and $T : X \to Y$ be a closed linear operator. Then $T \in \mathcal{L}(X, Y)$.*

(See [128], Theorem 3.10.)

Lemma 1.4.16 *Let X, $\tilde{X}$, Y, $\tilde{Y}$ be Banach spaces, $\tilde{X} \hookrightarrow X$, $\tilde{Y} \hookrightarrow Y$. Let $T \in \mathcal{L}(X, Y)$. If $T(\tilde{X}) \subset \tilde{Y}$, then $T \in \mathcal{L}(\tilde{X}, \tilde{Y})$.*

Proof Let $x_n \to x$ in $\tilde{X}$, $Tx_n \to y$ in $\tilde{Y}$. Then $x_n \to x$ in X, $Tx_n \to y$ in Y. So, $Tx = \lim Tx_n = y$ and the operator T is a closed operator from $\tilde{X}$ to $\tilde{Y}$. Theorem 1.4.15 gives $T \in \mathcal{L}(\tilde{X}, \tilde{Y})$.

Proposition 1.4.17 *Let X and Y be Banach space and Z be a dense subspace of X, $\{T_k; k \in N\} \subset \mathcal{L}(X, Y)$. Suppose that*

- $\sup\{\|T_n(x)\|; n \in N\} < \infty$ *for each* $x \in X$.
- *For each $x \in Z$, there exists $y \in Y$ such that $T_n x \to y$ as $n \to \infty$.*

Then there exists $T \in \mathcal{L}(X, Y)$ such that $T_n x \to Tx$ as $n \to \infty$ for all $x \in X$. Moreover,

$$\|T\| \leq \liminf_{n \to \infty} \|T_n\| \leq \sup_{n \in N} \|T_n\| < \infty.$$

(See [43], Chapter II, §3, Theorem 6.)

Theorem 1.4.18 *Let X be a Banach space and $T : X \to X$ be a contraction, i.e., there exists a number $0 < q < 1$ such that $\|Tx - Ty\| \leq q\|x - y\|$ for all $x, y \in X$. Then the equation $Tx = x$ has exactly one solution $x \in X$. Fix $x_0 \in X$ and put*

$$x_k = Tx_{k-1}, \qquad k \in N.$$

Then $x_k \to x$ in X and

$$\|x - x_k\| \leq \frac{q^k}{1 - q} \|x_1 - x_0\|.$$

(See [40], Satz 1.24.)

1.5 The Dual Space

A special role in linear operators plays linear functions (so-called linear functionals). The dual space X' is the space of all continuous linear functionals on X. This section is devoted to continuous linear functionals. Hahn–Banach's theorem shows that a continuous linear functional on a subspace can be continuously extended onto the whole space. For a continuous linear operator $T : X \to Y$, we define the adjoint operator $T' : Y' \to X'$. The adjoint operator T' has similar properties like T.

In application we are sometimes not able to prove the convergence $x_n \to x$ in X. But we can prove weaker convergence $T(x_n) \to T(x)$ for $T \in X'$. This weak convergence is defined in this section.

Definition 1.5.1 Let X be a normed vector space. A functional on X is a map from X to the scalars. The dual X' of X is the Banach space of bounded linear functionals on X; that is, $X' = \mathcal{L}(X, Y)$, where Y is the Banach space of scalars with absolute value taken as norm.

Theorem 1.5.2 (Hahn–Banach's Theorem) *Let X be a normed vector space and Y be a subspace of X. Let T be a linear functional on Y and M be a positive constant such that*

$$|Tx| \le M\|x\| \qquad \forall x \in Y.$$

Then there is $\tilde{T} \in X'$ with $\|\tilde{T}\| \le M$ such that $\tilde{T}x = Tx$ for all $x \in Y$.

(See [128], Theorem 2.5 or [40], Korollar 3.12.)

Proposition 1.5.3 *Let X be a normed vector space and Y be a closed subspace of X. If $x \in X \setminus Y$, then there exists $T \in X'$ such that $\|T\| = 1$, $Ty = 0$ for all $y \in Y$ and $Tx = \operatorname{dist}(x, Y) > 0$.*

(See [40], Satz 3.14.)

Lemma 1.5.4 *Let X be a normed vector space and Y be a closed subspace of X. Denote $Z = \{T \in X'; T(x) = 0\ \forall x \in Y\}$. Then $Y = \{x \in X; T(x) = 0\ \forall T \in Z\}$.*

(See [128], Lemma 3.5.)

Definition 1.5.5 Let X be a normed vector space. Define the linear mapping $J : X \to (X')'$ by $Jx(T) = T(x)$ for $T \in X'$. Then J is an injection of X into $(X')'$. If $J(X) = (X')'$, we say that X is reflexive.

Lemma 1.5.6 *Let X be a reflexive Banach space. If Y is a closed subspace of X, then Y is a reflexive Banach space.*

(See [43], Chapter II, §3, Theorem 23 or [40], Satz 3.30.)

Definition 1.5.7 A sequence $\{x_n\}$ of elements of a Banach space X is said to converge weakly to an element $x \in X$ if $T(x_n) \to T(x)$ as $n \to \infty$ for each $T \in X'$.

Lemma 1.5.8 *If* $\{x_n\}$ *is a weakly convergent sequence in a Banach space* X*, then* $\{x_n\}$ *is bounded and*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

(See [149], Chapter V, §1, Theorem 1 or [40], Lemma 3.23.)

Theorem 1.5.9 *Let* X *be a normed vector space. If* $\{F_n\}$ *is a bounded sequence in* X'*, then there exists a subsequence* $F_{n(k)}$ *of* F_n *and* $F \in X'$ *such that* $F_{n(k)}(x) \to F(x)$ *as* $k \to \infty$ *for each* $x \in X$*.*

(See [141], Chapter 4, Theorem 4.61.A.)

Theorem 1.5.10 *Let* X *be a Banach space. Then* X *is reflexive if and only if every bounded sequence has a weakly convergent subsequence.*

(See [149], Chapter V, §4.)

Proposition 1.5.11 *Let* X *and* Y *be Banach spaces,* $T \in \mathcal{L}(X, Y)$*. If* $y' \in Y'$*, then there is unique* $(T'y') \in X'$ *such that*

$$(T'y')(x) = y'(Tx)$$

for each $x \in X$*. The operator* T' *is called the adjoint (or conjugate) of* T*. The following holds:* $T' \in \mathcal{L}(X, Y)$ *and* $\|T'\| = \|T\|$*.*

(See [128], Theorem 3.3.)

Lemma 1.5.12 *Let* X*,* Y*, and* Z *be Banach spaces. If* $T \in \mathcal{L}(X, Y)$ *and* $S \in \mathcal{L}(Y, Z)$*, then* $(ST)' = T'S'$*.*

(See [43], Chapter VI, §2, Lemma 4.)

Proposition 1.5.13 *Let* X *and* Y *be Banach spaces,* $T \in \mathcal{L}(X, Y)$*. If* $T(X)$ *is closed, then* $T(X) = \{y \in Y; f(y) = 0 \ \forall f \in \operatorname{Ker} T'\}$*.*

(See [128], Theorem 3.7.)

Proposition 1.5.14 *Let* X *be a normed vector space and* M *be a closed convex subset of* X*. If* $\{x_n\} \subset M$ *and* x_n *converges weakly to* x*, then* $x \in M$*.*

(See [43], Chapter V, §3, Theorem 13 or [40], Satz 3.33.)

1.6 Spectrum

If T is a continuous linear operator on a complex Banach space, we are interested for what complex λ there exists a continuous inverse of $\lambda I - T$. The complement of such λ in the complex plane is the spectrum of T.

Definition 1.6.1 Let X be a Banach space and $T \in \mathcal{L}(X)$. A scalar λ for which $\mathrm{Ker}(\lambda I - T) \neq \{0\}$ is called an eigenvalue of T. Any element $x \neq 0$ of X such that $Tx = \lambda x$ is called an eigenvector of T corresponding to λ.

Definition 1.6.2 Let X be a Banach space, $\mathcal{D}(T)$ be a dense subspace of X, and T be a closed linear operator from $\mathcal{D}(T)$ to X. The scalars λ for which $T - \lambda I$ has an inverse in $\mathcal{L}(X)$ comprise the resolvent set $\rho(T)$ of T. The spectrum $\sigma(T)$ of T consists of all scalars not in $\rho(T)$.

Theorem 1.6.3 *If X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is one-to-one and onto, then T^{-1} exists and is bounded.*

(See [128], Theorem 3.8.)

Proposition 1.6.4 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $\sigma(T') = \sigma(T)$.*

(See [128], Theorem 6.24 or [141], Theorem 5.3.A.)

Lemma 1.6.5 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $\sigma(T)$ is a closed set.*

(See [128], Theorem 6.3.)

Proposition 1.6.6 *Let X be a Banach space, $T \in \mathcal{L}(X)$, and λ be a scalar. If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$ and*

$$(\lambda I - T)^{-1} = \sum_{k=0}^{\infty} \lambda^{-k-1} T^k.$$

(See [141], Theorem 3.2.A.)

Proposition 1.6.7 *Let X be a Banach space, $T \in \mathcal{L}(X)$. Denote*

$$r(T) = \inf_n \|T^n\|^{1/n}.$$

Then

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

and $\lambda \in \rho(T)$ for each scalar λ such that $|\lambda| > r(T)$. If X is a complex Banach space, then

$$r(T) = \max_{\lambda\in\sigma(T)} |\lambda|$$

and $r(T)$ is called the spectral radius of T.

(See [128], Theorem 6.4 and [128], Theorem 6.13.)

Proposition 1.6.8 (Spectral Mapping Theorem) *Let X be a complex Banach space and $T \in \mathcal{L}(X)$. If P is a polynomial, then $\sigma(P(T)) = P(\sigma(T)) \equiv \{P(\lambda); \lambda \in \sigma(T)\}$.*

(See [141], Theorem 5.71-B.)

1.7 Compact Operators

This section is devoted to compact linear operators. The essential property of a compact linear operator $K : X \to X$ is that for $\lambda \neq 0$ there exists a continuous $(\lambda I - K)^{-1}$ if and only if the equation $(\lambda I - K)x = 0$ has only trivial solution.

We also prove another result useful in applications. If $T : X \to Y$ is a continuous linear operator and $K : X \to Y$ is a compact linear operator such that $\|Tx\|_Y + \|Kx\|_Y$ is an equivalent norm on X, then we are able to prove some crucial properties of T.

Definition 1.7.1 Let X and Y be normed vector spaces and $K \in \mathcal{L}(X, Y)$. The operator K is called compact if $K(B(0; 1))$ is relatively compact.

Lemma 1.7.2 *Let X and Y be normed vector spaces, $K \in \mathcal{L}(X, Y)$. The operator K is compact if and only if from each bounded sequence $\{x_n\} \subset X$ we can choose a subsequence $\{z_k\}$ such that $Kz_k \to y$ as $k \to \infty$ for some $y \in Y$.*

Proof This proposition is an easy consequence of Lemma 1.2.7.

Definition 1.7.3 Let X and Y be normed vector spaces, $K \in \mathcal{L}(X, Y)$. We say that K is of finite rank if $K(X)$ is a finite-dimensional vector space.

Clearly every operator of finite rank is compact.

Theorem 1.7.4 *Let X, Y, and Z be normed vector spaces, $P, T \in \mathcal{L}(X, Y)$, and $S \in \mathcal{L}(Y, Z)$. If one of the operators T, S is compact, then ST is a compact operator, too. If α, β are scalars and the operators P, T are compact, then the operator $\alpha P + \beta T$ is compact. If $T_n \in \mathcal{L}(X, Y)$ are compact, $T_n \to T$ as $n \to \infty$, then T is compact.*

(See [149], Chapter X, §2.)

Theorem 1.7.5 *Let X and Y be Banach spaces, $T \in \mathcal{L}(X, Y)$. Then T is compact if and only if T' is compact.*

(See [121], § 15, Theorem 4.)

Theorem 1.7.6 *Let X be a complex Banach space and $T \in \mathcal{L}(X)$ be a compact operator. If $\lambda \in C \setminus \{0\}$, then $(T - \lambda I)(X)$ is closed.*

(See [121], § 15, Theorem 7.)

Theorem 1.7.7 *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a compact operator. Then $\sigma(T)$ is at most countable. If $\lambda \in \sigma(T) \setminus \{0\}$, then λ is an isolated point of*

$\sigma(T)$ and

$$0 < \dim \operatorname{Ker}(T - \lambda I) = \dim \operatorname{Ker}(T' - \lambda I)$$

$$= \operatorname{codim}(T - \lambda I)(X) = \operatorname{codim}(T' - \lambda I)(X) < \infty.$$

(See [141], §5.5.)

Proposition 1.7.8 *Let X and Y be Banach spaces and $K \in \mathcal{L}(X, Y)$ be compact. If x_n converges weakly to x in X, then Kx_n converges to Kx in Y.*

(See [121], Chapter III, § 15, Corollary 6.)

Lemma 1.7.9 *Let X, Y, $Z_1, \ldots, Z_k$ be Banach spaces and $T \in \mathcal{L}(X, Y)$. Let $K_j \in \mathcal{L}(X, Z_j)$ be compact operators. Suppose that*

$$\|x\|_K = \|Tx\|_Y + \sum_{j=1}^{k} \|K_j x\|_{Z_j}$$

represents a norm on X equivalent to the original norm $\|x\|_X$. Then one has the following properties:

- *The kernel of T is finite dimensional.*
- *The range $T(X)$ of T is closed.*
- *There exists a constant C such that if W is a normed space and $L \in \mathcal{L}(X, W)$ satisfies $Lx = 0$ whenever $Tx = 0$, then one has $\|Lx\|_W \le C\|L\| \cdot \|Tx\|_Y$ for all $x \in X$.*
- *If W is a normed space and $M \in \mathcal{L}(X, W)$ satisfies $Mx \neq 0$ whenever $Tx = 0$ and $x \neq 0$, then $\|x\|_M \equiv \|Tx\|_Y + \|Mx\|_W$ represents a norm on X equivalent to the original norm.*

Proof Let $\{x_n\}$ be a sequence of points from $\operatorname{Ker} T$ such that $\|x_n\|_K = 1$. Since $K_1, \ldots, K_k$ are compact operators, we can choose a subsequence y_n of x_n such that $K_j y_n \to z_j \in Z_j$ as $n \to \infty$ (see Lemma 1.7.2). Since $\{K_j y_n\}$ is a Cauchy sequence and

$$\| y_n - y_l\|_K = \|Ty_n - Ty_l\|_X + \sum_{j=1}^{k} \|K_j y_n - K_j y_l\|_{Z_j} = \sum_{j=1}^{k} \|K_j y_n - K_j y_l\|_{Z_j},$$

the sequence y_n is a Cauchy sequence of the Banach space X. Therefore there is $y \in X$ such that $y_n \to y$ as $n \to \infty$. Since $\operatorname{Ker} T$ is a closed subspace of X by Lemma 1.4.11, we have $y \in \operatorname{Ker} T$. Since $\{x \in \operatorname{Ker} T; \|x\|_K = 1\}$ is a compact set (see Lemma 1.2.7), Lemma 1.3.11 gives that $\operatorname{Ker} T$ is a finite-dimensional space.

Suppose now that $M \in \mathcal{L}(X, W)$ satisfies $Mx \neq 0$ whenever $Tx = 0$ and $x \neq 0$. If $x \in X$, then $\|x\|_M \le [\|M\| + \|T\|]\|x\|_X$. Suppose that there exist $x_n \in X$ with $\|x_n\|_K = 1$ such that $\|x_n\|_M \le 1/n$. Then $\|Tx_n\| \le 1/n$. Since $\{x_n\}$ is a bounded sequence and K_j are compact operators, we can suppose that $K_j x_n \to z_j \in Z_j$. (We can choose a subsequence of x_n.) If $l \geq n$ then

$$\|x_l - x_n\|_K \leq \frac{2}{n} + \sum_{j=1}^{k} \|K_j x_n - K_j x_l\|_{Z_j}.$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is a Banach space, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Then

$$\|x\|_K = \lim_{n\to\infty} \|x_n\|_K = 1, \quad \|x\|_M = \lim_{n\to\infty} \|x_n\|_M = 0.$$

Thus $x \neq 0$ and $Tx = Mx = 0$, what is a contradiction. Hence there is a positive constant α such that $\|x\|_M \geq \alpha\|x\|_K$ for each $x \in X$ and $\|\cdot\|_M$ represents a norm on X equivalent to the original norm.

Ker T is a closed subspace of X by Lemma 1.4.11. Since Ker T is a finite dimensional subspace of X, there is a closed subspace V of X such that X is the direct sum of Ker T and V (see Proposition 1.3.9). Denote by P the projection of X onto V along Ker T. Then $I - P \in \mathcal{L}(X, \operatorname{Ker} T)$ by Corollary 1.4.9 and satisfies $(I - P)x \neq 0$ whenever $Tx = 0$ and $x \neq 0$. Therefore the norms $\|x\|_X$ and $\|x\|_{(I-P)} = \|Tx\|_Y + \|(I - P)x\|_X$ are equivalent. Let now $Tx_n \to y \in Y$ as $n \to \infty$. Then $\|Px_k - Px_j\|_{(I-P)} = \|TPx_n - TPx_j\|_Y + \|(I-P)Px_n - (I-P)Px_j\|_X = \|Tx_n - Tx_j\|$, and the sequence Px_k is a Cauchy sequence. Since X is a Banach space, there is $x \in X$ such that $Px_n \to x$ as $n \to \infty$. Since

$$Tx = \lim_{n\to\infty} TPx_n = \lim_{k\to\infty} Tx_n = y$$

we deduce that the range of T is closed. Since T is one-to-one bounded linear operator from the Banach space X onto the Banach space $T(X)$, Theorem 1.6.3 gives that there is $\beta > 0$ such that $\|Tx\|_Y \geq \beta\|x\|_X$ for each $x \in V$.

Suppose now that $L \in \mathcal{L}(X, W)$ satisfies $Lx = 0$ whenever $Tx = 0$. If $z \in X$, then $\|Lz\|_W = \|LPz\|_W \leq \|L\| \cdot \|Pz\|_X \leq \|L\| \cdot \beta^{-1}\|TPz\|_Y = \beta^{-1}\|L\| \cdot \|Tz\|_Y$.

Lemma 1.7.10 *Let X, Y, Z, and W be reflexive Banach spaces and* $T \in \mathcal{L}(X, Y)$*. Let* $K \in \mathcal{L}(X, Z)$ *and* $Q \in \mathcal{L}(X, W)$ *be compact operators. Suppose that* $\|x\|_K = \|Tx\|_Y + \|Kx\|_Z$ *represents a norm on X equivalent to the original norm* $\|x\|_X$ *and* Ker$(K) \subset$ Ker(Q). *Then for every* $\epsilon > 0$ *there exists* $C(\epsilon) > 0$ *such that*

$$\|Qx\|_W \leq \epsilon\|Tx\|_Y + C(\epsilon)\|Kx\|_Z, \qquad \forall x \in X.$$

Proof Suppose the opposite. Then there are $\epsilon > 0$ and $x_n \in X$ with $\|x_n\|_K = 1$ such that

$$\|Qx_n\|_W > \epsilon\|Tx_n\|_Y + n\|Kx_n\|_Z, \qquad \forall x \in X.$$

Since Qx_n is bounded, we infer that $Kx_n \to 0$ in Z. Since $\|x_n\|_K = 1$, we obtain $\|Tx_n\|_Y \to 1$ as $n \to \infty$. According to Theorem 1.5.10, we can suppose that x_n converges weakly to $x \in X$. Proposition 1.7.8 gives that $Kx_n \to Kx$. Thus $Kx = 0$. The relation $\mathrm{Ker}(K) \subset \mathrm{Ker}(Q)$ forces $Qx = 0$. Since $Qx_n \to Qx$ by Proposition 1.7.8, we have

$$\|Qx\|_W = \lim_{n\to\infty} \|Qx_n\|_W \geq \lim_{n\to\infty} \epsilon \|Tx_n\|_Y = \epsilon > 0,$$

what is a contradiction.

1.8 Fredholm Operators

A generalization of operators of the type $\lambda I - K$ for $\lambda \neq 0$ and $K : X \to X$ compact are Fredholm operators and more generally semi-Fredholm operators. We study these operators. In particular we show that if $T_j : X_j \to Y_j$ are Fredholm operators with the same index, X_1 is a dense subset of X_2, Y_1 is a dense subset of Y_2, $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$, then T_1 and T_2 have the same kernel.

Definition 1.8.1 Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Denote $\alpha(T) = \dim \mathrm{Ker}\, T$, $\beta(T) = \dim \mathrm{Ker}\, T'$. We say that:

1. T is upper semi-Fredholm if $T(X)$ is a closed and $\alpha(T) < \infty$.
2. T is lower semi-Fredholm if $\beta(T) < \infty$.
3. T is Fredholm if $\alpha(T) < \infty$ and $\beta(T) < \infty$.

If T is semi-Fredholm, then the index of T is defined by $i(T) = \alpha(T) - \beta(T)$.

Theorem 1.8.2 *Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ with closed range. Then $\alpha(T') = \beta(T)$ and $\beta(T') = \alpha(T)$. Thus:*

- *T is a Fredholm operator if and only if T' is a Fredholm operator.*
- *T is an upper semi-Fredholm operator if and only if T' is a lower semi-Fredholm operator.*
- *T is a lower semi-Fredholm operator if and only if T' is an upper semi-Fredholm operator.*

If T is semi-Fredholm, then $i(T') = -i(T)$.

(See [121], § 16, Theorem 4.)

Theorem 1.8.3 *Let X and Y be Banach spaces, $T, K \in \mathcal{L}(X, Y)$, and K be a compact operator. If T is a semi-Fredholm operator, then $T + K$ is a semi-Fredholm operator and $i(T + K) = i(T)$.*

(See [121], § 16, Theorem 16.)

Lemma 1.8.4 *Let X_1, X_2, Y_1, Y_2 be Banach spaces. Let $T_j \in \mathcal{L}(X_j, Y_j)$ be Fredholm operators, $i(T_1) = i(T_2)$. Let $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$, X_1 be a dense subset of X_2, and Y_1 be a dense subset of Y_2. Let $T_1 = T_2$ on X_1. Then* $\operatorname{Ker} T_1 = \operatorname{Ker} T_2$. *If $x \in X_2$ and $T_2 x = y$, then $x \in X_1$ if and only if $y \in Y_1$. The operator T_1 is an isomorphism if and only if T_2 is an isomorphism.*

Proof Clearly, $X_2' \subset X_1'$, $Y_2' \subset Y_1'$, and $T_2' = T_1'$ on Y_2'. Since $\operatorname{Ker} T_2' \subset \operatorname{Ker} T_1'$, we have $\beta(T_2) \le \beta(T_1)$. On the other hand, $\operatorname{Ker} T_1 \subset \operatorname{Ker} T_2$, and thus $\alpha(T_1) \le \alpha(T_2)$. From $\alpha(T_1) - \beta(T_1) = \alpha(T_2) - \beta(T_2)$, we infer that $\alpha(T_1) = \alpha(T_2)$, $\beta(T_1) = \beta(T_2)$, and thus $\operatorname{Ker} T_1 = \operatorname{Ker} T_2$, $\operatorname{Ker} T_1' = \operatorname{Ker} T_2'$.

Let $T_2 x = y$. If $x \in X_1$, then $T_2 x = T_1 x \in Y_1$. Let now $y \in Y_1$. We have $f(y) = 0$ for each $f \in \operatorname{Ker}(T_2')$ by Proposition 1.5.13. Since $\operatorname{Ker} T_1' = \operatorname{Ker} T_2'$, Proposition 1.5.13 gives that there exists $z \in Y_1$ such that $T_1 z = y$. Since $x - z \in \operatorname{Ker} T_2 = \operatorname{Ker} T_1 \subset X_1$ we infer that $x = z + (x - z) \in X_1$.

If T_1 or T_2 is an isomorphism, then $\operatorname{Ker} T_1 = \operatorname{Ker} T_2 = \{0\}$. Since T_1, T_2 are Fredholm operators with index 0 they are onto. Theorem 1.6.3 gives that T_1, T_2 are isomorphisms.

Proposition 1.8.5 *Let X and Y be Banach spaces. The index is constant on every component of connectivity of the set of semi-Fredholm operators from $\mathcal{L}(X, Y)$.*

(See [121],§ 18, Corollary 3.)

Theorem 1.8.6 *Let X, Y, and Z be Banach spaces, $T \in \mathcal{L}(X, Y)$, and $S \in \mathcal{L}(Y, Z)$ be Fredholm operators. Then $ST \in \mathcal{L}(X, Z)$ is a Fredholm operator and $i(ST) = i(S) + i(T)$.*

(See [128], Theorem 5.7.)

Lemma 1.8.7 *Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ be Fredholm. Then* $\operatorname{codim} T(X) = \beta(T) < \infty$.

Proof $T(X) = \{x \in X; f(x) = 0\ \forall f \in \operatorname{Ker} T'\}$ by Proposition 1.5.13. If $\operatorname{Ker} T' = \{0\}$, then $T(X) = X$ and $\operatorname{codim} T(X) = \beta(T) = 0$.

Let now $k = \beta(T) > 0$. Let $f_1, \dots, f_k$ be a basis of $\operatorname{Ker} T'$. First we show that $\operatorname{codim} T(X) \le \beta(T)$. If it is not true, then there exists a subspace Z of X with basis $z_1, \dots, z_{k+1}$ such that $Z \cap T(X) = \{0\}$. Denote $a_i^j = f_i(z_j)$, $a^j = [a_1^j, \dots, a_k^j]$. The vectors $a^1, \dots, a^{k+1}$ are linearly dependent. Therefore there exist scalars c_j such that $|c_1| + \dots + |c_{k+1}| > 0$ and $c_1 a^1 + \dots + c_{k+1} a^{k+1} = 0$. Put $z = c_1 z_1 + \dots + c_{k+1} z_{k+1}$. Then $z \ne 0$, $z \in Z$. But $f_j(z) = c_1 f_j(z_1) + \dots + c_{k+1} f_j(z_{k+1}) = c_1 a_j^1 + \dots c_{k+1} a_j^{k+1} = 0$. Thus $f(z) = 0$ for all $f \in \operatorname{Ker} T'$. Proposition 1.5.13 gives that $z \in T(X)$. Since $Z \cap T(X) = \{0\}$, we deduce that $z = 0$, what is a contradiction.

Since $\operatorname{codim} T(X) \le k$, there exists a subspace Z of X with a basis $z_1, \dots, z_n$ such that $X = Z \oplus T(X)$. We have proved that $n \le k$. Suppose that $n < k$. Denote $b_i^j = f_j(z_i)$, $b^j = [b_1^j, \dots, b_n^j]$. The vector $b^1, \dots, b^k$ are linearly dependent. Therefore there exist scalars $c_1, \dots, c_k$ with $|c_1| + \dots + |c_k| > 0$ such that $c_1 b^1 + \dots + c_k b^k = 0$. Put $f = c_1 f_1 + \dots + c_k f_k$. Then $f \in \operatorname{Ker} T'$, $f \ne 0$. We have $f(z_i) = c_1 f_1(z_i) + \dots c_k f_k(z_i) = c_1 b_i^1 + \dots + c_k b_i^k = 0$. If $x \in T(X)$, then

$f(x) = 0$ by Proposition 1.5.13. Since $X = Z \oplus T(X)$, we deduce that $f = 0$, what is a contradiction.

Lemma 1.8.8 *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a Fredholm operator. Suppose that $\tilde{X}$ is a closed subspace of X of finite codimension and $T(\tilde{X}) \subset \tilde{X}$. Denote by $\tilde{T}$ the restriction of T onto $\tilde{X}$. Then $\tilde{T}$ is a Fredholm operator and $i(\tilde{T}) = i(T)$. If T is an isomorphism, then $\tilde{T}$ is an isomorphism.*

Proof According to Lemma 1.8.7, there exists a subspace Y of X such that $X = \tilde{X} \oplus Y$ and $\dim Y = \beta(T)$. Let P be the projection of X onto $\tilde{X}$ along Y (see Lemma 1.4.8). Clearly, P is a Fredholm operator with $i(P) = 0$. Theorem 1.8.6 gives that TP is a Fredholm operator and $i(TP) = i(T)$. So, $\tilde{T}(\tilde{X}) = TP(X)$ is closed. Clearly, $\alpha(TP) = \alpha(\tilde{T}) + \dim Y$. Since $\beta(\tilde{T})$ is a codimension of $\tilde{T}(\tilde{X})$ in $\tilde{X}$, and $\beta(TP)$ is the codimension of $TP(X) = T(\tilde{X})$ in X (see Lemma 1.8.7), we obtain $\beta(TP) = \beta(\tilde{T}) + \dim Y$. This forces $i(\tilde{T}) = i(TP) = i(T)$.

Let T now be an isomorphism. Since the kernel of $\tilde{T}$ is trivial and $i(\tilde{T}) = 0$, we infer that $\tilde{T}(\tilde{X}) = \tilde{X}$. Theorem 1.6.3 gives that $\tilde{T}$ is an isomorphism.

1.9 Hilbert Spaces

Hilbert spaces are Banach spaces endowed by a scalar product. These spaces have better properties than other Banach spaces. We gather essential results for Hilbert spaces and continuous linear operators on them, especially Schwarz's inequality, theorem on orthogonal projection, and Riesz representation theorem. We also study spectral properties of normal operators, self-adjoint operators, and positive operators on Hilbert spaces.

Definition 1.9.1 Let H be a vector space over the real or complex numbers. An inner product on H is a scalar valued function $(\cdot, \cdot)$ defined on the Cartesian product $H \times H$ with the following properties:

- $(\alpha u, v) = \alpha(u, v)$ for $u, v \in H$ and a scalar α,
- $(u + v, w) = (u, w) + (v, w)$ for $u, v \in H$,
- $(v, u) = \overline{(u, v)}$ for $u, v \in H$,
- $(u, u) > 0$ whenever $u \neq 0$.

H, together with an inner product, is called inner-product space.

Theorem 1.9.2 *Given inner-product space H, $\|u\| = \sqrt{(u, u)}$ defines a norm on H.*

(See [57], Theorem I.7.5.)

Proposition 1.9.3 (Schwarz's Inequality) *Let H be an inner-product space. Then for all u and $v \in H$*

$$|(u, v)| \leq \|u\| \|v\|.$$

Equality holds if and only if u and v are linearly dependent.

(See [57], Theorem I.7.4.)

Definition 1.9.4 A Hilbert space is an inner-product space which is also a Banach space with the norm $\|u\| = \sqrt{(u, u)}$.

Definition 1.9.5 Let H be a Hilbert space and $M \subset H$. We define the orthogonal complement of M via $M^{\perp} = \{x \in H; (x, y) = 0 \text{ for all } y \in M\}$.

Clearly, for an arbitrary $M \subset H$, its orthogonal complement $M^{\perp}$ is closed.

Theorem 1.9.6 (Theorem on Orthogonal Projection) *Let X be a closed linear subspace of the Hilbert space H. Then each element $z \in H$ possesses the following representation:*

$$z = x + y, \qquad x \in X, y \in X^{\perp}.$$

Here the elements x and y are uniquely determined.

(See [41], Theorem 3.21.)

Corollary 1.9.7 *If M is a subspace of the Hilbert space H, then $(M^{\perp})^{\perp} = \overline{M}$.*

(See [41], Corollary 3.22.)

Definition 1.9.8 A subset $S = \{x_\alpha; \alpha \in A\}$ of a Hilbert space H is called orthonormal if $(x_\alpha, x_\beta) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in A$. An orthonormal set S is called complete (or an orthonormal basis of H) if $\{a_1x_1 + \cdots + a_kx_k; x_1, \ldots, x_k \in S, a_1, \ldots, a_k \text{ scalars}\}$ is dense in H.

Theorem 1.9.9 *Let $\{x_j\}$, $j = 1, 2, \ldots$ represent a complete orthonormal sequence in a Hilbert space H. If $x, y \in H$, then*

$$\sum_{j=1}^{\infty} |(x, x_j)|^2 = \|x\|^2,$$

$$x = \sum_{j=1}^{\infty} (x, x_j)x_j,$$

$$(x, y) = \sum_{j=1}^{\infty} (x, x_j)(y, x_j).$$

(See [128], Chapter 1, Theorem 1.6 and [128], Chapter 1, Theorem 1.7.)

Theorem 1.9.10 (Riesz Representation Theorem) *Let H be a Hilbert space and $L \in H'$. Then there exists unique $y \in H$ such that $Lx = (x, y)$ for all $x \in H$. Moreover, $\|y\|_H = \|L\|_{H'}$. On the other hand, if $y \in H$ and we denote $L_y(x) = (x, y)$ for $x \in H$, then $L_y \in H'$ and $\|L_y\|_{H'} = \|y\|_H$.*

(See [149], Chapter III, §6.)

Definition 1.9.11 Let H be a Hilbert space. A bilinear form (or sesquilinear functional) $a(x, y)$ on H is an assignment of a scalar to each pair of vectors $x, y \in H$ in such a way that:

- $a(\gamma x + \beta y, z) = \gamma a(x, z) + \beta a(y, z)$ for $x, y, z \in H$ and scalars γ, β,
- $a(z, \gamma x + \beta y) = \overline{\gamma} a(z, x) + \overline{\beta} a(z, y)$ for $x, y, z \in H$ and scalars γ, β.

We say that a is symmetric if $a(x, y) = \overline{a(y, x)}$ for each $x, y \in H$. We say that a is bounded if there is a constant M such that $|a(x, y)| \le M\|x\|\|y\|$ for each $x, y \in H$.

If a is a bounded bilinear form and $x_n \to x$ and $y_n \to y$, then $a(x_n, y_n) \to a(x, y)$, because

$$|a(x, y) - a(x_n, y_n)| \le |a(x, y - y_n)| + |a(x - x_n, y_n)|.$$

Definition 1.9.12 If T is a bounded linear operator on a Hilbert space H, then the adjoint of T, denoted T^*, is a linear operator on H satisfying $(Tx, y) = (x, T^*y)$ for $x, y \in H$.

Theorem 1.9.13 *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then there is unique T^*. Moreover:*

- $T^* \in \mathcal{L}(H)$ *and* $\|T^*\| = \|T\|$.
- $(T^*)^{-1} = (T^{-1})^*$ *for invertible* $T \in \mathcal{L}(H)$.
- $\|T\|^2 = \|T^*T\|$ *for* $T \in \mathcal{L}(H)$.

(See [141], §4.9.)

Lemma 1.9.14 *Let H be a Hilbert space. If $T, S \in \mathcal{L}(H)$, then $(TS)^* = S^*T^*$.*

(See [140], §4.9.)

Definition 1.9.15 Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then:

- T is normal if $\|T^*x\| = \|Tx\|$ for all $x \in H$;
- T is self-adjoint if $T = T^*$;
- T is positive if $(Tx, x) \ge 0$ for $x \in H$.

Proposition 1.9.16 *Let H be a Hilbert space with an inner product $(\ ,\ )$ and $T \in \mathcal{L}(H)$ be self-adjoint. If $x \in H$, then $(Tx, x) \in R$. Put*

$$m(T) = \inf\{(Tx, x); x \in H, \|x\| = 1\},$$

$$M(T) = \sup\{(Tx, x); x \in H, \|x\| = 1\}.$$

Then $m(T), M(T) \in \sigma(T)$. If $\lambda \in \sigma(T)$, then $\lambda \in R$ and $m(T) \le \lambda \le M(T)$. Moreover,

$$\|T\| = r(T) = \max(|m(T)|, |M(T)|).$$

If $x \in H$, $\|x\| = 1$ *and* $\lambda = (Tx, x)$ *with* $\lambda = m(T)$ *or* $\lambda = M(T)$, *then* $Tx = \lambda x$.

(See [141], § 6.11 and § 6.2.)

Proposition 1.9.17 *Let* H *be a Hilbert space with an inner product* $(\ ,\ )$ *and* $T \in \mathcal{L}(H)$ *be self-adjoint. If* $x, y \in H$, $Tx = \lambda x$, $Ty = \mu y$, *and* $\lambda \neq \mu$, *then* $(x, y) = 0$.

(See [141], Theorem 6.11-A.)

Proposition 1.9.18 *Let* H *be a Hilbert space with an inner product* $(\ ,\ )$, $T \in \mathcal{L}(H)$ *be compact and self-adjoint, and* $T \neq 0$. *Then there exists an orthonormal set* $\{x_j; j \in J\}$ *with* J *at most countable such that* $Tx_j = \lambda_j x_j$ *for* $j \in J$ *and*

$$Tx = \sum_{j \in J} \lambda_j (x, x_j) x_j, \qquad \forall x \in X.$$

If J *is infinite, then* $\lambda_j \to 0$ *as* $j \to \infty$.

(See [141], Theorem 6.4-B and [141], Theorem 6.11-A.)

Corollary 1.9.19 *Let* H *be an infinite-dimensional Hilbert space and* $T \in \mathcal{L}(H)$ *be compact, self-adjoint, and one-to-one. Then there exists a sequence of eigenvalues* $\{\lambda_j\}$ *with* $\lambda_j \to 0$ *and a sequence of corresponding eigenfunctions* $\{x_j\}$ *which forms a complete orthonormal system in* H. *If* λ *is an eigenvalue, then* $\lambda \in \{\lambda_j\}$ *and* $\{u_j; \lambda_j = \lambda\}$ *is a basis of* $\operatorname{Ker}(T - \lambda I)$.

Proof Let $\{\lambda_j\}$ and $\{x_j\}$ be from Proposition 1.9.18. If $\{x_j\}$ is not complete, then there exists a nontrivial $x \in H$ such that $(x, x_j) = 0$ for all j. But for such x, we have $Tx = 0$, what is a contradiction.

Clearly, $Tx_j = \lambda_j x_j$. Let λ now be an eigenvalue with an eigenfunction x. If $\lambda \notin \{\lambda_j\}$, then $(x, x_j) = 0$ for all j by Proposition 1.9.17. Thus $Tx = 0$ what is a contradiction. So, $\lambda \in \{\lambda_j\}$. Denote $L = \{j; \lambda_j = \lambda\}$. Then $(x, x_j) = 0$ for $j \notin L$ by Proposition 1.9.17. Thus

$$\lambda x = \lambda \sum_{j \in L} (x, x_j) x_j.$$

Since $\lambda \neq 0$, we deduce that $\{u_j; j \in L\}$ is a basis of $\operatorname{Ker}(T - \lambda I)$.

Proposition 1.9.20 *Let* H *be a complex Hilbert space and* $T \in \mathcal{L}(H)$. *Then* T *is self-adjoint if and only if* (Tx, x) *is real for* $x \in H$.

(See [41], Proposition 4.12.)

Proposition 1.9.21 *Let* H *be a Hilbert space and* $T \in \mathcal{L}(H)$ *be normal. Then* $\operatorname{Ker} T = T(X)^{\perp}$ *and* $X = \operatorname{Ker} T \oplus \overline{T(X)}$.

(See [141], Theorem 6.2-G.)

1.10 Interpolation Spaces

The interpolation theory is very useful in studying solutions of boundary value problems. If we know that some boundary value problem is uniquely solvable in some space of functions X and in a larger space of functions Y, then the interpolation theory allows us to deduce that this boundary value problem is uniquely solvable in an intermediate space of functions Z.

Let E_0 and E_1 be two normed vector spaces over the same scalars which are subspaces of a vector space E, $E_0 \hookrightarrow E$, and $E_1 \hookrightarrow E$. (We say that (E_0, E_1) is a compatible couple of normed spaces.) Then $E_0 \cap E_1$ is equipped with the norm

$$\|a\|_{E_0 \cap E_1} = \max(\|a\|_{E_0}, \|a\|_{E_1}),$$

and $E_0 + E_1 = \{a_0 + a_1; a_0 \in E_0, a_1 \in E_1\}$ is equipped with the norm

$$\|a\|_{E_0+E_1} = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in E_0, a_1 \in E_1}} (\|a_0\|_{E_0} + \|a_1\|_{E_1}).$$

If E_0 and E_1 are Banach spaces, then $E_0 \cap E_1$ and $E_0 + E_1$ are Banach spaces, too (see [16], Chapter 3, Theorem 1.3).

For $a \in E_0 + E_1$ and $t > 0$, one defines

$$K(t, a) = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in E_0, a_1 \in E_1}} (\|a_0\|_{E_0} + t\|a_1\|_{E_1}).$$

Denote $\mu = t^{-1}\mathcal{H}_1$ on $(0, \infty)$. For $0 < \theta < 1$ and $1 \le p \le \infty$ (or for $p = \infty$ and $\theta \in \{0, 1\}$), one writes

$$\|a\|_{(E_0,E_1)_{\theta,p}} = \|t^{-\theta} K(t, a)\|_{L^p(\mu)}, \qquad a \in E_0 + E_1,$$

$$(E_0, E_1)_{\theta,p} = \{a \in E_0 + E_1; \|a\|_{(E_0,E_1)_{\theta,p}} < \infty\}.$$

Proposition 1.10.1 *Suppose that (E_0, E_1) is a compatible couple of Banach spaces. Let Y, Z be closed subspaces of $E_0 + E_1$ such that $E_0 + E_1 = Y \oplus Z$. Suppose that the projection of $E_0 + E_1$ onto Y along Z belongs to $\mathcal{L}(E_0)$ and to $\mathcal{L}(E_1)$. Then $(E_0 \cap Y, E_1 \cap Y)$ is a compatible couple of Banach spaces. If $0 < \theta < 1$ and $1 \le p \le \infty$, then $(E_0 \cap Y, E_1 \cap Y)_{\theta,p} = (E_0, E_1)_{\theta,p} \cap Y$.*

(See [144], §1.17.1, Theorem 1.)

Proposition 1.10.2 *Suppose that (E_0, E_1) is a compatible couple of Banach spaces. Let Y, Z be closed subspaces of $E_0 + E_1$ such that $E_0 + E_1 = Y \oplus Z$. Suppose that the projection of $E_0 + E_1$ onto Y along Z belongs to $\mathcal{L}(E_0)$ and to $\mathcal{L}(E_1)$. Then*

$(E_0/E_0 \cap Z, E_1/E_1 \cap Z)$ is a compatible couple of Banach spaces. If $0 < \theta < 1$ and $1 \le p \le \infty$, then $(E_0/E_0 \cap Z, E_1/E_1 \cap Z)_{\theta,p} = (E_0, E_1)_{\theta,p}/(E_0, E_1)_{\theta,p} \cap Z$.

(See [144], §1.17.2, Theorem.)

Proposition 1.10.3 *Suppose that (E_0, E_1) and (F_0, F_1) are compatible couples of normed vector spaces and $0 < \theta < 1$, $1 \le p \le \infty$. If T is a linear operator from $E_0 + E_1$ into $F_0 + F_1$ and maps E_j into F_j with*

$$\|Tx\|_{F_j} \le M_j \|x\|_{E_j}$$

for $j = 0, 1$, then T is a bounded linear operator from $(E_0, E_1)_{\theta,p}$ into $(F_0, F_1)_{\theta,p}$ with

$$\|Tx\|_{(F_0,F_1)_{\theta,p}} \le M_0^{1-\theta} M_1^{\theta} \|x\|_{(E_0,E_1)_{\theta,p}}.$$

(See [140], Lemma 22.3.)

Proposition 1.10.4 *Let X_0, X_1, Y_0, and Y_1 be Banach spaces. Let $X_0 \hookrightarrow X_1$, $Y_0 \hookrightarrow Y_1$, X_0 be a dense subset of X_1, and Y_0 be a dense subset of Y_1. Then $X_1 + X_0 = X_1$ and the corresponding norms are equivalent; $X_1 \cap X_0 = X_0$ and the corresponding norms are equivalent. Let $T_j \in \mathcal{L}(X_j, Y_j)$ be Fredholm operators, $i(T_1) = i(T_0)$. Let $T_1 = T_0$ on X_0. Fix $\theta \in (0, 1)$ and $1 \le p \le \infty$. Denote by $T_{\theta,p}$ the restriction of T_1 onto $(X_0, X_1)_{\theta,p}$. Then $T_{\theta,p} : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is a bounded Fredholm operator, $i(T_{\theta,p}) = i(T_1)$, $\operatorname{Ker} T_{\theta,p} = \operatorname{Ker} T_1 = \operatorname{Ker} T_0$, $T_{\theta,p}((X_0, X_1)_{\theta,p}) = (Y_0, Y_1)_{\theta,p} \cap T_1(X_1)$. If one of the operators T_0 and T_1 is an isomorphism, then T_0 and T_1 are isomorphisms, and $T_{\theta,p} : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is an isomorphism.*

Proof Since $X_0 \hookrightarrow X_1$, there exists a positive constant c such that $\|x\|_1 \le c\|x\|_0$ for all $x \in X_0$. If $x \in X_0 \cap X_1 = X_0$, then

$$\|x\|_0 \le \|x\|_{X_0 \cap X_1} \le \max(1, c)\|x\|_0.$$

If $x \in X_0 + X_1 = X_1$, then

$$\min(c, 1)\|x\|_1 \le \|x\|_{X_0 + X_1} \le max(c, 1)\|x\|_1.$$

$T_{\theta,p} : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is a bounded linear operator by Proposition 1.10.3. Suppose first that one of the operators T_0, T_1 is an isomorphism. Then T_0, T_1 are isomorphisms by Lemma 1.8.4. Clearly, $T_1^{-1} = T_0^{-1}$ on Y_0. So, $T_1^{-1} \in \mathcal{L}((Y_0, Y_1)_{\theta,p}, (X_0, X_1)_{\theta,p})$ by Proposition 1.10.3. Since $T_{\theta,p} T_1^{-1} = T_1 T_1^{-1} = I$, $T_1^{-1} T_{\theta,p} = T_1^{-1} T_1 = I$, the operator $T_{\theta,p} : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}$ is an isomorphism.

Let $T_j \in \mathcal{L}(X_j, Y_j)$ be Fredholm operators and $i(T_1) = i(T_0)$. Lemma 1.8.4 gives $\operatorname{Ker} T_0 = \operatorname{Ker} T_1$, $T_0(X_0) = Y_0 \cap T_1(X_1)$. Since $i(T_1) = i(T_0)$ and $\alpha(T_1) = \alpha(T_0)$, we have $\beta(T_1) = \beta(T_0)$. There exists a subspace Z of Y_0 such that $\dim Z = \beta(T_0)$ and $Y_0 = T_0(X_0) \oplus Z$. Since $Z \cap T_1(X_1) = Z \cap (X_0 \cap T_1(X_1)) = Z \cap T_0(X_0) = \{0\}$ and $\beta(T_1) = \dim Z$, we deduce that $Y_1 = T_1(X_1) \oplus Z$. The operator T_j is an isomorphism from $X_j / \operatorname{Ker} T_0$ onto $Y_j \cap T_1(X_1)$. The definition of the interpolation spaces gives $X_0 \subset (X_0, X_1)_\theta \subset X_1$, $Y_0 \subset (Y_0, Y_1)_\theta \subset Y_1$. By virtue of Propositions 1.10.1 and 1.10.2, we obtain that $T_{\theta,p}$ is an isomorphism from $(X_0, X_1)_{\theta,p} / \operatorname{Ker} T_0$ to $(Y_0, Y_1)_{\theta,p} \cap T_1(X_1)$. Hence $\operatorname{Ker} T_{\theta,p} = \operatorname{Ker} T_0 = \operatorname{Ker} T_1$, $(Y_0, Y_1)_{\theta,p} = T_{\theta,p}((X_0, X_1)_{\theta,p}) \oplus Z$, $T_{\theta,p}((X_0, X_1)_{\theta,p})$ is a closed subspace of $(Y_0, Y_1)_{\theta,p}$. Since $\beta(T_{\theta,p}) = \dim Z = \beta(T_1)$, the operator $T_{\theta,p}$ is Fredholm with $i(T_{\theta,p}) = i(T_1) = i(T_0)$. Lemma 1.8.4 gives $T_{\theta,p}((X_0, X_1)_{\theta,p}) = (Y_0, Y_1)_{\theta,p} \cap T_1(X_1)$.

Proposition 1.10.5 *Let E_0, E_1, F_0, and F_1 be normed spaces, $E_0 \subset E_1$, and $F_0 \subset F_1$. Let A be a possibly nonlinear operator from E_1 to F_1 such that $A(E_0) \subset F_0$. Suppose that*

$$\|Au - Av\|_{F_1} \le M_1 \|u - v\|_{E_1} \qquad \forall u, v \in E_1,$$

$$\|Au\|_{F_0} \le M_0 \|u\|_{E_0} \qquad \forall u \in E_0.$$

If $0 < \theta < 1$, $1 \le p \le \infty$, then A maps $(E_0, E_1)_{\theta,p}$ into $(F_0, F_1)_{\theta,p}$ and

$$\|Au\|_{(F_0,F_1)_{\theta,p}} \le C \|u\|_{(E_0,E_1)_{\theta,p}} \qquad \forall u \in (E_0, E_1)_{\theta,p}.$$

(See [140], Lemma 28.1.)

Lemma 1.10.6 *Let (E_0, E_1) be a compatible couple of normed vector spaces. If $0 < \theta < 1$ and $1 \le q \le \infty$, then $(E_0, E_1)_{\theta,q} = (E_1, E_0)_{1-\theta,q}$ (with equal norms).*

(See [17], Theorem 3.4.1.)

1.11 Measures

The theory of measure and integration is a fundamental tool in mathematics. We remember it in the next three sections. We begin with positive measures.

Definition 1.11.1 A system $\mathcal{A}$ of subsets of a set X is called σ-algebra if we have the following properties:

- $X \in \mathcal{A}$.
- With $B \in \mathcal{A}$, its complement satisfies $(X \setminus B) \in \mathcal{A}$ as well.
- For each sequence of sets $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{A}$ their denumerable union $\cup_{j=1}^{\infty} B_j$ belongs to $\mathcal{A}$ as well.

Definition 1.11.2 Let X be a metric space. The σ-algebra of Borel sets is the smallest σ-algebra containing all open subsets of X.

Definition 1.11.3 Let X be a set with a σ-algebra $\mathcal{A}$. We name the function $\mu : \mathcal{A} \to \langle 0, \infty \rangle$ on a σ-algebra $\mathcal{A}$ a nonnegative measure if the following conditions are fulfilled:

- $\mu(\emptyset) = 0$.
- For all mutually disjoint sets $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{A}$, we have

$$\mu\Bigl(\bigcup_{j=1}^{\infty} B_j\Bigr) = \sum_{j=1}^{\infty} \mu(B_j).$$

We call this measure finite if $\mu(X) < \infty$ holds true.

Definition 1.11.4 Let X be a set with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ on $\mathcal{A}$. A set $A \subset X$ is named null set if $A \in \mathcal{A}$ and $\mu(A) = 0$. A property holds true μ almost everywhere in X (symbolically μ-a.e.), if there exists a null set $A \subset X$ such that this property is valid for all points $x \in X \setminus A$. A measure μ is called complete if any subset of a μ-null set lies in $\mathcal{A}$.

Let X be a set with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ on $\mathcal{A}$. Put $\mathcal{B} = \{A \cup M; A \in \mathcal{A}, M \subset B \in \mathcal{A}, \mu(B) = 0\}$. Then $\mathcal{B}$ is a σ-algebra. If $A \in \mathcal{A}, M \subset B \in \mathcal{A}, \mu(B) = 0\}$ put $\tilde{\mu}(A \cup M) = \mu(A)$. Then $\tilde{\mu}$ is a complete nonnegative measure on $\mathcal{B}$ such that $\tilde{\mu} = \mu$ on $\mathcal{A}$.

Definition 1.11.5 Let X be a set with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ on $\mathcal{A}$. We say that μ is finite if $\mu(X) < \infty$. We say that μ is σ-finite if there exist a subsequence of sets $X_j \in \mathcal{A}$ such that $\mu(X_j) < \infty$ and $X = \cup X_j$.

Definition 1.11.6 Let X be a metric space with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ on $\mathcal{A}$. Suppose that $\mathcal{A}$ contains all Borel sets. We say that μ is regular if, for every $B \in \mathcal{A}$,

$$\mu(B) = \inf\{\mu(G); B \subset G, G \text{ open}\} = \sup\{\mu(K); K \subset B, K \text{ compact}\}.$$

If every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$, we say μ is locally finite. Finally, if $\mathcal{A}$ is the system of all Borel sets, we call μ the Borel measure on X.

Definition 1.11.7 Let μ be a nonnegative Borel measure on a metric space X. Then the support of μ is the closed set $\operatorname{spt} \mu = X \setminus \cup\{V \text{ open}; \mu(V) = 0\}$.

Example 1.11.8 For fixed a in a metric space X, define

$$\delta_a(A) = \begin{cases} 1, & a \in A, \\ 0, & a \notin A, \end{cases} \tag{1.1}$$

for $A \subset X$. Then δ_a is the Dirac measure on X with support $\{a\}$.

Definition 1.11.9 Suppose X is a σ-compact metric space with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ. Suppose that $\mathcal{A}$ contains all Borel sets. We say that μ is a Radon measure if it is regular and locally finite.

Notation 1.11.10 *Let X be a set with a σ-algebra $\mathcal{A}$ and a nonnegative measure μ on $\mathcal{A}$. Fix $A \in \mathcal{A}$. Define on $\mathcal{A}$ the nonnegative measure $\tilde{\mu}$ by $\tilde{\mu}(B) = \mu(A \cap B)$. Then $\tilde{\mu}$ is called the restriction of μ onto A and denoted by $\mu|A$.*

Hausdorff Measures 1.11.11 Let $s \geq 0$ be fixed. Fix a positive constant $\alpha(s)$. In case s is a positive integer, suppose that $\alpha(s)$ is the volume of the unit ball in R^s. For each $\epsilon > 0$ and a set $E \subset R^m$, let

$$\mathcal{H}_{s,\epsilon}(E) = \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) 2^{-s} \operatorname{diam}(A_j)^s ; E \subset \cup_{j=1}^{\infty} A_j, \operatorname{diam} A_j < \epsilon \right\}.$$

Because $\mathcal{H}_{s,\epsilon}(E)$ is nondecreasing in ϵ, we may define the s-dimensional Hausdorff measure of E as

$$\mathcal{H}_s(E) = \lim_{\epsilon \to 0} \mathcal{H}_{s,\epsilon}(E).$$

Put $\mathcal{A} = \{E \subset R^m; \mathcal{H}_s(D) = \mathcal{H}_s(D \cap E) + \mathcal{H}_s(D \setminus E)$ for every $D \subset R^m\}$. Then $\mathcal{A}$ is a σ-algebra containing all Borel sets, and $\mathcal{H}_s$ is a complete nonnegative measure on $\mathcal{A}$ (see [5], Chapter IX, Theorem 4.2 and [5], Chapter IX, Theorem 4.3). Remark that $\mathcal{H}_m$ is the Lebesgue measure in R^m (see [150], Theorem 1.4.2). The Lebesgue measure is regular (see [5], Chapter IX, Theorem 5.4). If $0 \leq s < r < \infty$ and $\mathcal{H}_s(A) < \infty$, then $\mathcal{H}_r(A) = 0$ (see [49], Chapter 2, § 2.10).

Definition 1.11.12 Let X be a set with a σ-algebra $\mathcal{A}$. Let ν, μ be nonnegative measures on $\mathcal{A}$. We say that ν is absolutely continuous with respect to μ if it holds

$$A \in \mathcal{A}, \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

1.12 Real and Complex Measures

This section is devoted to finite real and complex measures. We explain the relation between these measures and positive measures.

Definition 1.12.1 Let X be a set with a σ-algebra $\mathcal{A}$. We name the function $\mu : \mathcal{A} \to \langle -\infty, \infty \rangle$ on a σ-algebra $\mathcal{A}$ a real measure if the following conditions are fulfilled:

- $\mu(\emptyset) = 0$.
- μ takes on only one of the two values $+\infty$, $-\infty$.

- For all mutually disjoint sets $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{A}$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mu(B_j).$$

Theorem 1.12.2 (Jordan Decomposition) *Let μ be a real measure on a measurable space $(X, \mathcal{A})$. Then there are two disjoint sets $G^+, G^- \in \mathcal{A}$ such that:*

- $X = G^+ \cup G^-$.
- $M \in \mathcal{A}, M \subset G^+ \Rightarrow \mu(M) \geq 0$;
- $M \in \mathcal{A}, M \subset G^- \Rightarrow \mu(M) \leq 0$.

Denote $\mu^+(M) = \mu(M \cap G^+)$, $\mu^-(M) = -\mu(M \cap G^-)$ for all $M \in \mathcal{A}$. Then μ^+ and μ^- are nonnegative measures on $\mathcal{A}$ and $\mu(M) = \mu^+(M) - \mu^-(M)$ for each $M \in \mathcal{A}$.

(See [62], Chapter VI, §29.)

We write $\mu = \mu^+ - \mu^-$. The total variation $|\mu|$ of μ is defined by $|\mu| = \mu^+ + \mu^-$. We denote $\operatorname{spt} \mu = \operatorname{spt} |\mu|$ the support of μ. Remark that one of the measures μ^+, μ^- is finite. If both μ^+ and μ^- are finite measures, μ is said to be of bounded variation on X, in which case the norm of μ is defined by $\|\mu\| = |\mu|(X)$. We say that μ is a Radon measure if μ^+, μ^- are Radon measures.

Definition 1.12.3 Let X be a set with a σ-algebra $\mathcal{A}$. We name the function $\mu : \mathcal{A} \to C$ on a σ-algebra $\mathcal{A}$ a complex measure if the following conditions are fulfilled:

- $\mu(\emptyset) = 0$.
- For all mutually disjoint sets $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{A}$, we have

$$\mu\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mu(B_j).$$

Definition 1.12.4 Let X be a set with a σ-algebra $\mathcal{A}$. Let μ be a nonnegative measure on $\mathcal{A}$ and ν be a real or complex measure on $\mathcal{A}$. We say that ν is absolutely continuous with respect to μ if it holds

$$A \in \mathcal{A}, \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

1.13 Integrals

In this section, we show how to construct integrals on the basis of the measure theory. Then we state the basic propositions: Lusin's theorem, Lebesgue lemma, Theorem on monotone convergence, and Fatou's lemma.

In this paragraph, we shall suppose that X is a set with a σ-algebra $\mathcal{A}$ and a complete nonnegative measure μ on $\mathcal{A}$. The function $f : X \to \langle -\infty, \infty\rangle$ is named measurable if $\{x \in X; f(x) > a\} \in \mathcal{A}$ for all $a \in R$.

We name a function $f : X \to (-\infty, \infty)$ simple if there exist finitely many mutually disjoint sets $A_1, \dots, A_k \in \mathcal{A}$ with $\mu(A_j) < \infty$ for each $j = 1, \dots, k$ and numbers $c_1, \dots, c_k$, such that the following representation holds true in X:

$$f = \sum_{j=1}^{k} c_j \chi_{A_j}.$$

Put

$$\int_X f\, d\mu = \sum_{j=1}^{k} c_j \mu(A) < \infty.$$

Let $f : X \to \langle 0, \infty\rangle$ be a measurable function. Then there is an increasing sequence of nonnegative simple functions f_j such that $f_j \to f$ as $j \to \infty$ (see [5], Chapter X, Theorem 1.12). Put

$$\int_X f\, d\mu \equiv \lim_{j\to\infty} \int_X f_j\, d\mu.$$

Remark that this limit does not depend on the choice of f_j (see [5], Chapter X, Corollary 3.2).

Let now $f : X \to \langle -\infty, \infty\rangle$ be measurable. Put $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. If one of the values $\int f^+\, \mathrm{d}\mu$, $\int f^-\, \mathrm{d}\mu$ is finite, define

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu.$$

We say that f is μ-integrable (denote $f \in L^1(\mu)$) if

$$\int_X f\, d\mu \in R^1.$$

Remark that $f \in L^1(\mu)$ if f is measurable and $|f| \in L^1(\mu)$. If f is measurable and $u \le f \le h$ with $u, h \in L^1(\mu)$, then $f \in L^1(\mu)$.

If ν is a real measure with the Jordan decomposition $\nu = \nu^+ - \nu^-$ and f is a measurable function, then define

$$\int f\, \mathrm{d}\nu = \int f\, \mathrm{d}\nu^+ - \int f\, \mathrm{d}\nu^-$$

whenever the right side makes sense.

Theorem 1.13.1 (Lusin's Theorem) *Suppose* X *is a* σ*-compact metric space,* μ *is a nonnegative complete Radon measure on* X*, and* f *is a* μ *measurable real function. Then for every* μ*-measurable set* A *of finite measure and for every* $\epsilon > 0$*, there is a compact subset* K *of* X *such that* $\mu(A \setminus K) < \epsilon$ *and the restriction of* f *onto* K *is continuous.*

(See [49], Lusin's theorem 2.3.5.)

Lemma 1.13.2 (Lebesgue Lemma) *Let* μ *be a nonnegative measure. Let* f_n *be a sequence of* μ*-measurable functions. Suppose that* $f_n(x) \to f(x)$ *as* $n \to \infty$ *for* μ *almost all* x*. If there is* $g \in L^1(\mu)$ *such that* $|f_n| \le g$ *for all* n*, then* $f, f_n \in L^1(\mu)$ *for all* n *and*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

(See [19], Chapter IV, §4, Predlozenie 3 or [62], §26, Theorem 4 or [5], Chapter X, Theorem 3.12.)

Theorem 1.13.3 (On Monotone Convergence) *Suppose that* $\{f_j\}$ *is an increasing sequence of nonnegative functions from* $L^1(\mu)$*. Then*

$$\int \lim_{j\to\infty} f_j \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu.$$

(See [5], Theorem 3.4.)

Corollary 1.13.4 *Suppose that* $\{f_j\}$ *is a monotone sequence of functions from* $L^1(\mu)$*. Then*

$$\int \lim_{j\to\infty} f_j \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu.$$

Proof Without loss of generality, we can suppose that f_j is an increasing sequence. According to Theorem 1.13.3,

$$\int \lim_{j\to\infty} f_j \, d\mu = \int \lim_{j\to\infty} (f_j - f_1) \, d\mu + \int f_1 \, d\mu$$

$$= \lim_{j\to\infty} \int (f_j - f_1) \, d\mu + \int f_1 \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu.$$

Lemma 1.13.5 (Fatou's Lemma) *For every sequence of nonnegative* μ*-measurable functions* f_j*, we have*

$$\int_X \liminf_{j\to\infty} f_j \, d\mu \le \liminf_{j\to\infty} \int_X f_j \, d\mu.$$

(See [5], Chapter X, Theorem 3.7 or [49], Fatou's Lemma 2.4.6.)

Theorem 1.13.6 (On Continuity of Parametrized Integrals) *Suppose* Y *is a metric space and* $f : X \times Y \to R$ *satisfies the following:*

- $f(\cdot, y) \in L^1(\mu)$ *for every* $y \in Y$;
- $f(x, \cdot) \in \mathcal{C}^0(Y)$ *for* μ*-almost every* $x \in X$;
- *there exists* $g \in L^1(\mu)$ *such that* $|f(x, y)| \leq g(x)$ *for* $(x, y) \in X \times Y$.

Then

$$F(y) = \int_X f(x, y)\, d\mu(x)$$

is well defined and continuous.

(See [5], Chapter X, Theorem 3.17.)

Theorem 1.13.7 (On Differentiability of Integrals) *We suppose that* $\Omega \subset R^m$ *is open and* $f : X \times \Omega \to R$ *satisfies the following:*

- $f(\cdot, y) \in L^1(\mu)$ *for every* $y \in \Omega$;
- $f(x, \cdot) \in \mathcal{C}^1(\Omega)$ *for* μ*-almost every* $x \in X$;
- *there exists* $g \in L^1(\mu)$ *such that*

$$\left| \frac{\partial f(x, y)}{\partial y_j} \right| \leq g(x) \text{ for } (x, y) \in X \times \Omega \text{ and } j = 1, \ldots, m.$$

Then

$$F(y) = \int_X f(x, y)\, d\mu(x)$$

is continuously differentiable and

$$\frac{\partial F(y)}{\partial y_j} = \int_X \frac{\partial f(x, y)}{\partial y_j}\, d\mu(x) \text{ for } y \in \Omega \text{ and } j = 1, \ldots, m.$$

(See [5], Chapter X, Theorem 3.18.)

Theorem 1.13.8 (Riesz Representation Theorem) *Let* $K \subset R^m$ *be compact and* T *be a linear functional on* $\mathcal{C}^0(K)$. *Suppose that* $Tf \geq 0$ *for* $f \geq 0$. *Then there is unique Radon measure* μ *on* K *such that* $Tf = \int f\, \mathrm{d}\mu$ *for each* $f \in \mathcal{C}^0(K)$.

(See [125], Theorem 2.14.)

Proposition 1.13.9 *Let $f \in L^1_{loc}(R^m)$. Then*

$$\lim_{r \searrow 0} \oint_{B(x;r)} |f(y) - f(x)| \, \mathrm{d}\mathcal{H}_m(y) = 0 \tag{1.2}$$

for almost all $x \in R^m$. (The points at which (1.2) holds are called Lebesgue points.)

(See [136], Chapter I, §1.8.)

Theorem 1.13.10 *Let X be a set with a σ-algebra $\mathcal{A}$ and μ be a σ-finite nonnegative measure on $\mathcal{A}$. If ν is a finite nonnegative measure on $\mathcal{A}$, which is absolutely continuous with respect to μ, then there exists $f \in L^1(\mu)$ such that*

$$\nu(E) = \int_E f \, \mathrm{d}\mu \qquad \forall\, E \in \mathcal{A}.$$

(See [43], Chapter III, §10, Theorem 2.)

1.14 L^p-spaces

The next section studies Lebesgue spaces $L^p(\mu)$, i.e., spaces of measurable functions f with μ-integrable $|f|^p$. These spaces are Banach spaces, and we characterize their dual spaces. We also remark interpolation properties of Lebesgue spaces.

In this paragraph, we shall suppose that X is a set with a σ-algebra $\mathcal{A}$ and a complete nonnegative measure μ on $\mathcal{A}$.

Let $1 \le p < \infty$. Denote by $L^p(\mu)$ the set of all μ-measurable functions f such that

$$\|f\|_{L^p(\mu)} \equiv \left[\int |f|^p \, d\mu \right]^{1/p} < \infty. \tag{1.3}$$

If we identify f with all functions, which are equal to f μ-almost everywhere, then $L^p(\mu)$ equipped with the norm (1.3) is a Banach space (see [149], Chapter 1, §9). $L^2(\mu)$ is a real Hilbert space with respect to the inner product

$$(f, g) = \int_X fg \, d\mu$$

(see [5], Chapter X, Corollary 4.11).

Denote by $L^\infty(\mu)$ a set of all μ-measurable functions f such that

$$\|f\|_{L^\infty(\mu)} \equiv \inf\{\lambda; \mu(\{x; |f(x)| > \lambda\}) = 0\} < \infty. \tag{1.4}$$

If we identify f with all functions, which are equal to f μ-almost everywhere, then $L^\infty(\mu)$ equipped with the norm (1.4) is a Banach space (see [149], Chapter 1, §9).

If $\Omega \subset R^m$ is a Borel set, $1 \le p \le \infty$ denote $L^p(\Omega) = L^p(\mathcal{H}_m|\Omega)$ and by $L^p_{loc}(\Omega)$ the space of all Borel measurable functions on Ω which are from $L^p(\mathcal{H}_m|K)$ for each compact subset K of Ω.

Proposition 1.14.1 (Hölder's Inequality) *Let μ be σ-finite measure, $1 \le p, q \le \infty$, $f \in L^p(\mu)$, $h \in L^q(\mu)$, and $1/p + 1/q = 1$. (Denote $1/\infty = 0$.) Then*

$$\|fh\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)}\|h\|_{L^q(\mu)}.$$

(See [5], Chapter X, Theorem 4.2 or [49], Hölder's inequality 2.4.14.)

Corollary 1.14.2 *Let μ be a finite nonnegative measure on X. If $1 \le q < p \le \infty$, then the identity operator is a bounded linear operator from $L^p(\mu)$ into $L^q(\mu)$.*

Proof If $p = \infty$ and $\|f\|_{L^\infty(\mu)} \le 1$, then $\|f\|_{L^q(\mu)} \le [\mu(X)]^{1/q}$. Let now $p < \infty$. Put $p' = p/(p-1)$. If $\|f\|_{L^p(\mu)} \le 1$, then Hölder's inequality (Proposition 1.14.1) gives

$$\|f\|^q_{L^q(\mu)} \le \|f\|_{L^p(\mu)}\left[\int_X 1^{p'}\, \mathrm{d}\mu\right]^{1/p'} \le \mu(X)^{1/p'}.$$

Theorem 1.14.3 *Let μ be a nonnegative measure, and $1 \le p < \infty$, $(1/p) + (1/q) = 1$. Then $(L^p(\mu))'$ is equivalent to $L^q(\mu)$, in which $x' \in (L^p(\mu))'$ is related to the corresponding $g \in L^q(\mu)$ by*

$$x'f = \int gf\, d\mu, \qquad f \in L^p(\mu)$$

and

$$\|x'\| = \|g\|_{L^q(\mu)}.$$

(See [43], Chapter IV, §8 or [4], Chapter VI, §2, Theorem 1.)

So, if $1 < p < \infty$, then $L^p(\mu)$ is a reflexive Banach space.

Corollary 1.14.4 *Let μ be a nonnegative measure, $1 < p < \infty$, $(1/p)+(1/q) = 1$, and $n \in N$. Define on $[L^p(\mu)]^n$ a norm*

$$\|[f_1,\dots,f_n]\|_{[L^p(\mu)]^n} = \left[\int \sum_{j=1}^{n} |f_j|^p \, d\mu\right]^{1/p}.$$

If T is a bounded linear functional on $[L^p(\mu)]^n$, then there are $g_1,\dots,g_n \in L^q(\mu)$ such that

$$T[f_1,\dots,f_n] = \int \sum_{j=1}^{n} f_j g_j \, d\mu \tag{1.5}$$

and

$$\|T\| = \|[g_1,\dots,g_n]\|_{[L^p(\mu)]^n}. \tag{1.6}$$

Proof Put

$$\tilde{X} = \bigcup_{j=1}^{n}\{j\}\times X, \tilde{\mathcal{A}} = \{\bigcup_{j=1}^{n}\{j\}\times A_j; A_j \in \mathcal{A}\}, \tilde{\mu}(\bigcup_{j=1}^{n}\{j\}\times A_j) = \sum_{j=1}^{n}\mu(A_j).$$

Since $[L^p(\mu)]^n = L^p(\tilde{\mu})$, we can suppose that $n = 1$. Theorem 1.14.3 gives that there is $g \in L^q(\mu)$ such that (1.5) holds. Hölder's inequality (Proposition 1.14.1) gives

$$\|T\| \le \|g\|_{L^q(\mu)}.$$

If we put $f(x) = 0$ for $g(x) = 0$ and $f(x) = |g(x)|^{q-2}g(x)$ elsewhere, then

$$\|f\|_{L^p(\mu)} = \left[\int (|g|^{q-1})^{q/(q-1)} \, d\mu\right]^{1/p} = \left[\int |g|^q \, d\mu\right]^{1-1/q} < \infty,$$

$$Tf = \int |g(x)|^q = \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}.$$

Hence (1.6) holds.

Proposition 1.14.5 *If $\Omega \subset R^m$ is open and $1 \le p(0) < p(1) \le \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p(0) + \theta/p(1)$, then*

$$L^p(\Omega) = (L^{p(0)}(\Omega), L^{p(1)}(\Omega))_{\theta,p},$$

and the corresponding norms are equivalent.

(See [1], Corollary 7.27.)

Proposition 1.14.6 *Let μ be a massive Radon measure on a σ-compact metric space X and $1 < p < \infty$. Then the space of continuous functions on X with compact support is a dense vector subspace of $L^p(\mu)$.*

(See [5], Chapter X, Theorem 4.18.)

Proposition 1.14.7 *Let $\Omega \subset R^m$ be a bounded open set and $1 \leq p < \infty$. Then $C_c^\infty(\Omega)$ is a dense subset of $L^p(\Omega)$.*

(See [94],Theorem 2.6.1.)

Proposition 1.14.8 *Let μ be σ-finite and $1 \leq p \leq \infty$, and suppose $f_j, f \in L^p(\mu)$ satisfy $f_j \to f$ in $L^p(\mu)$. Then there exists a subsequence f_{j_k} of f_j converging μ-a.e. to f.*

(See [5], Chapter X, Corollary 4.7.)

Theorem 1.14.9 (Marcinkiewicz's Interpolation Theorem) *Let A be a Borel subset of R^m with $\mathcal{H}_m(A) < \infty$, $K = R$ or $K = C$, $1 \leq p(1), p(2), q(1)$, and $q(2) \leq \infty$ and T be a bounded linear operator from $L^{p(j)}(A, K)$ to $L^{q(j)}(A, K)$ for $j = 1, 2$. Fix $\theta \in (0, 1)$. Let $1 \leq p, q \leq \infty$ be such that*

$$\frac{1}{p} = \frac{1-\theta}{p(1)} + \frac{\theta}{p(2)}, \qquad \frac{1}{q} = \frac{1-\theta}{q(1)} + \frac{\theta}{q(2)}.$$

Then T is a bounded linear operator from $L^p(A, K)$ to $L^q(A, K)$. Moreover, if T is a compact operator from $L^{p(1)}(A, K)$ to $L^{q(1)}(A, K)$, then T is a compact operator from $L^p(A, K)$ to $L^q(A, K)$.

(See [87], Theorem 2.4. and [87], Theorem 3.10.)

Definition 1.14.10 For $f \in L^1_{loc}(R^m)$ the maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x;r)} |f(y)| \, \mathrm{d}\mathcal{H}_m(y), \qquad x \in R^m. \tag{1.7}$$

Lemma 1.14.11 *Let $1 < p \leq \infty$. If $f \in L^p(R^m)$, then $Mf \in L^p(R^m)$ and*

$$\|Mf\|_{L^p(R^m)} \leq C\|f\|_{L^p(R^m)}$$

with a constant C depending only on p and m.

(See [140], Lemma 27.2.)

1.15 Convolution

The convolution is a tool used to approximate a function by smooth functions.

For two functions $u, v \in L^1_{loc}(R^m)$, their convolution is defined by

$$u * v(x) = \int_{R^m} u(x-y)v(y)\, d\mathcal{H}_m(y)$$

whenever this integral has a sense.

Lemma 1.15.1 (Young's Inequality) *Suppose* $1 \le p, q, r \le \infty$, *and* $1/p + 1/q = 1 + 1/r$. *If* $f \in L^p(R^m)$ *and* $g \in L^q(R^m)$, *then* $f * g \in L^r(R^m)$ *and*

$$\|f * g\|_{L^r(R^m)} \le \|f\|_{L^p(R^m)} \|g\|_{L^q(R^m)}.$$

(See [1], Corollary 2.25.)

If $g \in L^1_{loc}(R^m)$, $\phi \in \mathcal{C}^k(R^m)$ has compact support, then $g * \phi \in \mathcal{C}^k(R^m)$ and $\partial^\alpha(g * \phi) = g * (\partial^\alpha \phi)$ for each multiindex α with $|\alpha| \le k$. (See [140], Lemma 2.3.)

Let $\phi \in \mathcal{C}^\infty(R^m)$ be a nonnegative function supported in $B(0; 1)$ such that

$$\int_{R^m} \phi(x)\, d\mathcal{H}_m(x) = 1.$$

For $\epsilon > 0$ put $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^m$. For $u \in L^1_{loc}(R^m)$ fixed, put

$$u_\epsilon = u * \phi_\epsilon.$$

Then $u_\epsilon \in \mathcal{C}^\infty(R^m)$ and $\partial^\alpha u_\epsilon = u * \partial^\alpha \phi_\epsilon$ for each multiindex α. If $u \in L^p(R^m)$ and $1 \le p \le \infty$, then $u_\epsilon \in L^p(R^m)$ and

$$\|u_\epsilon\|_{L^p(R^m)} \le \|u\|_{L^p(R^m)}.$$

Moreover $u_\epsilon \to u$ in $L^p(R^m)$ as $\epsilon \to 0$. If $u \in \mathcal{C}^0(\Omega)$ for an open set Ω, then u_ϵ converges uniformly to u on each compact subset of Ω. (See [105], p. 8.)

Lemma 1.15.2 *Suppose that* $f, g \in L^1_{loc}(R^m)$ *are convolvable and* f *has compact support. Then*

$$\operatorname{spt} f * g \subset \operatorname{spt} f + \operatorname{spt} g.$$

(See [5], Theorem 7.10.)

1.16 Continuous and Hölder Continuous Functions

In this section, we gather basic properties of continuous and Hölder continuous functions.

If X is a metric space, then $\mathcal{C}^0(X)$ denotes the set of all real continuous functions on X. By $\mathcal{C}^{0,0}(X)$ denote the set of all bounded continuous functions on X equipped with the norm

$$\|f\|_{\mathcal{C}^{0,0}(X)} = \sup_{x \in X} |f(x)|.$$

$\mathcal{C}^{0,0}(X)$ is a Banach space (see [55], Corollary 9.32). If $K \subset R^m$ is a compact set, then $\mathcal{C}^0(K) = \mathcal{C}^{0,0}(K)$ (see [43], Chapter IV, §6).

Theorem 1.16.1 (The Arzela–Ascoli Theorem) *Let X be a compact metric space and $F \subset \mathcal{C}^{0,0}(X)$. Then F is relatively compact if and only if F is bounded and equicontinuous, i.e., for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in K$, $|x - y| < \delta$ and for every $f \in F$, we have $|f(x) - f(y)| < \epsilon$.*

(See [55], Theorem 6.49.)

Theorem 1.16.2 (The Stone–Weierstrass Theorem) *Let X be a compact metric space, F be a closed subset of $\mathcal{C}^{0,0}(X)$, and $f \equiv 1 \in F$. Suppose that for each $f, g \in F$ and $a, b \in R^1$, we have $f \cdot g \in F$ and $af + bg \in F$. Then $F = \mathcal{C}^{0,0}(X)$ if and only if for each $x, y \in X$, $x \neq y$, there exists $f \in F$ such that $f(x) \neq f(y)$.*

(See [149], Introduction, §2.)

Theorem 1.16.3 *Let X be a metric space. Denote by $\mathcal{C}'(X)$ the space of all finite real regular Borel measures on X equipped with the norm $\|\mu\| = |\mu|(X)$. Then $\mathcal{C}'(X)$ is the dual space of $\mathcal{C}^{0,0}(X)$. Each continuous linear functional on $\mathcal{C}^{0,0}(X)$ has a form*

$$T_\mu(f) = \int_X f \,\mathrm{d}\mu, \tag{1.8}$$

where $\mu \in \mathcal{C}'(X)$. If $T_\mu(f) \geq 0$ for each nonnegative $f \in \mathcal{C}^{0,0}(X)$, then $\mu \geq 0$.

(See [43] Chapter VI, §6, Theorem 2.)

Lemma 1.16.4 *Let $K \subset R^m$ be compact. Then the set of polynomials with rational coefficients is dense in $\mathcal{C}^{0,0}(K)$.*

(See [40], Anhang, Satz A.8.)

If $0 < \beta \leq 1$ and $K \subset R^m$, denote by $\mathcal{C}^{0,\beta}(K)$ the set of all functions $f \in \mathcal{C}^{0,0}(K)$ for which

$$\|f\|_{\mathcal{C}^{0,\beta}(K)} \equiv \|f\|_{\mathcal{C}^{0,0}(K)} + \sup_{\substack{x, y \in K \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Clearly, if $f \in \mathcal{C}^{0,\beta}(K)$, then $f \in \mathcal{C}^{0,\beta}(\overline{K})$, and the norm of f in both spaces is equal. A function from $\mathcal{C}^{0,\beta}(K)$ is called β-Hölder continuous function. A function from $\mathcal{C}^{0,1}(K)$ is called a Lipschitz function. $\mathcal{C}^{0,\beta}(K)$ is a Banach space (see [55], Proposition 9.47).

Proposition 1.16.5 *Let K be a compact subset of R^m and $0 \leq \beta < \alpha \leq 1$. Then the identity operator I is a compact operator from $\mathcal{C}^{0,\alpha}(K)$ to $\mathcal{C}^{0,\beta}(K)$.*

(See [60], Theorem 1.3.27.)

Proposition 1.16.6 *Let $F \subset R^m$ be closed. Then there exists an extension Ef of all functions f on F such that the following assertions hold:*

- *If $f \in \mathcal{C}^0(F)$, then $Ef \in \mathcal{C}^0(R^m) \cap \mathcal{C}^\infty(R^m \setminus F)$.*
- *If $0 < \lambda \leq 1$, then $E : f \mapsto Ef$ is a bounded linear operator from $\mathcal{C}^{0,\lambda}(F)$ to $\mathcal{C}^{0,\lambda}(R^m)$.*

(See [136], Chapter VI, §2.)

Theorem 1.16.7 (Rademacher's Theorem) *If f is a Lipschitz function on R^m, then f is differentiable $\mathcal{H}_m$-almost everywhere, i.e.,*

$$\lim_{y \to x} \frac{f(y) - f(x) - (y - x) \cdot \nabla f(x)}{|x - y|} = 0,$$

and the partial derivatives of f are Lebesgue measurable functions.

(See [86], Theorem 5.1.11 or [112], Chapter 15, §3, Lemma 1.)

Lemma 1.16.8 *Let $K \subset R^m$ and $0 < \alpha \leq 1$. If $f, g \in \mathcal{C}^{0,\alpha}(K)$, then $fg \in \mathcal{C}^{0,\alpha}(K)$ and $\|fg\|_{\mathcal{C}^{0,\alpha}(K)} \leq 2\|f\|_{\mathcal{C}^{0,\alpha}(K)}\|g\|_{\mathcal{C}^{0,\alpha}(K)}$. If K is compact and $g > 0$, then $f/g \in \mathcal{C}^{0,\alpha}(K)$.*

Proof

$$|f(x)g(x) - f(y)g(y)| \leq |[f(x) - f(y)]g(x)| + |f(y)[g(x) - g(y)]|$$

$$\leq 2\|f\|_{\mathcal{C}^{0,\alpha}(K)}\|g\|_{\mathcal{C}^{0,\alpha}(K)}|x - y|^\alpha.$$

Let K now be compact and $g > 0$. Then there exists $\delta > 0$ such that $g \geq \delta$:

$$\left|\frac{f(x)}{g(x)} - \frac{f(z)}{g(z)}\right| = \frac{|f(x)g(z) - f(z)g(x)|}{|g(x)g(z)|}$$

$$\leq \frac{|f(x)||g(z) - g(x)| + |g(x)||f(x) - g(z)|}{\delta^2} \leq \frac{2\|f\|_{\mathcal{C}^{0,\alpha}}\|g\|_{\mathcal{C}^{0,\alpha}}}{\delta^2}.$$

Remark 1.16.9 Let $\Omega \subset R^m$ be open, $k \in N$, and $0 < \lambda < 1$. Then $f \in \mathcal{C}^{k,\lambda}(\Omega)$ if and only if for each multiindex β with $|\beta| \leq k$ there exists a function f_β on F such that $f = f_0$ and there exists a constant M such that $|f_\alpha| \leq M$,

$$\left| f_\alpha(x) - \sum_{|\alpha+\beta| \leq k} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta \right| \leq M|x-y|^{k+\lambda-|\alpha|}, \quad x, y \in F \tag{1.9}$$

for each multiindex α with $|\alpha| \leq k$. (In fact $f_\alpha = \partial^\alpha f$.) The same is true for $F = \overline{\Omega}$.

Definition 1.16.10 Let $F \subset R^m$ be closed, $k \in N$, and $0 < \lambda < 1$. We say that $f \in \mathcal{C}^{k,\lambda}(F)$ if for each multiindex β with $|\beta| \leq k$ there exists a function f_β on F such that $f = f_0$ and there exists a constant M such that $|f_\alpha| \leq M$, and (1.9) holds true for each multiindex α with $|\alpha| \leq k$. The norm of an element f is the smallest M for which (1.9) holds.

Proposition 1.16.11 *Let $F \subset R^m$ be closed, $k \in N$, and $0 < \lambda < 1$. Then there exists a bounded linear operator E from $\mathcal{C}^{k,\lambda}(F)$ to $\mathcal{C}^{k,\lambda}(R^m)$ such that $Ef = f$ on F.*

(See [136], Chapter VI, §2, Theorem 4.)

1.17 Domains

In this section, we define some classes of domains: domains with Lipschitz boundary, domains with boundary of class $\mathcal{C}^{k,\alpha}$, and starshaped domains. We also prove several auxiliary lemmas we shall need in the future.

Definition 1.17.1 Let $m \geq 2$ and $\Phi \in \mathcal{C}^{0,0}(R^{m-1})$. We say that $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x')\}$ is a graph domain corresponding to Φ. If Φ is of class $\mathcal{C}^{k,\beta}$, we say that Ω is a graph domain of class $\mathcal{C}^{k,\beta}$. If Φ is Lipschitz we say that Ω is a Lipschitz graph domain.

Definition 1.17.2 Let $\Omega \subset R^m$ be an open set, $x \in \partial\Omega$, $k \in N_0$, and $0 \leq \beta \leq 1$. We say that Ω has boundary of class $\mathcal{C}^{k,\beta}$ in a neighborhood of x if there exists a coordinate system, $r > 0$, and a graph domain D of class $\mathcal{C}^{k,\beta}$ such that $B(x;r) \cap \Omega = B(x;r) \cap D$. If Ω has boundary of class $\mathcal{C}^{k,\beta}$ in a neighborhood of each point of $\partial\Omega$, we say that Ω has boundary of class $\mathcal{C}^{k,\beta}$. If Ω has boundary of class $\mathcal{C}^{0,0}$, we say that Ω has boundary of class $\mathcal{C}$. If Ω has boundary of class $\mathcal{C}^{0,1}$, we say that Ω has Lipschitz boundary.

Lemma 1.17.3 *Let $\Omega \subset R^m$ be an open set with boundary of class $\mathcal{C}^{1,0}$ and $m \geq 2$. If $z \in \partial\Omega$ and $\delta > 0$, then there exists a coordinate system, $r > 0$ and $\Phi \in \mathcal{C}^{1,0}(R^{m-1})$ with $|\Phi| < \delta$, $|\nabla\Phi| < \delta$ such that $\Omega \cap B(z;r) = B(z;r) \cap \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x')\}$.*

Proof We can choose a coordinate system, $g \in \mathcal{C}^{1,0}(R^{m-1})$ and $\rho > 0$ such that $x = 0$, $\Omega \cap B(0;\rho) = B(0;\rho) \cap \{[x', x_m]; x' \in R^{m-1}, x_m > g(x')\}$, $g(0) = 0$, and $\nabla g(0) = 0$. Choose $\varphi \in \mathcal{C}_c^\infty(R^{m-1})$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B(0;1)$, and $\operatorname{spt}\varphi \subset B(0;2)$. Let $|\nabla\varphi| \leq M, M \geq 1$. Fix $r \in (0,\rho)$ such that $|g| + |\nabla g| < \delta/(2M)$ on $B(0;2r)$. Put $\Phi(x') = g(x')\varphi(x'/r)$. Then $\Omega \cap B(z;r) = B(z;r) \cap \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x')\}$. Clearly $|\Phi| < \delta$. If $|x'| \geq 2r$, then $|\nabla\Phi(x')| = 0$. Let now $|x'| < 2r$. Then $|g(x')| = |g(x') - g(0)| \leq |x'||\nabla g(\theta x')| < r\delta/(2M)$ for some $\theta \in (0,1)$. Hence $|\nabla\Phi(x')| \leq |\varphi(x'/r)||\nabla g(x')| + |g(x')||\nabla\varphi(x'/r)|/r < \delta/2 + [r\delta/(2M)]M/r = \delta$.

Definition 1.17.4 Let Ω be an open subset of R^m and $x = [x', x_m] \in \partial\Omega$. Suppose that $\Phi \in \mathcal{C}^0(R^{m-1})$ be such that $x_m = \Phi(x')$ and in some neighborhood of x the set Ω lies under the graph of Φ and $R^m \setminus \overline{\Omega}$ lies above the graph of Φ. If there is $\nabla\Phi(x')$, we denote

$$n^\Omega(x) = [-\nabla\Phi(x'), 1]/\sqrt{|\nabla\Phi(x')|^2 + 1}$$

the outward unit normal of Ω at x.

Remark 1.17.5 Let $G \subset R^{m-1}$ be open, $\Phi \in \mathcal{C}^{0,1}(G)$, and $M \subset G$ be Borel. Then $\nabla\Phi(x')$ exists at almost all $x' \in G$ and

$$\mathcal{H}_{m-1}(\{[x', \Phi(x')]; x' \in M\}) = \int_M \sqrt{1 + |\nabla\Phi(x')|^2}\, d\mathcal{H}_{m-1}(x').$$

(See [86], Theorem 5.1.1 and Theorem 1.17.6.)

Corollary 1.17.6 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. Then there is a constant M such that $\mathcal{H}_{m-1}(B(x;r) \cap \partial\Omega) \leq Mr^{m-1}$ for each $x \in R^m$ and $r > 0$.*

Lemma 1.17.7 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. Then there exist $\Phi \in \mathcal{C}_c^\infty(R^m, R^m)$ and $\epsilon > 0$ such that $\Phi \cdot n^\Omega \geq \epsilon$ on $\partial\Omega$.*

(See [58], Lemma 1.5.19.)

Lemma 1.17.8 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $C^{1,\alpha}$ and $0 < \alpha \leq 1$. Then there is a constant M such that $|n^\Omega(x) - n^\Omega(y)| \leq M|x - y|^\alpha$ for each $x, y \in \partial\Omega$.*

Proof Fix $z \in \partial\Omega$. Then there are $r > 0$, a coordinate system centered at z, and a function $\Phi \in \mathcal{C}^{1,\alpha}(R^{m-1})$ such that $\Phi(0,\ldots,0) = 0$, $\Omega \cap B(z;r)$ lies under the graph of Φ and $(R^m \setminus \Omega) \cap B(z;r)$ lies above the graph of Φ. There is a constant K such that for each $x', y' \in R^{m-1}$, $|x'| \leq r$, $|y'| \leq r$ we have

$$|\Phi(x')| + |\nabla\Phi(x')| \leq K, \quad |\nabla\Phi(x') - \nabla\Phi(y')| \leq K|x' - y'|^\alpha.$$

Fix $0 < \delta < r/(1+K)$. If $x', y' \in R^{m-1}$, $|x'| < \delta$, $|y'| < \delta$ we have

$$|n^\Omega(x) - n^\Omega(y)| \leq \frac{|[-\nabla\Phi(x'), 1] - [-\nabla\Phi(y'), 1]|}{\sqrt{|\nabla\Phi(x')|^2 + 1}}$$

$$+\left|\frac{[-\nabla\Phi(y'), 1]}{\sqrt{|\nabla\Phi(x')|^2 + 1}} - \frac{[-\nabla\Phi(y'), 1]}{\sqrt{|\nabla\Phi(y')|^2 + 1}}\right| \leq K|x' - y'|^\alpha$$

$$+\frac{(1+K)|[|\nabla\Phi(x')|^2 + 1] - [|\nabla\Phi(y')|^2 + 1]|}{\sqrt{|\nabla\Phi(x')|^2 + 1}\sqrt{|\nabla\Phi(y')|^2 + 1}(\sqrt{|\nabla\Phi(x')|^2 + 1} + \sqrt{|\nabla\Phi(y')|^2 + 1})}$$

$$\leq [K + (1+K)]|x' - y'|^\alpha.$$

So, for each $z \in \partial\Omega$, there are positive constants $\delta(z)$ and $K(z)$ such that $|n^\Omega(x) - n^\Omega(y)| \leq K(z)|x - y|^\alpha$ for each $x, y \in B(z; 2\delta(z)) \cap \partial\Omega$. Since $\partial\Omega$ is compact and $\partial\Omega \subset \cup\{B(z;\delta(z)); z \in \partial\Omega\}$, we can choose $z^1, \dots, z^k \in \partial\Omega$ such that $\partial\Omega \subset B(z^1;\delta(z^1)) \cup \cdots \cup B(z^k;\delta(z^k))$. Put $\delta = \min(\delta(z^1), \dots, \delta(z^k))$, $C = \max(K(z^1), \dots, K(z^k))$. Suppose now that $x, y \in \partial\Omega$. If $|x - y| \geq \delta$, then $|n^\Omega(x) - n^\Omega(y)| \leq 2\delta^{-\alpha}|x - y|^\alpha$. If $|x - y| < \delta$, then there is z^j such that $|x - z^j| < \delta$. Since $|y - z^j| < 2\delta$, we obtain $|n^\Omega(x) - n^\Omega(y)| \leq C|x - y|^\alpha$.

Lemma 1.17.9 *Let Ω be a bounded open set in R^m with boundary of class $C^{1,\alpha}$ and $0 < \alpha \leq 1$. Then there is a constant C such that $|n^\Omega(y) \cdot (y - x)| \leq C|x - y|^{1+\alpha}$ for each $x, y \in \partial\Omega$.*

Proof Fix $z \in \partial\Omega$. Then there are $r > 0$, a coordinate system centered at z, and a function $\Phi \in \mathcal{C}^{1,\alpha}(R^{m-1})$ such that $\Phi(0, \dots, 0) = 0$, $\Omega \cap B(z; r)$ lies under the graph of Φ and $(R^m \setminus \Omega) \cap B(z; r)$ lies above the graph of Φ. Further, there is a constant L such that for each $x', y' \in R^{m-1}$, $|x'| \leq r$, $|y'| \leq r$ we have

$$|\Phi(x')| + |\nabla\Phi(x')| \leq L, \quad |\nabla\Phi(x') - \nabla\Phi(y')| \leq L|x' - y'|^\alpha.$$

Fix $0 < \delta < r/(1+L)$. If $x', y' \in R^{m-1}$, $|x'| < \delta$, $|y'| < \delta$ we have

$$|n^\Omega(x', \Phi(x')) \cdot ([y', \Phi(y')] - [x', \Phi(x')])|$$

$$= \left|\frac{[-\nabla\Phi(x'), 1]}{\sqrt{|\nabla\Phi(x')|^2 + 1}} \cdot \left[y' - x', \int_0^1 (y' - x') \cdot \nabla\Phi(x' + t(y' - x'))\, dt\right]\right|$$

$$= \frac{1}{\sqrt{|\nabla\Phi(x')|^2 + 1}}\left|\int_0^1 (y' - x') \cdot [\nabla\Phi(x' + t(y' - x')) - \nabla\Phi(x')]\, dt\right|$$

$$\leq L|x' - y'|^{1+\alpha}.$$

So, for each $z \in \partial\Omega$, there are positive constants $\delta(z)$ and $L(z)$ such that $|n^{\Omega}(x) \cdot (y - x)| \leq L(z)|x - y|^{\alpha+1}$ for each $x, y \in B(z; 2\delta(z)) \cap \partial\Omega$. Since $\partial\Omega$ is compact and $\partial\Omega \subset \cup\{B(z; \delta(z)); z \in \partial\Omega\}$, we can choose $z^1, \dots, z^k \in \partial\Omega$ such that $\partial\Omega \subset B(z^1; \delta(z^1)) \cup \dots \cup B(z^k; \delta(z^k))$. Put $\delta = \min(\delta(z^1), \dots, \delta(z^k))$, $C = \max(L(z^1), \dots, L(z^k))$. Suppose now that $x, y \in \partial\Omega$. If $|x - y| \geq \delta$ then $|n^{\Omega}(x) \cdot (y - x)| \leq \delta^{-\alpha-1}|x - y|^{1+\alpha}$. If $|x - y| < \delta$, then there is z^j such that $|x - z^j| < \delta$. Since $|y - z^j| < \delta$, we obtain $|n^{\Omega}(x) \cdot (y - x)| \leq C|x - y|^{1+\alpha}$.

Definition 1.17.10 We say that an open set $\Omega \subset R^m$ satisfies an interior ball condition at $x \in \partial\Omega$ if there exist $z \in R^m$ and $r > 0$ such that $B(z; r) \subset \Omega$, $x \in \partial B(z; r)$.

Lemma 1.17.11 *Let $\Omega \subset R^m$ be an open set with boundary of class $\mathcal{C}^2$ in a neighborhood of $x \in \partial\Omega$, $m \geq 2$. Then Ω satisfies an interior ball condition at x.*

Proof We can choose a coordinate system in a such way that $x = [0, \dots, 0]$ and $n^{\Omega}(x) = (0, \dots, 0, 1)$. Choose $\delta > 0$ and a function $\varphi \in \mathcal{C}^2(R^{m-1})$ such that $B(0; \delta) \cap \Omega = B(0; \delta) \cap \{[y', y_m]; y' \in R^{m-1}, y_m > \varphi(y')\}$. Since $\varphi(0, \dots, 0) = 0$, $\partial_j\varphi(0, \dots, 0) = 0$, there exists a constant C such that $|\varphi(y')| \leq C\|y'\|^2$ for $|y'| \leq \delta$. Fix $0 < r < \min(\delta, (2C)^{-1})$. Put $z = [0, \dots, 0, r]$. If $y = [y', y_m] \in R^m$, $|y'| < r$, $y_m \leq \varphi(y')$, then $\varphi(y') < r$,

$$\|y - z\|^2 \geq \|y'\|^2 + |\varphi(y') - r|^2 \geq r^2 + \|y'\|^2 - 2r\varphi(y')$$

$$\geq r^2 + \|y'\|^2 - 2rC\|y'\|^2 \geq r^2.$$

Thus $B(z; r) \subset \Omega$. Clearly $x = [0, \dots, 0] \in \partial B(z; r)$.

Notation 1.17.12 *Let $\Omega \subset R^m$ be a domain with Lipschitz boundary. If $x \in \partial\Omega$, $a > 0$ denote the nontangential approach regions of opening a at the point x by*

$$\Gamma_a(x) = \{y \in \Omega; |x - y| < (1 + a)\,\mathrm{dist}(y, \partial\Omega)\}.$$

Lemma 1.17.13 *If $\Omega \subset R^m$ is an open set with compact Lipschitz boundary, then there is a sequence of open sets Ω_j with boundaries of class C^{∞} such that*

- $\overline{\Omega_j} \subset \Omega$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(y) \in \Gamma_a(y)$ for each j and each $y \in \partial\Omega$ and* $\sup\{|y - \Lambda_j(y)|; y \in \partial\Omega\} \to 0$ *as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and from infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\Lambda_j(E))$, and so that $\omega_j \to 1$ pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \leq s < \infty$.*
- *The normal vectors to Ω_j, $n(\Lambda_j(y))$, converge pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \leq s < \infty$, to $n(y)$.*

If Ω is connected, then Ω_j are connected.

(See [147], Theorem 1.12.)

Lemma 1.17.14 *Let* $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \varphi(x')\}$, φ *be a Lipschitz function with Lipschitz constant* M. *If* $x = [x', x_m] \in \partial\Omega$ *then* $\{y = [y', y_m]; y_m - x_m > M|x' - y'|\} \subset \Omega$, $\{y = [y', y_m]; y_m - x_m < -M|x' - y'|\} \subset R^m \setminus \overline{\Omega}$.

Proof Fix $x = [x', x_m] \in \partial\Omega$. If $y = [y', y_m] \in \partial\Omega \setminus \{x\}$, then $|y_m - x_m| = |\varphi(x') - \varphi(y')| \le M|x' - y'|$. The set $\{y = [y', y_m]; |y_m - x_m| > M|x' - y'|\} \subset R^m \setminus \partial\Omega$ has two components. Since $\{[x', t]; t > \varphi(x')\} \subset \Omega$ and $\{[x', t]; t > \varphi(x')\} \subset R^m \setminus \overline{\Omega}$, we deduce that $\{y = [y', y_m]; y_m - x_m > M|x' - y'|\} \subset \Omega$, $\{y = [y', y_m]; y_m - x_m < -M|x' - y'|\} \subset R^m \setminus \overline{\Omega}$.

Proposition 1.17.15 (Whitney Decomposition) *Let* $\Omega \subset R^m$ *be an open set,* $\partial\Omega \neq \emptyset$. *Then there exists a sequence of open disjoint cubes* Q_j, *the edges of which are parallel to the coordinate axes,* $\operatorname{diam} Q_j \le \operatorname{dist}(Q_j, \partial\Omega) \le 4 \operatorname{diam} Q_j$ *and*

$$\Omega = \bigcup_{j=1}^{\infty} \overline{Q_j}.$$

Fix $0 < \epsilon < 1/4$. *Let* x^j *be a center of* Q_j. *Denote* $\tilde{Q}_j = \{x^j + (1+\epsilon)(y - x^j); y \in Q_j\}$. *Then each point of* Ω *is an element of at most* 12^m *cubes* $\tilde{Q}_j$.

(See [136], Chapter VI, §1.)

Definition 1.17.16 A subset Ω of R^m is starshaped with respect to a set $M \subset \Omega$ if $\{tx + (1-t)y; 0 \le t \le 1\} \subset \Omega$ for each $x \in M$ and each $y \in \partial\Omega$. We say that Ω is starshaped if it is starshaped with respect to a point.

Lemma 1.17.17 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. Then there exist starshaped domains* $\Omega_1, \dots, \Omega_k$ *with Lipschitz boundary such that* $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$.

(See [105], Chapter I, §1.3.2, §1.3.3.)

1.18 Partition of Unity

If we study regularity of solutions of boundary value problems, we restrict ourselves first to graph domains. For the regularity of solutions on domains with compact boundary, we use localization. For this we need the following lemma about the partition of unity.

Lemma 1.18.1 *Let* $E \subset R^m$ *and let* $\mathcal{G}$ *be a collection of open sets* U *such that* $E \subset \{\cup U; U \in \mathcal{G}\}$. *Then, there exists a family* $\mathcal{F}$ *of nonnegative functions* $f \in \mathcal{C}_c^\infty(R^m)$ *such that* $0 \le f \le 1$ *and:*

- *for each* $f \in \mathcal{F}$, *there exists* $U \in \mathcal{G}$ *such that* $\operatorname{spt} f \subset U$,

- *if* $K \subset E$ *is compact, then* $\operatorname{spt} f \cap K \neq$ *for only finitely many* $f \in \mathcal{F}$*,*
- $\sum_{f \in \mathcal{F}} f(x) = 1$ *for each* $x \in E$*.*

(See [150], Lemma 2.3.1 or [5], Theorem 7.16.)

1.19 The Gauss–Green Theorem

An essential role in the definition of distributions plays the following Gauss–Green theorem.

Theorem 1.19.1 (Gauss–Green Theorem) *Let* $\Omega \subset R^m$ *be an open set,* G *be a bounded domain with Lipschitz boundary, and* $\overline{G} \subset \Omega$*. If* $v_1, \ldots, v_m \in \mathcal{C}^1(\Omega)$*, then*

$$\int_G \sum_{j=1}^m \partial_j v_j \, d\mathcal{H}_m = \int_{\partial G} n^G(x) \cdot v(x) \, d\mathcal{H}_{m-1}(x)$$

where $n^G(x)$ *denotes the outward unit normal of* G *at* x*.*

(See [150], Theorem 5.8.2.)

1.20 Distributions

It is well known that not all boundary value problems for partial differential equations possess a classical solution. For that reason, distributions were invented as a generalization of functions. In this section, we define distributions and gather their fundamental properties. We shall see that for all distributions there are derivatives in the sense of distributions. We explain what it means that $u_n \to u$ in the sense of distributions. We define also a convolution of distributions as a generalization of a convolution of functions. Then we define a special class of distributions, so-called tempered distributions. For these distributions, we define the Fourier transform as a generalization of the Fourier transform of functions.

Let $\Omega \subset R^m$ be an open set. Denote by $\mathcal{C}_c^\infty(\Omega)$ the set of all $\varphi \in \mathcal{C}^\infty(R^m)$ with compact support in Ω. If $\{\varphi_j\}$ is a sequence of functions from $\mathcal{C}_c^\infty(\Omega)$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we say that φ_j converges to φ in $\mathcal{C}_c^\infty(\Omega)$ if:

- there exists a compact set $K \subset \Omega$ such that $\operatorname{spt} \varphi \subset K$ and $\operatorname{spt} \varphi_j \subset K$ for every j,
- $\lim_{j \to \infty} \partial^\alpha \varphi_j = \partial^\alpha \varphi$ uniformly on K for each multiindex α.

A distribution in Ω is a linear functional T on the space $\mathcal{C}_c^\infty(\Omega)$ such that $\langle T, \varphi_j \rangle \to \langle T, \varphi \rangle$ if $\varphi_j \to \varphi$ in $\mathcal{C}_c^\infty(\Omega)$. Denote by $\mathcal{D}'(\Omega)$ the set of all distributions in Ω.

If μ is a Radon measure in Ω, then we can define the distribution

$$\langle T_\mu, \varphi \rangle = \int_\Omega \varphi \, d\mu, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

We shall identify μ and T_μ.

Similarly, if $f \in L^1_{loc}(\Omega)$, we shall identify f with the distribution

$$\varphi \mapsto \int_\Omega f\varphi \, d\mathcal{H}_n, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

If $f_1, f_2 \in L^1_{loc}(\Omega)$ and $\langle f_1, \varphi \rangle = \langle f_2, \varphi \rangle$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $f_1 = f_2$ almost everywhere in Ω (see [25], Theorem 2.3).

If $\{T_j\}$ is a sequence of distributions in Ω and T is a distribution in Ω, we say that T_j converges to T in $\mathcal{D}'(\Omega)$ if $\langle T_j, \varphi \rangle \to \langle T, \varphi \rangle$ for each $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Instead of a sequence, we can also take a family T_ϵ of distributions that depend on a real parameter ϵ. We then say that T_ϵ converges in $\mathcal{D}'(\Omega)$ to T as $\epsilon \to \epsilon_0$ if $\langle T_\epsilon, \varphi \rangle \to \langle T, \varphi \rangle$ as $\epsilon \to \epsilon_0$ for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$. If $T \in \mathcal{D}'(\Omega)$, then there exist $\Phi_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\Phi_n \to T$ in $\mathcal{D}'(\Omega)$ as $n \to \infty$ (see [42], Corollary 11.7).

If $f \in \mathcal{C}^1(\Omega)$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we choose a bounded open set G with smooth boundary such that $\operatorname{spt} \varphi \subset G \subset \overline{G} \subset \Omega$. According to Gauss–Green theorem (Theorem 1.19.1),

$$\int_\Omega \varphi \partial_j f \, \mathrm{d}\mathcal{H}_m = \int_G [\partial_j(f\varphi) - f\partial_j\varphi] \, \mathrm{d}\mathcal{H}_m$$

$$= \int_{\partial G} n_j^G (f\varphi) \, \mathrm{d}\mathcal{H}_{m-1} - \int_G f\partial_j\varphi \, \mathrm{d}\mathcal{H}_m = -\int_\Omega f\partial_j\varphi \, \mathrm{d}\mathcal{H}_m.$$

If $f \in \mathcal{C}^k(\Omega)$ and α is a multiindex with $|\alpha| \leq k$, then

$$\langle \partial^\alpha f, \varphi \rangle = \int_\Omega \varphi \partial^\alpha f \, \mathrm{d}\mathcal{H}_m = (-1)^{|\alpha|} \int_\Omega f \partial_\alpha \varphi \, \mathrm{d}\mathcal{H}_m = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle.$$

If $T \in \mathcal{D}'(\Omega)$ and α is a multiindex, then the corresponding partial derivative of T is defined as

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Remark that $\partial_j \partial_k T = \partial_k \partial_j T$ for each $T \in \mathcal{D}'(\Omega)$ because $\partial_j \partial_k \varphi = \partial_k \partial_j \varphi$ for each $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Clearly, if $T_n \to T$, then $\partial^\alpha T_n \to \partial^\alpha T$ for each multiindex α (see [149], Chapter II, §3). If $f \in \mathcal{C}^1(\Omega)$, then $\partial_j f$ in the classical sense and in the sense of distributions coincide by the Gauss–Green theorem (Theorem 1.19.1). On the other hand, if $T \in \mathcal{D}'(\Omega)$, $\partial_j T \in \mathcal{C}^0(\Omega)$ for $j = 1, \dots, m$, then there exists $f \in \mathcal{C}^1(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_\Omega f\varphi \, \mathrm{d}\mathcal{H}_m, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega)$$

(see [130], Chapter 1, §7, Theorem 2).

If $T, S \in \mathcal{D}'(\Omega)$, then $T + S$ is the distribution defined by

$$\langle T + S, \varphi \rangle = \langle T, \varphi \rangle + \langle S, \varphi \rangle.$$

If $f \in L^1_{loc}(\Omega)$, $\psi \in \mathcal{C}^\infty(\Omega)$, $\varphi \in \mathcal{C}_c^\infty(\Omega)$ then

$$\langle \psi f, \varphi \rangle = \int_\Omega \psi f \varphi \, \mathrm{d}\mathcal{H}_m = \langle f, \psi\varphi \rangle, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

If $T \in \mathcal{D}'(\Omega)$, $\psi \in \mathcal{C}^\infty(\Omega)$, then ψT is the distribution defined by

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle, \qquad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

It holds $\partial_j(\psi T) = \psi \partial_j T + (\partial_j \psi) T$ (see [149], Chapter 1, §8).

Let G, Ω_1, Ω_2 be open subsets of R^m such that $G \subset \Omega_1 \cap \Omega_2$. Let $T_j \in \mathcal{D}'(\Omega_j)$ for $j = 1, 2$. We say that $T_1 = T_2$ on G if $\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle$ for each $\varphi \in \mathcal{C}_c^\infty(G)$.

The support of $T \in \mathcal{D}'(\Omega)$, denoted by $\operatorname{spt} T$, is the set of all $x \in \Omega$ for which there exists no open neighborhood U of x such that $T = 0$ on U.

Let $T \in \mathcal{D}'(\Omega)$ have compact support. Let φ be a function not necessarily defined on the whole Ω but of class C^∞ on a neighborhood G of $\operatorname{spt} T$. We can define T on φ as follows: Choose $\psi \in \mathcal{C}_c^\infty(\Omega)$ such that $\operatorname{spt} \psi \subset G$ and $\psi = 1$ on a neighborhood of $\operatorname{spt} T$. Then define

$$\langle T, \varphi \rangle \equiv \langle T, \varphi\psi \rangle.$$

Remark that $\langle T, \varphi \rangle$ does not depend on the choice of ψ (see [149], Chapter 1, §13). Moreover, there exist constants C and k such that

$$|\langle T, \varphi \rangle| \le C \sum_{|\alpha| \le k} \sup_{x \in \operatorname{spt} \psi} |\partial^\alpha \varphi(x)|, \quad \varphi \in \mathcal{C}^\infty(\Omega)$$

(see [149], Chapter 1, §13, Proposition 5).

If $f, g \in L^1(R^m)$, then $f * g \in L^1(R^m)$. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then Fubini's theorem gives

$$\langle f * g, \varphi\rangle = \int\limits_{R^m}\int\limits_{R^m} \varphi(x)f(y)g(x-y)\,\mathrm{d}\mathcal{H}_m(y)\mathcal{H}_m(x)$$

$$= \int\limits_{R^m} f(y)\int\limits_{R^m} \varphi(x)g(x-y)\,\mathrm{d}\mathcal{H}_m(x)\mathcal{H}_m(y)$$

$$= \int\limits_{R^m} f(y)\int\limits_{R^m} \varphi(x+y)g(x)\,\mathrm{d}\mathcal{H}_m(x)\mathcal{H}_m(y).$$

This motivates definition of convolution of distributions.

If $\varphi \in \mathcal{C}_c^\infty(R^m)$, $T \in \mathcal{D}'(R^m)$, denote

$$\varphi_T(x) = \langle T, \varphi(\cdot + x)\rangle.$$

Then $\varphi_T \in \mathcal{C}^\infty(R^m)$. If T has compact support, then φ_T has compact support, too. If $T, S \in \mathcal{D}'(R^m)$ and one of the distributions T and S has compact support, define

$$\langle S * T, \varphi\rangle \equiv \langle S, \varphi_T\rangle, \qquad \varphi \in \mathcal{C}_c^\infty(R^m),$$

the convolution of distributions S and T. The distribution $S * T$ is well defined, $T * S = S * T$ and $\operatorname{spt}(S * T) \subset \operatorname{spt} T + \operatorname{spt} S$ (see [42], Theorem 11.17). Remark that $\delta_0 * T = T * \delta_0 = T$ (see [42], Example 11.16). (Recall that $\delta_a(\varphi) = \varphi(a)$.) Moreover, $\partial_j(T * S) = (\partial_j T) * S = T * (\partial_j S)$ (see [42], Theorem 11.17). If $T_k \to T$ in the sense of distributions and either S has compact support or there is a compact set K such that $\operatorname{spt} T_k \subset K$ for each k, then $T_k * S \to T * S$ in the sense of distributions. (See [130], Chapter II, § 12.) If $\Phi \in \mathcal{C}_c^\infty(R^m)$ is such that

$$\int\limits_{R^m} \Phi\,\mathrm{d}\mathcal{H}_m = 1,$$

we put $\Phi_\epsilon(x) = \Phi(x/\epsilon)/\epsilon^m$ for $\epsilon > 0$. If $T \in \mathcal{D}'(R^m)$, then $T * \Phi_\epsilon \in \mathcal{C}^\infty(R^m)$ and $T * \Phi_\epsilon \to T$ as $\epsilon \to 0_+$ (see [42], Lemma 11.6).

The Schwartz space $\mathcal{S}(R^m)$ of rapidly decreasing test functions consists of functions $\varphi \in \mathcal{C}^\infty(R^m)$ such that for each multiindexes α, β

$$\sup_{x\in R^m} |x^\alpha \partial^\beta \varphi(x)| < \infty.$$

If $\{\varphi_j\}$ is a sequence of functions from $\mathcal{S}(R^m)$ and $\varphi \in \mathcal{S}(R^m)$, we say that φ_j converges to φ in $\mathcal{S}(R^m)$ if $\lim_{j\to\infty} x^\beta \partial^\alpha \varphi_j = x^\beta \partial^\alpha \varphi$ uniformly for every multiindexes α, β. Remark that $\mathcal{C}_c^\infty(R^m)$ is a dense subset of $\mathcal{S}(R^m)$ (see [42], Lemma 14.7). A tempered distribution in R^m is a linear functional B on the space $\mathcal{S}(R^m)$ such that $\langle B, \varphi_j\rangle \to \langle B, \varphi\rangle$ if $\varphi_j \to \varphi$ in $\mathcal{S}(R^m)$. Denote by $\mathcal{S}'(R^m)$ the set of all tempered distributions in R^m. If $\{B_j\}$ is a sequence of tempered distributions in R^m and B is a tempered distribution in R^m, we say that B_j converges to B in $\mathcal{S}'(R^m)$ if $\langle B_j, \varphi\rangle \to \langle B, \varphi\rangle$ for each $\varphi \in \mathcal{S}(R^m)$. Clearly, $\mathcal{S}'(R^m) \subset \mathcal{D}'(R^m)$. On the other hand, if $T \in \mathcal{D}'(R^m)$ has compact support, then $T \in \mathcal{S}'(R^m)$. If $T \in \mathcal{S}'(R^m)$ and α is an multiindex, then $\partial^\alpha T \in \mathcal{S}'(R^m)$ (see [42], Theorem 14.21). If $f \in L^p(R^m)$, $1 \leq p \leq \infty$, then $f \in \mathcal{S}'(R^m)$ (see [42], Example 14.22). Remark that $\mathcal{C}_c^\infty(R^m)$ is a dense subset of $\mathcal{S}'(R^m)$ (see [42], p. 189). A distribution in $\mathcal{D}'(R^m)$ belongs to $\mathcal{S}'(R^m)$ if and only if it is a sum of derivatives of a function in $\mathcal{C}^0(R^m)$ that is at most polynomial growth at infinity (see [42], Theorem 17.10).

Denote $\mathcal{S}(R^m, C) = \{f + ig; f, g \in \mathcal{S}(R^m)\}$ the space of complex rapidly decreasing test functions and $\mathcal{S}'(R^m, C) = \{F + iG; F, G \in \mathcal{S}'(R^m)\}$ the space of complex tempered distributions in R^m. If $h = f + ig \in \mathcal{S}(R^m, C)$ and $H = F + iG \in \mathcal{S}'(R^m, C)$, then

$$\langle H, h\rangle = \langle F, f\rangle - \langle G, g\rangle + i\langle F, g\rangle + i\langle G, f\rangle.$$

If $h \in \mathcal{S}(R^m, C)$, define the Fourier transform $\hat{h}$ of h

$$\hat{h}(x) = (2\pi)^{-m/2} \int_{R^m} h(y) e^{-ix\cdot y}\, d\mathcal{H}_m(y). \tag{1.10}$$

Then the mapping $h \mapsto \hat{h}$ is an isomorphism on $\mathcal{S}(R^m, C)$ (see [24], Theorem 3.5.1 or [42], Lemma 14.12) and

$$h(x) = (2\pi)^{-m/2} \int_{R^m} \hat{h}(y) e^{ix\cdot y}\, d\mathcal{H}_m(y), \qquad h \in \mathcal{S}(R^m, C).$$

Remark that $\widehat{f * g} = (2\pi)^{m/2} \tilde{f} \cdot \tilde{g}$, $(2\pi)^{m/2} \widehat{f \cdot g} = \tilde{f} * \tilde{g}$, where $\tilde{f}(x) = \hat{f}(-x)$ (see [149], Chapter VI, §1).

If $F \in \mathcal{S}'(R^m, C)$, define the Fourier transform $\hat{F} \in \mathcal{S}'(R^m, C)$ of F by

$$\langle \hat{F}, \varphi\rangle = \langle F, \hat{\varphi}\rangle, \qquad \varphi \in \mathcal{S}(R^m, C).$$

If $F \in L^1(R^m)$, then $\hat{F}$ is given by (1.10) (see [149], Chapter VI, §2). If $T \equiv 1$, we have $\widehat{\delta_0} = (2\pi)^{-m/2} T$, $\hat{T} = (2\pi)^{m/2} \delta_0$ (see [149], Chapter VI, §2). The mapping $F \mapsto \hat{F}$ is an isomorphism on $\mathcal{S}'(R^m, C)$ (see [42], Theorem 14.24). Remark that $L^2(R^m, C) \subset \mathcal{S}'(R^m)$. Moreover, if $h \in L^2(R^m, C)$, then $\hat{h} \in L^2(R^m, C)$ and $\|\hat{h}\|_{L^2(R^m)} = \|h\|_{L^2(R^m)}$. If $T \in \mathcal{S}'(R^m, C)$, then $\widehat{(\partial_j T)} = ix_j \hat{T}$ and $\widehat{(ix_j T)} = -\partial_j \hat{T}$. (See [149], Chapter VI, §2.) If $f \in L^1(R^m)$, then $\hat{f} \in \mathcal{C}^{0,0}(R^m)$ and $\hat{f}(x) \to 0$ as $|x| \to \infty$

(see [42], Theorem 14.2). If $f \in \mathcal{S}'(R^m)$ has compact support, then $\hat{f} \in \mathcal{C}^\infty(R^m)$ (see [42], Lemma 14.4).

Lemma 1.20.1 *Let $\Omega \subset R^m$ be a domain. If $T \in \mathcal{D}'(\Omega)$ satisfies $\partial_j T = 0$ for $j = 1, \dots, m$, then T is constant, i.e., there exists $a \in R^1$ such that*

$$\langle T, \varphi \rangle = a \int_\Omega \varphi \, d\mathcal{H}_m$$

for each $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

(See [140], Lemma 6.4.)

Definition 1.20.2 Let $\Omega \subset R^m$ be an open set and $F, G \in \mathcal{D}'(\Omega)$. We say that $F \le G$ if $\langle F, \varphi \rangle \le \langle G, \varphi \rangle$ for each nonnegative $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Lemma 1.20.3 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, and $T \in \mathcal{D}'(\Omega)$. If $\langle T, \varphi \rangle \ge 0$ for each nonnegative $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then T is a nonnegative Borel measure.*

(See [67], Theorem 2.1.7.)

Lemma 1.20.4 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, and $T \in \mathcal{D}'(\Omega)$. If* $\operatorname{spt} T \subset \{x\}$, *then there exist $k \in N$ and constants a_α such that*

$$T = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_x.$$

(See [130], Chapter II, §10.)

Lemma 1.20.5 *Let T be a distribution with compact support in R^m. If $\Omega \subset R^m$ is an open set such that* $\operatorname{spt} T \subset \Omega$, *then there exist $k \in N$ and $f_\alpha \in L^1(R^m)$ with $f_\alpha = 0$ on $R^m \setminus \Omega$ such that*

$$T = \sum_{|\alpha| \le k} \partial^\alpha f_\alpha.$$

(See [130], Chapter II, §5.)

1.21 Sobolev Spaces

The introduction of distributions enables to define more appropriate spaces of functions in which we shall look for solutions of boundary value problems for the Laplace equation. Sobolev spaces are such function spaces. This section contains all about Sobolev spaces we shall need in the future.

Let $1 \le p \le \infty$, Ω be an open subset of R^m, $k \ge 0$ integer. Denote by $W^{k,p}(\Omega)$ the space of all functions $f \in L^p(\Omega)$ such that $\partial^\alpha f \in L^p(\Omega)$ in the sense of distributions for each multiindex α with $|\alpha| \leq k$ equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha| \leq k} \int_{\Omega} |\partial^{\alpha} u|^{p} \, d\mathcal{H}_m \right]^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|\partial^{\alpha} u\|_{L^{\infty}(\Omega)}.$$

Then $W^{k,p}(\Omega)$ is a Banach space (see [149], Chapter 1, §9, Proposition 5). For $p = 2$, $W^{k,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v) = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^{\alpha} u \partial^{\alpha} v \, d\mathcal{H}_m.$$

(See [140], Lemma 5.2.) If $u \in L^{p}_{loc}(\Omega)$, then

$$u(x) = \lim_{r \downarrow 0} \fint_{B(x;r)} u \, \mathrm{d}\mathcal{H}_m \tag{1.11}$$

for almost all $x \in \Omega$ (see Proposition 1.13.9). If $u \in W^{k,p}(\Omega)$, we shall suppose that (1.11) holds whenever the limit on the right side exists.

Let $\Omega \subset R^m$ be an open set, $s = k + \lambda$, $k \in N_0$, $0 < \lambda < 1$, and $1 < p < \infty$. Denote $W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega); \|u\|_{W^{s,p}(\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,p}(\Omega)} = \sqrt[p]{\|u\|^{p}_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_{\Omega \times \Omega} \frac{|\partial^{\alpha} u(x) - \partial^{\alpha} u(y)|^{p}}{|x - y|^{m+p\lambda}} \, \mathrm{d}\mathcal{H}_{2m}(x, y)}.$$

Let $\Omega \subset R^m$ be an open set, $0 \leq s < \infty$, and $1 < p < \infty$. Denote by $W^{s,p}_{loc}(\Omega)$ the set of all Lebesgue measurable functions f in Ω such that $f \in W^{s,p}(G)$ for each bounded open subset G of Ω with $\overline{G} \subset \Omega$ and by $W^{s,p}_{loc}(\overline{\Omega})$ the set of all Lebesgue measurable functions f in Ω such that $f \in W^{s,p}(G)$ for each bounded open subset G of Ω. Denote by $\overset{\circ}{W}{}^{s,p}(\Omega)$ the closure of $\mathcal{C}^{\infty}_{c}(\Omega)$ in $W^{s,p}(\Omega)$. The dual space of $\overset{\circ}{W}{}^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ where $p' = p/(p-1)$.

Theorem 1.21.1 *Let $\Omega \subset R^m$ be an open set, $u \in L^p(\Omega)$, and $1 \leq p \leq \infty$. Then $u \in W^{1,p}(\Omega)$ if and only if u has a representative $\tilde{u}$ that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes and whose (classical) partial derivatives belong to $L^p(\Omega)$.*

(See [150], Theorem 2.1.4.)

Theorem 1.21.2 *Let* $1 < p < \infty$. *If* $f \in W^{1,p}(R^m)$ *and* $h \in R^m \setminus \{0\}$, *then*

$$|h|^{-1}\|f(x+h) - f(x)\|_{L^p(R^m)} \le \|\nabla f\|_{L^p(R^m)}.$$

On the other hand, if $f \in L^p(R^m)$ *and*

$$\sup_{|h|>0} |h|^{-1}\|f(x+h) - f(x)\|_{L^p(R^m)} < \infty,$$

then $f \in W^{1,p}(R^m)$

(See [150], Theorem 2.1.6 and the proof of this theorem in [150], p. 46.)

Lemma 1.21.3 *Let* $\Omega \subset R^m$ *be a bounded open set with continuous boundary. If* $1 \le p < \infty$, *then* $\mathcal{C}_c^\infty(R^m)$ *is dense in* $W^{1,p}(\Omega)$.

(See [140], Lemma 12.3.)

Lemma 1.21.4 *If* $1 \le p < \infty$, $k \in N_0$, *then* $W^{k,p}(R^m) = \overset{\circ}{W}{}^{k,p}(R^m)$.

(See [140], Lemma 6.5.)

Theorem 1.21.5 *Let* $\Omega \subset R^m$ *be an open set,* $u, v \in W^{1,p}(\Omega)$, $1 \le p \le \infty$. *Then* $\max(u,v), \min(u,v) \in W^{1,p}(\Omega)$ *and*

$$\nabla \max(u,v) = \begin{cases} \nabla u \text{ a.e. on } \{u \ge v\}, \\ \nabla v \text{ a.e. on } \{v \ge u\}, \end{cases}$$

$$\nabla \min(u,v) = \begin{cases} \nabla u \text{ a.e. on } \{u \le v\}, \\ \nabla v \text{ a.e. on } \{v \le u\}. \end{cases}$$

In particular, $\nabla u = \nabla v$ *a.e. on the set* $\{x \in \Omega; u(x) = v(x)\}$.

(See [101], Corollary 1.43 or [98], Corollary 6.18.)

Lemma 1.21.6 *Let* $m, k \in N$, $1 \le p < m/k$, $q = mp/(m-kp)$. *Then there exists a constant* C *such that*

$$\|u\|_{L^q(R^m)} \le C\|\nabla^k u\|_{L^p(R^m)} \qquad \forall u \in \overset{\circ}{W}{}^{k,p}(R^m).$$

(See [105], Chapter I, §1.8, Corollary 1 or [150], Theorem 2.4.2.)

Lemma 1.21.7 *Let* $1 \le p \le \infty$, $k \in N$, $l \in N_0$, $l < k$, *and* $\Omega \subset R^m$ *be a bounded open set. Then the injection of* $\overset{\circ}{W}{}^{1,p}(\Omega)$ *into* $L^p(\Omega)$ *is compact. If* $kp < m$ *and* $1 \le q < mp/(m - mp)$, *then* $\overset{\circ}{W}{}^{k,p}(\Omega)$ *is compactly imbedded into* $L^q(\Omega)$. *If* $(k-l)p > m$, *then* $\overset{\circ}{W}{}^{k,p}(\Omega)$ *is compactly imbedded into* $\mathcal{C}^l(\overline{\Omega})$.

(See [140], Lemma 11.2 and [150], Lemma 2.5.1.)

Lemma 1.21.8 *If* $1 \leq p \leq \infty$ *and* $\Omega \subset R^m$ *is a bounded open set with boundary of class* $\mathcal{C}$*, then the injection of* $W^{1,p}(\Omega)$ *into* $L^p(\Omega)$ *is compact.*

(See [140], Lemma 18.4.)

Proposition 1.21.9 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $k \in N$*, and* $1 \leq p \leq \infty$*. Let* $T_1, \ldots, T_n$ *be bounded linear operators on* $W^{k,p}(\Omega)$*. Define*

$$\|u\| = \left[\sum_{|\alpha|=k} \|\partial^\alpha u\|^p_{L^p(\Omega)} + \sum_{j=1}^n |T_j(u)|^p \right]^{1/p} .$$

Suppose that for a polynomial u *of degree less or equal to* $k-1$ *the relation* $\|u\| = 0$ *implies that* $u = 0$*. Then* $\| \ \|$ *is an equivalent norm on* $W^{k,p}(\Omega)$*.*

(See [94], Theorem 5.11.2.)

Lemma 1.21.10 *Let* $1 \leq p \leq \infty$*,* $k \in N$*, and* $\Omega \subset R^m$ *be a bounded open set. Then* $\|\nabla^k u\|_{L^p(\Omega)}$ *is a norm in* $\overset{\circ}{W}{}^{k,p}(\Omega)$ *which is equivalent to the original norm.*

(See [150], Theorem 4.4.1.)

Proposition 1.21.11 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary,* $-\infty < s < \infty$*, and* $1 \leq p < \infty$*. Then* ∂_j *is a bounded linear operator from* $W^{s,p}(\Omega)$ *into* $W^{s-1,p}(\Omega)$ *unless* $s = 1/p$*.*

(See [58], Theorem 1.4.4.6.)

Lemma 1.21.12 *Let* $\Omega \subset R^m$ *be an open set. Suppose that there exist* $n \in N$*,* $M > 0$*, and* $\epsilon > 0$ *and a sequence of open sets* U_j *such that:*

- *If* $x \in \partial\Omega$*, then there exists* $j \in N$ *such that* $B(x; \epsilon) \subset U_j$*.*
- *If* $x \in R^m$*, then* x *is an element of at most* n *sets* U_j*.*
- *For each* j*, there exists a graph Lipschitz domain* Ω_j *such that* $\Omega \cap U_j = \Omega_j \cap U_j$*.*

Then there exists a linear operator E *mapping functions defined on* Ω *into functions defined on* R^m *and having the following properties:*

- $Eu = u$ *on* Ω*;*
- E *is a continuous linear operator from* $W^{k,p}(\Omega)$ *into* $W^{k,p}(R^m)$ *for all* $1 \leq p \leq \infty$*,* $k \in N_0$*.*

(See [136], Chapter VI, §3, Theorem 5.)

Theorem 1.21.13 *Let* $1 \leq p < \infty$*,* $k \in N$*,* $l \in N_0$*,* $l \leq k$*, and* $\Omega \subset R^m$ *be a domain. Suppose that* $W^{k,p}(\Omega)$ *can be continuously extended onto* $W^{k,p}(R^m)$*.*

- *If* $p(k-l) < m$, $l < k$, *and* $p \leq q \leq mp/[m-(k-l)p]$, *then* $W^{k,p}(\Omega)$ *is continuously imbedded into* $W^{l,q}(\Omega)$.
- *If* $p(k-l) = m$ *and* $p > 1$, *then* $W^{k,p}(\Omega)$ *is continuously imbedded into* $W^{l,q}(\Omega)$ *for every* $p \leq q < \infty$.
- *If* $p = 1$ *and* $k - l = m$, *then* $W^{k,p}(\Omega)$ *is continouously imbedded into* $\mathcal{C}^{l,0}(\overline{\Omega})$.
- *If* $p(k-l) > m$, $\lambda = k-l-m/p \in (0,1)$, *then* $W^{k,p}(\Omega)$ *is continuously imbedded into* $\mathcal{C}^{l,\lambda}(\overline{\Omega})$.

(See [105], Chapter I, §1.4.1, §1.8.1, §1.8.2.)

Proposition 1.21.14 *Let* $\Omega, \tilde{\Omega}, \omega, \tilde{\omega} \subset R^m$ *be open sets and* $\overline{\omega} \subset \Omega$, $\overline{\tilde{\omega}} \subset \tilde{\Omega}$, *and* ω, $\tilde{\omega}$ *be bounded. Suppose moreover that* $T = (T_1, \dots, T_m)$ *is a Lipschitz continuous injective mapping* Ω *onto* $\tilde{\Omega}$ *such that* T^{-1} *is Lipschitz continuous. If* $f \in W^{1,p}(\tilde{\omega})$ *and* $1 \leq p \leq \infty$, *then* $g = f \circ T \in W^{1,p}(\omega)$. *If* $y = T(x)$, *then*

$$\partial_k g(x) = \sum_{j=1}^{m} [\partial_j f(y)][\partial_k T_j(x)].$$

(See [150], Theorem 2.2.2.)

Proposition 1.21.15 *Let* $\Omega, \omega \subset R^m$ *be open. Let* Φ *be a one-to-one transformation of* ω *onto* Ω *having inverse* $\Psi = \Phi^{-1}$. *Let* $k \in N$ *and* $1 \leq p \leq \infty$. *If* $\Phi \in \mathcal{C}^{k,0}(\overline{\omega}, R^m)$ *and* $\Psi \in \mathcal{C}^{k,0}(\overline{\Omega}, R^m)$, *then the transform* $Tu = u \circ \Phi$ *is a bounded linear operator from* $W^{k,p}(\Omega)$ *to* $W^{k,p}(\omega)$ *and has a bounded inverse.*

(See [1], Theorem 3.41.)

Lemma 1.21.16 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. If* $1 \leq p < \infty$ *and* $0 \leq t < s < \infty$, *then* $W^{s,p}(\Omega) \hookrightarrow W^{t,p}(\Omega)$.

(See [122], Chap. 2, §5.4, Lemma 5.4.)

Lemma 1.21.17 *Let* $\Omega \subset R^m$ *be a bounded domain,* $k \in N_0$, $s = k + \lambda$, $0 < \lambda < 1$, *and* $1 \leq p < \infty$. *If* $u \in W^{s,p}(\Omega)$ *and* $v \in \mathcal{C}^{k+1}(\overline{\Omega})$, *then* $uv \in W^{s,p}(\Omega)$ *and*

$$\|uv\|_{W^{s,p}(\Omega)} \leq c\|u\|_{W^{s,p}(\Omega)},$$

where c does not depend on u.

(See [40], Lemma 6.36.)

Theorem 1.21.18 *Let* $k \in N_0$ *and* $\Omega \subset R^m$ *be a bounded domain of class* $\mathcal{C}^{k,1}$. *Then there exists an extension operator E such that E is a bounded linear operator from* $W^{s,p}(\Omega)$ *to* $W^{s,p}(R^m)$ *for all* $0 \leq s \leq k+1$, $1 \leq p < \infty$.

(See [40], Satz 6.40.)

1.22 Bessel-Potential Spaces

Further appropriate function spaces for solution of boundary value problems are Bessel-potential spaces. Sobolev spaces form a special class of these spaces.

For $s > 0$ and $x \in R^m$, denote

$$G_s(x) = (4\pi)^{-s/2}\Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty e^{-\pi|x|^2/t} e^{-t/4p} t^{(s-m)/2} \frac{\mathrm{d}}{t} .$$

Then $\|G_s\|_{L^1(R^m)} = 1$, $G_s * G_t = G_{t+s}$, and the Fourier transform $\hat{G}_s(x) = (1 + 4\pi|x|^2)^{-s/2}$ (see [136], Chapter V, §3). If $1 \le p \le \infty$ denote

$$L_s^p(R^m) = \{G_s * f; f \in L^p(R^m)\},$$

$$\|G_s * f\|_{L_s^p(R^m)} = \|f(x)\|_{L^p(R^m)}.$$

Set $L_0^p(R^m) = L^p(R^m)$ and $\|g\|_{L_0^p(R^m)} = \|g\|_{L^p(R^m)}$. Then $L_s^p(R^m) \subset L_t^p(R^m)$ and $\|g\|_{L_t^p(R^m)} \le \|g\|_{L_s^p(R^m)}$ for $s > t \ge 0$ (see [136], Chapter V, §3).

We shall write $H^s(R^m) = L_s^2(R^m)$. $H^s(R^m)$ are so-called Sobolev–Slobodecki spaces and $L_s^p(R^m)$ are Bessel-potential spaces. Denote the dual space of $H^s(R^m)$ by $H^{-s}(R^m)$.

If $\Omega \subset R^m$ is open and $s \ge 0$, denote $L_s^p(\Omega) = \{u|\Omega; u \in L_s^p(R^m)\}$ equipped with the norm

$$\|u\|_{L_s^p(\Omega)} = \inf\{\|v\|_{L_s^p(R^m)}; v|\Omega = u\}.$$

Theorem 1.22.1 *If $s \in N$ and $1 < p < \infty$, then $L_s^p(R^m) = W^{s,p}(R^m)$ and both norms are equivalent.*

(See [150], Chapter 2, Theorem 2.6.1.)

Theorem 1.22.2 *If α is a multiindex, then $\partial^\alpha : u \mapsto \partial^\alpha u$ is a bounded linear operator from $L_s^p(R^m)$ to $L_{s-|\alpha|}^p(R^m)$.*

(See [123], Chapter 3, Theorem 9.)

1.23 Besov and Lizorkin–Triebel Spaces

Besov spaces and Lizorkin–Triebel spaces are further important function spaces. This section treats relations between spaces, imbeddings, extensions, and interpolations of these spaces.

Let $\Phi \in \mathcal{S}(R^m)$. Suppose that the Fourier transform $\hat{\Phi}$ of Φ satisfies $\operatorname{spt} \hat{\Phi} \subset B(0; 1)$ and $\hat{\Phi} = 1$ on $B(0; 1/2)$. For integer n set

$$\Phi_n(x) = 2^{mn}\Phi(2^n x), \quad \phi_n(x) = \Phi_n(x) - \Phi_{n-1}(x).$$

If $u \in \mathcal{S}'(R^m)$, $s \in R$, $1 \le p \le \infty$, and $1 \le q < \infty$, denote

$$\|u\|_{B_s^{p,q}} = \|\Phi * u\|_{L^p(R^m)} + \left[\sum_{n=1}^{\infty}(2^{ns}\|\phi_n * u\|_{L^p(R^m)})^q\right]^{1/q},$$

$$\|u\|_{B_s^{p,\infty}} = \|\Phi * u\|_{L^p(R^m)} + \sup_{n\in N} 2^{ns}\|\phi_n * u\|_{L^p(R^m)}.$$

For $s \in R$, $1 \le p, q \le \infty$, define the Besov spaces

$$B_s^{p,q}(R^m) = \{u \in \mathcal{S}'(R^m); \|u\|_{B_s^{p,q}} < \infty\}.$$

If $u \in \mathcal{S}'(R^m)$, $s \in R$, and $1 \le p, q < \infty$, denote

$$\|u\|_{F_s^{p,q}} = \|\Phi * u\|_{L^p(R^m)} + \left\{\int_{R^m}\left[\sum_{n=1}^{\infty}(2^{ns}|\phi_n * u(x)|)^q\right]^{p/q} \mathrm{d}\mathcal{H}_m(x)\right\}^{1/p},$$

$$\|u\|_{F_s^{p,\infty}} = \|\Phi * u\|_{L^p(R^m)} + \left\{\int_{R^m}\left[\sup_{n\in N} 2^{ns}|\phi_n * u(x)|\right]^p \mathrm{d}\mathcal{H}_m(x)\right\}^{1/p},$$

$$\|u\|_{F_s^{\infty,q}} = \|\Phi * u\|_{L^\infty(R^m)} + \operatorname{essup}_{x\in R^m}\left[\sum_{n=1}^{\infty}(2^{ns}|\phi_n * u(x)|)^q\right]^{1/q},$$

$$\|u\|_{F_s^{\infty,\infty}} = \|\Phi * u\|_{L^\infty(R^m)} + \operatorname{essup}_{x\in R^m} \sup_{n\in N} 2^{ns}|\phi_n * u(x)|.$$

For $s \in R$, $1 \le p < \infty$, and $1 \le q < \infty$, define the Lizorkin–Triebel spaces

$$F_s^{p,q}(R^m) = \{u \in \mathcal{S}'(R^m); \|u\|_{F_s^{p,q}} < \infty\}.$$

$B_s^{p,q}(R^m)$ and $F_s^{p,q}(R^m)$ do not depend on the choice of Φ, and the corresponding norms are equivalent (see [143], §2.3.2, Proposition 1).

If $\Omega \subset R^m$ is open and $s \in R^1$, $1 \le p, q \le \infty$, denote by $B_s^{p,q}(\Omega)$ the set of all $u \in \mathcal{D}'(\Omega)$ for which there exists $v \in B_s^{p,q}(R^m)$ such that $u = v$ on Ω,

$$\|u\|_{B_s^{p,q}(\Omega)} = \inf\{\|v\|_{B_s^{p,q}(R^m)}; u = v \text{ on } \Omega\}.$$

Since $B_s^{p,q}(\Omega)$ is the factor space $B_s^{p,q}(R^m)/\{f \in B_s^{p,q}(R^m); f = 0 \text{ on } \Omega\}$, it is a Banach space (see Proposition 1.3.13). Similarly, if $p, q < \infty$, denote by $F_s^{p,q}(\Omega)$ the set of all $u \in \mathcal{D}'(\Omega)$ for which there exists $v \in F_s^{p,q}(R^m)$ such that $u = v$ on Ω,

$$\|u\|_{F_s^{p,q}(\Omega)} = \inf\{\|v\|_{F_s^{p,q}(R^m)}; u = v \text{ on } \Omega\}.$$

$F_s^{p,q}(\Omega)$ is a Banach space.

We denote by $\overset{\circ}{B}{}_s^{p,q}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $B_s^{p,q}(\Omega)$ (and therefore in $B_s^{p,q}(R^m)$) and by $\overset{\circ}{F}{}_s^{p,q}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $F_s^{p,q}(\Omega)$ (and therefore in $F_s^{p,q}(R^m)$).

Theorem 1.23.1 *Let $s \in R$ and $1 \le p, q \le \infty$:*

- *The Besov spaces $B_s^{p,q}(R^m)$ are Banach spaces. If $p, q < \infty$, then $F_s^{p,q}(R^m)$ are Banach spaces.*
- *$\mathcal{S}(R^m) \subset B_s^{p,q}(R^m) \subset \mathcal{S}'(R^m)$. If $s > 0$, then $B_s^{p,q}(R^m) \hookrightarrow L^p(R^m)$.*
- *If $p, q < \infty$, then $\mathcal{S}(R^m)$ is a dense subset of $B_s^{p,q}(R^m)$.*
- *If $p, q < \infty$, then $\mathcal{S}(R^m) \subset F_s^{p,q}(R^m) \subset \mathcal{S}'(R^m)$ and $\mathcal{S}(R^m)$ is a dense subset of $F_s^{p,q}(R^m)$.*
- *If $p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, then $[B_s^{p,q}(R^m)]' = B_{-s}^{p',q'}(R^m)$. If $q > 1$, then $[F_s^{p,q}(R^m)]' = F_{-s}^{p',q'}(R^m)$.*

(See [2], Chapter IV, Theorem 4.1.3; [145], §2.3.3; [145], §2.11.2, Theorem.)

Theorem 1.23.2 *Let $k \in N_0$, $0 < \lambda \le 1$, and $s = k + \lambda$. Then $B_s^{\infty,\infty}(R^m) = \mathcal{C}^{k,\lambda}(R^m)$ and the corresponding norms are equivalent.*

(See [145], §2.5.7, Theorem or [145], §2.7.1, Remark 3.)

Theorem 1.23.3 *Let $k \in N_0$, $0 < \lambda < 1$, $s = k + \lambda$, and $1 \le p < \infty$. Then $B_s^{p,p}(R^m) = W^{s,p}(R^m)$ and the corresponding norms are equivalent.*

(See [145], §2.5.7, Theorem.)

Theorem 1.23.4 *Let $1 < p < \infty$. Then $F_0^{p,2}(R^m) = L^p(R^m)$ and the corresponding norms are equivalent.*

(See [145], §2.5.8, Theorem 1 and [145], §2.2.2.)

Theorem 1.23.5 *Let $s \in R$ and $1 < p < \infty$. Then $F_s^{p,2}(R^m)$ coincides with $L_s^p(R^m)$ and the corresponding norms are equivalent.*

(See [2], Theorem 4.2.2.)

Lemma 1.23.6 *Let $s(1), s(2) \in R$, $1 \le p(1), p(2), q(1), q(2) \le \infty$.*

- *$B_{s(1)}^{p(1),q(1)}(R^m) = B_{s(2)}^{p(2),q(2)}(R^m)$ if and only if $s(1) = s(2)$, $p(1) = p(2)$, $q(1) = q(2)$.*
- *If $p(1), p(2) \in R$, then $F_{s(1)}^{p(1),q(1)}(R^m) = F_{s(2)}^{p(2),q(2)}(R^m)$ if and only if $s(1) = s(2)$, $p(1) = p(2)$, $q(1) = q(2)$.*

- *If* $p(1) \in R$, *then* $F_{s(1)}^{p(1),q(1)}(R^m) = B_{s(2)}^{p(2),q(2)}(R^m)$ *if and only if* $s(1) = s(2)$, $p(1) = p(2) = q(1) = q(2)$.

(See [145], §2.3.9, Theorem.)

Proposition 1.23.7 *The following imbeddings are continuous:*

- *If* $1 \le q(1) \le q(2) \le \infty$ *and* $s \in R$, *then* $B_s^{p,q(1)}(R^m) \hookrightarrow B_s^{p,q(2)}(R^m)$ *for* $1 \le p \le \infty$ *and* $F_s^{p,q(1)}(R^m) \hookrightarrow F_s^{p,q(2)}(R^m)$ *for* $1 \le p < \infty$.
- *If* $1 \le q(1), q(2) \le \infty$, $s \in R$, $\epsilon > 0$ *then* $B_{s+\epsilon}^{p,q(1)}(R^m) \hookrightarrow B_s^{p,q(2)}(R^m)$ *for* $1 \le p \le \infty$ *and* $F_{s+\epsilon}^{p,q(1)}(R^m) \hookrightarrow F_s^{p,q(2)}(R^m)$ *for* $1 \le p < \infty$.
- *If* $1 \le p < \infty$, $1 \le q \le \infty$, $s \in R$, *then*

$$B_s^{p,\min(p,q)}(R^m) \hookrightarrow F_s^{p,q}(R^m) \hookrightarrow B_s^{p,\max(p,q)}(R^m).$$

(See [145], §2.3.2, Proposition 2.)

Theorem 1.23.8 *If* α *is a multiindex, then* $\partial^\alpha : u \mapsto \partial^\alpha u$ *is a bounded linear operator from* $B_s^{p,q}(R^m)$ *to* $B_{s-|\alpha|}^{p,q}(R^m)$.

(See [123], Chapter 3, Theorem 9.)

Lemma 1.23.9 *Let* $s \in R$, $1 \le p(0) \le p(1) \le \infty$, *and* $1 \le q \le \infty$. *If* $g \in \mathcal{C}_c^\infty(R^m)$, *then* $f \mapsto fg$ *is a bounded linear operator from* $B_s^{p(1),q}(R^m)$ *into* $B_s^{p(0),q}(R^m)$. *If, moreover* $p(1), q \in R$, *then* $f \mapsto fg$ *is a bounded linear operator from* $F_s^{p(1),q}(R^m)$ *into* $F_s^{p(0),q}(R^m)$.

(See [145], §3.3.1, Lemma.)

Proposition 1.23.10 *If* $k(0), k(1) \in N_0$, $0 < \theta < 1$, $s = (1-\theta)k(0) + \theta k(1)$, *then* $(W^{k(0),p}(R^m), W^{k(1),p}(R^m))_{\theta,q} = B_s^{p,q}(R^m)$.

(See [123], Chapter 3, Corollary 3.)

Theorem 1.23.11 *Let* $0 < \theta < 1$, $1 \le q(1), q(2), p \le \infty$ *and* $s(1)$, $s(2)$ *be real numbers with* $s(1) \ne s(2)$, $s = (1-\theta)s(1) + \theta s(2)$. *Then* $(B_{s(1)}^{p,q(1)}(R^m), B_{s(2)}^{p,q(2)}(R^m))_{\theta,q} = B_s^{p,q}(R^m)$.

(See [145], §2.4.2, Theorem.)

Proposition 1.23.12 *Let* $s(0), s(1) \in R$, $1 < p, q < \infty$, $s(0) \ne s(1)$, *and* $0 < \theta < 1$, $s = (1-\theta)s(0) + \theta s(1)$. *Then* $[L_{s(0)}^p(R^m), L_{s(1)}^p(R^m)]_{\theta,q} = B_s^{p,q}(R^m)$.

(See [123], Chapter 3, Theorem 7.)

Proposition 1.23.13 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary,* $0 \le s < \infty$, $1 < p < \infty$.

- *If* $s \in N_0$, *then* $W^{s,p}(\Omega) = F_s^{p,2}(\Omega)$, *and the corresponding norms are equivalent.*
- *If* $s \notin N_0$, *then* $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$, *and the corresponding norms are equivalent.*

(See [118], Chapter 2, §2.4 and [38], Theorem 6.7.)

Theorem 1.23.14 *Let* $1 \le p(0), p(1), q(0), q(1) \le \infty$, $-\infty < s(1) < s(0) < \infty$, $s(0) - m/p(0) > s(1) - m/p(1)$, $\Omega \subset R^m$ *is open and bounded. Then the identity is a compact operator from* $B^{p(0),q(0)}_{s(0)}(\Omega)$ *to* $B^{p(1),q(1)}_{s(1)}(\Omega)$.

(See [146], Theorem 1.97.)

Proposition 1.23.15 *Let* $\Omega \subset R^m$ *be open.*

- *Let* $s \in R^1$, $1 \le p < \infty$, $1 \le q, r, t \le \infty$. *Then* $B^{p,r}_s(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \hookrightarrow B^{p,t}_s(\Omega)$ *if and only if* $r \le \min(p, q) \le \max(p, q) \le t$.
- *Let* $s, s(0), s(1) \in R^1$, $1 \le p(0) < p < p(1) \le \infty$, $s(0) - m/p(0) = s - m/p = s(1) - m/p(1)$ *and* $1 \le q, r, t \le \infty$. *Then* $B^{p(0),r}_{s(0)}(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \hookrightarrow B^{p(1),t}_{s(1)}(\Omega)$ *if and only if* $r \le p \le t$.
- *Let* $s(0), s(1) \in R^1$, $1 \le p(0) < p(1) < \infty$, $s(0) - m/p(0) = s(1) - m/p(1)$ *and* $1 \le r, t \le \infty$. *Then* $F^{p(0),r}_{s(0)}(\Omega) \hookrightarrow F^{p(1),t}_{s(1)}(\Omega)$.

(See [146], pp. 78–79.)

Theorem 1.23.16 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. Let* $s(0), s(1) \in R$, $s(0) \ne s(1)$, $1 \le q(0), q(1), q \le \infty$, $0 < \theta < 1$, $s = (1 - \theta)s(0) + \theta s(1)$. *Then*

$$(F^{p,q(0)}_{s(0)}(\Omega), F^{p,q(1)}_{s(1)}(\Omega))_{\theta,q} = B^{p,q}_s(\Omega), \quad 1 \le p < \infty,$$

$$(B^{p,q(0)}_{s(0)}(\Omega), B^{p,q(1)}_{s(1)}(\Omega))_{\theta,q} = B^{p,q}_s(\Omega), \quad 1 \le p \le \infty.$$

(See [146], Corollary 1.111.)

1.24 Traces

If u is a continuous function on the closure of a domain Ω, then the restriction of u onto the boundary of Ω is called the trace of u. We define Sobolev, Besov, and Bessel-potential spaces on $\partial\Omega$ and show how to extend the notion of trace to function spaces. We show under which condition the trace is a continuous linear operator or even a compact operator. We also characterize the space of traces for Sobolev spaces, Bessel-potential spaces, Besov spaces, and Lizorkin–Triebel spaces. We formulate Gauss–Green theorem for Sobolev spaces.

Suppose that $\Omega \subset R^m$ is an open set with Lipschitz boundary $\partial\Omega$, $1 < p \le \infty$, and $u \in W^{1,p}(\Omega)$. According to Lemma 1.21.12, there exists $\tilde{u} \in W^{1,p}(R^m)$ such that $u = \tilde{u}$ in Ω. We shall suppose that

$$\tilde{u}(x) = \lim_{r \downarrow 0} \oint_{B(x;r)} \tilde{u} \, \mathrm{d}\mathcal{H}_m \tag{1.12}$$

whenever this limit exists. This is true for $\mathcal{H}_{m-1}$ almost all points of $\partial\Omega$ (see [150], Theorem 3.3.3 and [150], Theorem 2.6.16). We define the trace of u setting $u = \tilde{u}$ on $\partial\Omega$. If $x \in \partial\Omega$ and (1.12) holds true, then there exists a Borel set $A \subset \Omega$ such that

$$\lim_{r\downarrow 0} \frac{\mathcal{H}_m(B(x;r) \cap \Omega \setminus A)}{\mathcal{H}_m(\Omega \cap B(x;r))} = 0$$

and the restriction of $\tilde{u}$ onto A is continuous at x. (See [150], Remark 4.4.5.) So, the trace of u does not depend on the choice of an extension $\tilde{u}$. We shall denote by γ_Ω the trace operator.

Let $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x'\}$) be a graph Lipschitz domain, $0 < s < \infty$, and $1 < p < \infty$. We say that $f \in W^{s,p}(\partial\Omega)$ if $f(x', \Phi(x')) \in W^{s,p}(R^{m-1})$. We equip $W^{s,p}(\partial\Omega)$ with the norm

$$\|f\|_{W^{s,p}(\partial\Omega)} = \|f(x', \Phi(x'))\|_{W^{s,p}(R^{m-1})}.$$

Let now $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. Let now $z_1, \dots, z_k \in \partial\Omega$ be such that $\partial\Omega \subset B(z_1, r_j) \cup \dots B(z_k, r_k)$. Let now $G_1, \dots, G_k$ be graph Lipschitz domains such that $\Omega \cap B(z_j; r_j) = G_j \cap B(z_j; r_j)$. Choose $\alpha_j \in \mathcal{C}_c^\infty(B(z_j; r_j)$ such that $\alpha_1 + \dots + \alpha_k = 1$ on a neighborhood of $\partial\Omega$ (see Lemma 1.18.1). Let $0 < s < \infty$, $1 < p < \infty$. We say that $f \in W^{s,p}(\partial\Omega)$ if $\alpha_j f \in W^{s,p}(\partial G_j)$ for $j = 1, \dots, k$. We equip the space $W^{s,p}(\partial\Omega)$ with the norm

$$\|f\|_{W^{s,p}(\partial\Omega)} = \sum_{j=1}^{k} \|\alpha_j f\|_{W^{s,p}(\partial G_j)}.$$

Similarly we define $L_s^p(\partial\Omega)$ and $B_s^{p,q}(\partial\Omega)$ with $1 < q < \infty$. If $p' = p/(p-1)$ and $q' = q/(q-1)$, we denote by $W^{-s,p'}(\partial\Omega)$, $L_{-s}^{p'}(\partial\Omega)$, and $B_{-s}^{p',q'}(\partial\Omega)$ the dual spaces of $W^{s,p}(\partial\Omega)$, $L_s^p(\partial\Omega)$, and $B_s^p(\partial\Omega)$, respectively.

Theorem 1.24.1 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p \leq \infty$, and $1 \leq q < \infty$. If $p < m$ suppose moreover that $q < p(m-1)/(m-p)$. Then γ_Ω is a compact bounded linear operator from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$.*

Proof For $p < m$ see [94], §6.10.5. Let now $p \geq m$. Put $\tilde{p} = 2mq/(2q+1)$. Then $\tilde{p} \leq m \leq p$ and Hölder's inequality (Proposition 1.14.1) gives that $W^{1,p}(\Omega)$ is continuously imbedded into $W^{1,\tilde{p}}(\Omega)$. Since $\tilde{p}(m-1)/(m-\tilde{p}) = 2q(m-1) > q$, we have that γ_Ω is a compact bounded linear operator from $W^{1,\tilde{p}}(\Omega)$ to $L^q(\partial\Omega)$. Now we use Theorem 1.7.4.

Theorem 1.24.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary and $1 < p < \infty$. Then the trace γ_Ω is a bounded linear operator from $W^{1,p}(\Omega)$ to $W^{1-1/p,p}(\partial\Omega)$. On the other hand, there exists a bounded linear operator E from $W^{1-1/p,p}(\partial\Omega)$ to $W^{1,p}(\Omega)$ such that $\gamma_\Omega Ef = f$.*

(See [94], Theorem 6.8.13 and [94], Theorem 6.9.2.)

Proposition 1.24.3 *If* $\Omega \subset R^m$ *is a bounded open set with Lipschitz boundary and* $p > 1$*, then* $\overset{\circ}{W}{}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega); u = 0 \text{ on } \partial\Omega\}$.

(See [140], Lemma 13.7.)

Proposition 1.24.4 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary. If* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $u \in W^{1,p}(\Omega)$*, and* $v \in W^{1,p'}(\Omega)$*, then for every* $j = 1, \ldots, m$*, one has*

$$\int\limits_{\Omega} \left(u \frac{\partial v}{\partial x_j} + v \frac{\partial u}{\partial x_j} \right) \mathrm{d}\mathcal{H}_m = \int\limits_{\partial\Omega} n_j^{\Omega} uv \ \mathrm{d}\mathcal{H}_{m-1}.$$

(See [140], Lemma 14.4.)

Theorem 1.24.5 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{k-1,1}$, $k \in N$*, and* $1 < p < \infty$*. If* α *is a multiindex with* $|\alpha| < k$*, then* ∂^α *is a bounded linear operator from* $W^{k,p}(\Omega)$ *to* $W^{k-|\alpha|-1/p,p}(\partial\Omega)$.

(See [122], Chap. 2, §5.4, Théorème 5.5.)

Theorem 1.24.6 *Let* $1 \le p < \infty$, $k \in N_0$, $0 < \lambda < 1$*, and* $s = k + \lambda + 1/p$*. Let* $\Omega \subset R^m$ *be a bounded open set with boundary of class* $\mathcal{C}^{l,1}$ *where* $s \le l + 1$*. Then the mapping*

$$T : u \mapsto \left[u, \frac{\partial u}{\partial n}, \ldots, \frac{\partial^k u}{\partial n^k} \right]$$

which is defined for $u \in \mathcal{C}^{l,1}(\overline{\Omega})$*, has a unique continuous extension from* $W^{s,p}(\Omega)$ *onto*

$$X = \prod_{j=0}^{k} W^{s-j-1/p,p}(\partial\Omega).$$

There exists a bounded linear operator $S : X \to W^{s,p}(\Omega)$ *such that* $TSu = u$ *for* $u \in W^{s,p}(\Omega)$*. This operator does not depend on* p.

(See [58], Theorem 1.5.1.2.)

Theorem 1.24.7 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary or a graph Lipschitz domain,* $1 \le p, q \le \infty$, $0 < s < 1$*. Then the following hold:*

- *The restriction to the boundary extends to a bounded linear trace operator* $\gamma_\Omega : B^{p,q}_{s+1/p}(\Omega) \to B^{p,q}_s(\partial\Omega)$*. Moreover,* γ_Ω *is onto and there is a bounded linear operator* $E : B^{p,q}_s(\partial\Omega) \to B^{p,q}_{s+1/p}(\Omega)$ *such that* $\gamma_\Omega E u = u$ *for* $u \in B^{p,q}_s(\partial\Omega)$.
- *If* $p, q < \infty$*, then the restriction to the boundary extends to a bounded linear trace operator* $\gamma_\Omega : F^{p,q}_{s+1/p}(\Omega) \to B^{p,q}_s(\partial\Omega)$*. Moreover,* γ_Ω *is onto, and there is*

a bounded linear operator $E : B_s^{p,q}(\partial\Omega) \to F_{s+1/p}^{p,q}(\Omega)$ *such that* $\gamma_\Omega E u = u$ *for* $u \in F_s^{p,q}(\partial\Omega)$.

(See [118], Theorem 2.5.2.)

Theorem 1.24.8 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary,* $1 < p < \infty$, *and* $0 < s < 1$. *The restriction to the boundary extends to a bounded linear trace operator* $\gamma_\Omega : W^{s+1/p,p}(\Omega) \to W^{s,p}(\partial\Omega)$. *Moreover,* γ_Ω *is onto, and there is a bounded linear operator* $E : W^{s,p}(\partial\Omega) \to W^{s+1/p,p}(\Omega)$ *such that* $\gamma_\Omega E u = u$ *for* $u \in W^{s,p}(\partial\Omega)$.

(See [58], Theorem 1.5.1.2.)

Theorem 1.24.9 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $1 < p, q < \infty$, *and* $1/p < s < 1 + 1/p$. *Denote by* $\overset{\circ}{B}{}_s^{p,q}(\Omega)$ *the closure of* $\mathcal{C}_c^\infty(\Omega)$ *in* $B_s^{p,q}(\Omega)$. *Then* $\overset{\circ}{B}{}_s^{p,q}(\Omega) = \{u \in B_s^{p,q}(\Omega); \gamma_\Omega u = 0\}$.

(See [118], Theorem 2.5.3.)

1.25 Homogeneous Sobolev Spaces

Boundary value problems for the Laplace equation are solvable in Sobolev spaces for a bounded domain with smooth boundary. But if we study boundary value problems on exterior domains, we see that they are not solvable in Sobolev spaces. The problem is that functions from these spaces have too nice behavior at infinity. To overcome this difficulty, we define homogeneous Sobolev spaces, the spaces of locally integrable functions for which the derivatives of a fixed order are p-integrable.

Let $\Omega \subset R^m$ be an open set, $1 \le p \le \infty$, and $k \in N_0$. The space $D^{k,p}(\Omega)$ consists of those functions in $L^p_{loc}(\Omega)$ for which all generalized derivatives of the order k are in $L^p(\Omega)$. The space $D^{k,p}(\Omega)$ is the space of distributions on Ω with derivatives of the order k in $L^p(\Omega)$ (see [105], §1.2.3, Theorem 1). If $u \in D^{k,p}(\Omega)$ and α is a multiindex with $|\alpha| < k$, then $D^\alpha u \in L^p_{loc}(\Omega)$ (see [105], §1.2.3).

Let Ω be a domain, $1 \le p \le \infty$, and $k \in N$. Fix a bounded open set ω such that $\overline{\omega} \subset \Omega$. Then $D^{k,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{D^{k,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla^k u\|_{L^p(\Omega)}.$$

Moreover, different choices of ω give equivalent norms. (See [105], §1.5.3, Corollary 2.)

Let $\Omega \subset R^m$ be an open set. Denote by $\overset{\circ}{D}{}^{k,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $D^{k,p}(R^m)$. If $1/p + 1/p' = 1$, denote by $D^{-k,p'}(\Omega)$ the dual space of $D^{k,p}(\Omega)$. Denote $\tilde{W}^{k,p}(\Omega) = \{\varphi|\Omega; \varphi \in \overset{\circ}{D}{}^{k,p}(R^m)\}$ equipped with the norm

$$\|f\|_{\tilde{W}^{k,p}(\Omega)} = \inf\{\|\varphi\|_{D^{k,p}(R^m)}; \varphi \in \overset{\circ}{D}{}^{k,p}(R^m), f = \varphi|\Omega\}.$$

Remark that $\tilde{W}^{k,p}(R^m) = \mathring{D}^{k,p}(R^m)$, $\mathring{D}^{k,p}(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $\tilde{W}^{k,p}(\Omega)$, and the corresponding norms are equivalent. Denote

$$N^{k,p}(\Omega) := \{u \in \mathring{D}^{k,p}(R^m); u = 0 \text{ in } \Omega\}.$$

Then $N^{k,p}(\Omega)$ is a closed subspace of $D^{k,p}(R^m)$. Since

$$\tilde{W}^{k,p}(\Omega) = \mathring{D}^{k,p}(R^m)/N^{k,p}(\Omega),$$

Proposition 1.3.13 gives that $\tilde{W}^{k,p}(\Omega)$ is a Banach space.

Denote $D^{k,p}(\overline{\Omega}) = \{\varphi|\Omega; \varphi \in D^{k,p}(R^m)\}$ equipped with the norm

$$\|f\|_{D^{k,p}(\overline{\Omega})} = \inf\{\|\varphi\|_{D^{k,p}(R^m)}; \varphi \in D^{k,p}(R^m), f = \varphi|\Omega\}.$$

Then $D^{k,p}(\overline{\Omega})$ is a Banach space. Clearly, $\tilde{W}^{k,p}(\Omega) \subset D^{k,p}(\overline{\Omega}) \subset D^{k,p}(\Omega)$.

Proposition 1.25.1 *Let $\Omega \subset R^m$ be an open set, $1 \le p \le \infty$, and $k \in N$. The space $D^{k,p}(\Omega)$ is the space of distributions on Ω with derivatives of order k in $L^p(\Omega)$.*

(See [105], §1.2.3, Theorem 1.)

Proposition 1.25.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 \le p \le \infty$, and $k \in N$. Then $W^{k,p}(\Omega) = \tilde{W}^{k,p}(\Omega) = D^{k,p}(\Omega) = D^{k,p}(\overline{\Omega})$, and the corresponding norms are equivalent.*

Proof $D^{k,p}(\Omega) = W^{k,p}(\Omega)$ and the corresponding norms are equivalent by [105], §1.5.2, [105], §1.5.3 and [105], §1.5.4. According to Lemma 1.21.12, there exists a bounded linear mapping E from $W^{k,p}(\Omega)$ to $W^{k,p}(R^m)$ such that $Ef = f$ on Ω. Since $W^{k,p}(R^m) = \mathring{W}^{k,p}(R^m)$ by Lemma 1.21.4, we obtain $W^{k,p}(\Omega) \subset \tilde{W}^{k,p}(\Omega)$. Hence $W^{k,p}(\Omega) = \tilde{W}^{k,p}(\Omega) = D^{k,p}(\Omega) = D^{k,p}(\overline{\Omega})$. If

$$\|f\|_{D^{k,p}(R^m)} = \|f\|_{L^p(\Omega)} + \| \, |\nabla^k f| \, \|_{L^p(R^m)},$$

then $\|f\|_{D^{k,p}(\Omega)} \le \|f\|_{D^{k,p}(R^m)}$ for each $f \in D^{k,p}(R^m)$. Thus $\|f\|_{D^{k,p}(\Omega)} \le \|f\|_{\tilde{W}^{k,p}(\Omega)} \le \|f\|_{\tilde{W}^{k,p}(\overline{\Omega})}$. All norms are equivalent by Proposition 1.3.6.

Lemma 1.25.3 *Let $1 < p < \infty$, $k \in N$, $\Omega \subset R^m$ be an open set. Then $D^{k,p}(\Omega)$, $\mathring{D}^{k,p}(\Omega)$, $D^{-k,p}(\Omega)$ are reflexive.*

Proof Fix a nonempty bounded open set $\omega \subset R^m$. Then $D^{k,p}(R^m)$ is isomorph to $\{[u|\omega, \nabla^k u]; u \in D^{k,p}(R^m)\}$, what is a closed subspace of the reflexive space $L^p(\omega) \times [L^p(R^m)]^n$ for some $n \in N$ (compare Theorem 1.14.3). So, $D^{k,p}(R^m)$ is a reflexive by Lemma 1.5.6. Since $\mathring{D}^{k,p}(\Omega)$ is a closed subspace of $D^{k,p}(R^m)$ it is reflexive by Lemma 1.5.6. Therefore its dual $D^{-k,p'}(\Omega)$ is reflexive. Here $p' = p/(p-1)$.

Lemma 1.25.4 *Let $\Omega \subset R^m$ be an open set, $\overline{\Omega} \neq R^m$, $1 \le p \le \infty$, $k \in N$. Then $\|\nabla^k u\|_{L^p(\Omega)}$ is an equivalent norm on $\mathring{D}^{k,p}(\Omega)$.*

Proof Let $\omega \subset R^m \setminus \overline{\Omega}$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $\|\varphi\|_{D^{k,p}(\Omega)} = \|\nabla^k \varphi\|_{L^p(\Omega)}$. The density of $\mathcal{C}_c^\infty(\Omega)$ in $\overset{\circ}{D}{}^{k,p}(\Omega)$ forces the proposition.

Lemma 1.25.5 *Let $\Omega \subset R^m$ be a domain, $1 \le p \le \infty$, and $k \in N$. Denote by $\tilde{P}_{k-1}(R^m)$ the space of polynomials of degree less than or equal to $k-1$. Then the factor space $D^{k,p}(\Omega)/\tilde{P}_{k-1}(R^m)$ is a Banach space equipped with the norm $\|\nabla^k u\|_{L^p(\Omega)}$.*

(See [105], §1.5.3.)

Proposition 1.25.6 *Let $1 \le p < \infty$, $k \in N$, X be a closed subspace of $D^{k,p}(R^m)$. If $\tilde{P}_{k-1}(R^m) \cap X = \{0\}$, then $\|\nabla^k u\|_{L^p(R^m)}$ is an equivalent norm in X.*

Proof Let $f_1, \dots, f_n$ be a basis of $\tilde{P}_{k-1}(R^m)$. If $j = 1, \dots, n$, then there exists a continuous linear functional T_j on $D^{k,p}(R^m)$ such that $T_j = 0$ on X, $T_j f_j \neq 0$ (see Proposition 1.5.3). Then $\operatorname{Ker} T_j$ is a closed subspace of $D^{k,p}(R^m)$ by Lemma 1.4.11 and $D^{k,p}(R^m) = \operatorname{Ker} T_j \oplus \{cf_j; c \in R\}$. Denote $Z = \operatorname{Ker} T_1 \cap \dots \operatorname{Ker} T_n$. Then Z is a closed subspace of $D^{k,p}(R^m)$ and $D^{k,p}(R^m) = Z \oplus \tilde{P}_{k-1}$. Define $Pf = \{f + v; v \in \tilde{P}_{k-1}(R^m)\}$. Then P is a bounded linear operator from Z into $D^{k,p}(R^m)/\tilde{P}_{k-1}(R^m)$ equipped with the norm $\|\nabla^k u\|_{L^p(R^m)}$. Moreover, P is one to one and onto. Since $D^{k,p}(R^m)/\tilde{P}_{k-1}(R^m)$ is a Banach space by Lemma 1.25.5, Theorem 1.6.3 gives that P is an isomorphism. So, there exists a constant C such that

$$\|u\|_{D^{k,p}(R^m)} \le C\|Pu\|_{D^{k,p}(R^m)/\tilde{P}_{k-1}(R^m)} = C\|\nabla^k u\|_{L^p(R^m)}, \qquad \forall u \in Z.$$

Since $X \subset Z$, we infer that $\|\nabla^k u\|_{L^p(R^m)}$ is an equivalent norm in X.

Lemma 1.25.7 *Let $m \ge 2$, $k \in N$, $l \in N_0$, $l < k$, $1 \le p < m/(k-l)$, and $q = mp/[m-(k-l)p]$. If $u \in \overset{\circ}{D}{}^{k,p}(R^m)$ and α is a multiindex with $|\alpha| = l$, then*

$$\|\partial^\alpha u\|_{L^q(R^m)} \le C\|\nabla^k u\|_{L^p(R^m)} \tag{1.13}$$

where C does not depend on u.

Proof According to Lemma 1.21.6, there exists a constant C depending only on m, $k-l$, and p such that (1.13) holds for all $u \in \mathcal{C}_c^\infty(R^m)$. Let $\varphi_n \in \mathcal{C}_c^\infty(R^m)$ and $\varphi_n \to u$ in $D^{k,p}(R^m)$. Since $\partial^\beta \varphi_n - \partial^\beta u \to 0$ in $L^p(R^m)$ for each multiindex β with $|\beta| = k$, $\partial^\alpha \varphi_n$ is a Cauchy sequence in $L^q(R^m)$. So there exists $f_\alpha \in L^q(R^m)$ such that $\partial^\alpha \varphi_n \to f_\alpha$ in $L^q(R^m)$. Let ω be a bounded domain with Lipschitz boundary. Since $\varphi_n \to u$ in $D^{k,p}(\Omega)$, we have that $\varphi_n \to u$ in $W^{k,p}(\Omega)$ by Proposition 1.25.2. Theorem 1.21.13 gives that $\partial^\alpha \varphi_n \to \partial^\alpha u$ in $L^q(\omega)$. So, $\partial^\alpha u = f_\alpha$. Therefore (1.13) holds.

Proposition 1.25.8 *Let $m, k \in N$, $l \in N_0$, $l < k$, and $1 < p < \infty$. If $p < m/k$, then $\tilde{P}_{k-1}(R^m) \cap \overset{\circ}{D}{}^{k,p}(R^m) = \{0\}$ and $\|\nabla^k u\|_{L^p(R^m)}$ is an equivalent norm in $\overset{\circ}{D}{}^{k,p}(R^m)$. If $m/(k-l) \le p$, then $\tilde{P}_l(R^m) \subset \overset{\circ}{D}{}^{k,p}(R^m)$.*

Proof Suppose first that $1 \le p < m/k$. Put $q = mp/(m - kp)$. If $u \in \tilde{P}_{k-1}(R^m) \cap \overset{\circ}{D}{}^{1,p}(R^m)$, then $u \in L^q(R^m)$ by Lemma 1.25.7. Thus $u \equiv 0$. Lemma 1.25.5 gives that $\|\nabla u^k\|_{L^p(R^m)}$ is an equivalent norm in $\overset{\circ}{D}{}^{k,p}(R^m)$.

Let now $m/(k-l) \le p$ and $u \in \tilde{P}_l(R^m)$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $B(0;1)$ and $\varphi = 0$ on $R^m \setminus B(0;2)$. Define $\varphi_n(x) = \varphi(x/n)$. Suppose that $|\partial^\alpha \varphi| \le K$ for $|\alpha| \le k$. Then $|\partial^\alpha \varphi_n| \le K n^{-|\alpha|}$. Fix a multiindex β with $|\beta| = k$. There are constants $a_{\alpha\gamma}$ (independent on n) such that

$$\partial^\beta(u\varphi_n) = \sum_{|\alpha|+|\gamma|=k, |\alpha| \le k} a_{\alpha\gamma} \partial^\alpha u \partial^\gamma \varphi_n.$$

Since $\varphi_n u = 0$ on $R^m \setminus B(0;2n)$, there exists a constant C independent of n such that

$$|\partial^\beta(u\varphi_n)| \le \sum_{|\alpha|+|\gamma|=k, |\alpha| \le k} |a_{\alpha\gamma}| C n^{l-|\alpha|} K n^{-|\gamma|}.$$

So, there exists a constant D independent of n such that $|\partial^\beta(u\varphi_n)| \le D n^{l-k}$. Since $p(k-l) \ge m$, we have

$$\int_{R^m} |\partial^\beta(u\varphi_n)|^p \, \mathrm{d}\mathcal{H}_m \le \int_{B(0;2n)} \left(D n^{l-k}\right)^p \, \mathrm{d}\mathcal{H}_m$$

$$\le \int_{B(0;2n)} D^p n^{-m} \, \mathrm{d}\mathcal{H}_m \le D^p \mathcal{H}_m(B(0;2)).$$

So, $\varphi_n u$ is a bounded sequence in $\overset{\circ}{D}{}^{k,p}(R^m)$. Since $\overset{\circ}{D}{}^{k,p}(R^m)$ is reflexive by Lemma 1.25.3, there is $v \in \overset{\circ}{D}{}^{k,p}(R^m)$ such that $u\varphi_n \to v$ weakly. (See Theorem 1.5.9.) In particular $\int wu\varphi_n \to \int wv$ for each $w \in \mathcal{C}_c^\infty(R^m)$. Since $\int wu\varphi_n \to \int wu$, we deduce that $\int wu = \int wv$ for each $w \in \mathcal{C}_c^\infty(R^m)$. Thus $u = v \in \overset{\circ}{D}{}^{k,p}(R^m)$.

Lemma 1.25.9 *If $u \in D^{1,p}(R^m)$ and $1 \le p \le \infty$, then there are constants C and k such that*

$$\int_{\partial B(0;r)} |u| \, \mathrm{d}\mathcal{H}_{m-1} \le C r^k, \qquad r > 1.$$

Therefore u is a tempered distribution.

Proof If $|y| = 1$, $r > 0$, and u is absolutely continuous on $\{ty; t > 0\}$, then

$$|u(ry)| = \left| u(y) + \int_1^r y \cdot \nabla u(ty)\,\mathrm{d}t \right| \le |u(y)| + \int_0^r |\nabla u(ty)|\,\mathrm{d}t.$$

u is absolutely continuous on $\{ty; t > 0\}$ for almost all $y \in \partial B(0;1)$ by Proposition 1.21.14 and Theorem 1.21.1. Moreover, $\|u\|_{L^1(\partial B(0;1))} < \infty$ by Theorem 1.24.1. If $r > 1$, we obtain by Fubini's theorem

$$\int_{\partial B(0;r)} |u|\,\mathrm{d}\mathcal{H}_{m-1} \le r^{m-1} \int_{\partial B(0;1)} |u|\,\mathrm{d}\mathcal{H}_{m-1}$$

$$+ r^{m-1} \int_{\partial B(0;1)} \int_1^r |\nabla u(ty)|\,\mathrm{d}t\mathrm{d}\mathcal{H}_{m-1}(y) \le r^{m-1} \|u\|_{L^1(\partial B(0;1))}$$

$$+ r^{m-1} \int_1^r t^{m-1} \int_{\partial B(0;1)} |\nabla u(ty)| \mathrm{d}\mathcal{H}_{m-1}(y)\,\mathrm{d}t$$

$$\le r^{m-1} \|u\|_{L^1(\partial B(0;1))} + r^{m-1} \|\nabla u\|_{L^1(B(0;r))}.$$

If $p = 1$ then

$$\int_{\partial B(0;r)} |u|\,\mathrm{d}\mathcal{H}_{m-1} \le r^{m-1} \left[\|u\|_{L^1(\partial B(0;1))} + \|\nabla u\|_{L^1(R^m)} \right].$$

If $p = \infty$ then

$$\int_{\partial B(0;r)} |u|\,\mathrm{d}\mathcal{H}_{m-1} \le r^{2m-1} \left[\|u\|_{L^1(\partial B(0;1))} + \mathcal{H}_m(B(0;1)) \|\nabla u\|_{L^\infty(R^m)} \right].$$

If $1 < p < \infty$, $p' = p/(p-1)$, then Hölder's inequality (Proposition 1.14.1) gives

$$\int_{\partial B(0;r)} |u| \le r^{m-1} \|u\|_{L^1(\partial B(0;1))} + r^{m-1} \|\nabla u\|_{L^p(B(0;r))} \mathcal{H}_m(B(0;r))^{1/p'}$$

$$\le r^{2m-1} \left[\|u\|_{L^1(\partial B(0;1))} + \mathcal{H}_m(B(0;1)) \|\nabla u\|_{L^p(R^m)} \right].$$

If $\varphi \in \mathcal{S}(R^m)$, then there exists a constant C_1 such that $|\varphi(x)| \le C_1|x|^{-k-2}$ for $|x| > 1$. Since $u \in L^1_{loc}(R^m)$ and

$$\int\limits_{\{|y|>1\}} |u(y)\varphi(y)|\,\mathrm{d}\mathcal{H}_m(y) = \int\limits_1^\infty \int\limits_{\partial B(0;r)} |u(y)\varphi(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)\mathrm{d}r$$

$$\le \int\limits_1^\infty C_1 r^{-k-2} C r^k\,\mathrm{d}r < \infty,$$

we deduce that $|\varphi u|\,\mathrm{d}\mathcal{H}_m < \infty$ and u is a tempered distribution.

Lemma 1.25.10 *Let $\Omega \subset R^m$ be an open set, $u_k \to u$ in $\tilde{W}^{1,p}(\Omega)$, and $1 \le p < \infty$. Then $u_k^+ \to u^+$ in $\tilde{W}^{1,p}(\Omega)$. If $u \in \overset{\circ}{D}{}^{1,p}(\Omega)$, then $u^+ \in \overset{\circ}{D}{}^{1,p}(\Omega)$, and there exist $\varphi_k \in \mathcal{C}_c^\infty(\Omega)$ and $\varphi_k \ge 0$ such that $\varphi_k \to u^+$ in $\overset{\circ}{D}{}^{1,p}(\Omega)$.*

Proof By the definition of $\tilde{W}^{k,p}(\Omega)$, we can suppose that $u_k, u \in \overset{\circ}{D}{}^{1,p}(R^m)$ and $u_k \to u$ in $\overset{\circ}{D}{}^{1,p}(R^m)$. Theorem 1.21.5 gives that $u_k^+, u^+ \in D^{1,p}(R^m)$. Fix $\epsilon > 0$. Then there exists a nonempty bounded domain $\omega \subset R^m$ such that $\|\nabla u\|_{L^p(R^m\setminus\omega)} < \epsilon/3$. Since $u_k \to u$, we can suppose that $\|\nabla u_k\|_{L^p(R^m\setminus\omega)} < \epsilon/3$ for each k. Clearly $u_k \to u$ in $W^{k,p}(\omega)$.

Put $L(s) = \{x \in \omega; u(x) > s\}$, $M_k^+ = \{x \in \omega; u_k(x) \ge 0\}$, $M_k^- = \{x \in \omega; u_k(x) \le 0\}$. If $s > 0$ then

$$\lim_{k\to\infty} \mathcal{H}_m(L(s) \cap M_k^-) \le \lim_{k\to\infty} \int\limits_{L(s)\cap M_k^-} \frac{|u-u_k|^p}{s^p}\,\mathrm{d}\mathcal{H}_m$$

$$\le \lim_{k\to\infty} \|u-u_k\|^p_{L^p(\omega)}/s^p = 0.$$

Thus

$$\chi_{M_k^-}(x) \to 0 \tag{1.14}$$

for almost all $x \in L(s)$. Therefore (1.14) holds for almost all $x \in L^+ \equiv \{y \in \omega; u(y) > 0\} = \cup\{L(1/n); n \in N\}$. Similarly,

$$\chi_{M_k^+}(x) \to 0 \tag{1.15}$$

for almost all $x \in L^- \equiv \{y \in \omega; u(y) < 0\}$. Theorem 1.21.5 gives $\nabla u_k^+ = \nabla u_k$ on $\omega \setminus M_k^-$, $\nabla u_k^+ = 0$ on M_k^-, $\nabla u^+ = \nabla u$ on L^+, $\nabla u_k^+ = 0$ on L^-, $\nabla u = 0$ on $\{x \in \omega; u(x) = 0\}$. Since

$$\chi_{M_k^-}|\nabla u|^p \le |\nabla u|^p \in L^1(L^+), \chi_{M_k^+}|\nabla u|^p \le |\nabla u|^p \in L^1(L^-),$$

we obtain using Lebesgue lemma (Lemma 1.13.2)

$$\lim_{k\to\infty} \|u^+ - u_k^+\|^p_{W^{1,p}(\omega)} \le \lim_{k\to\infty}\Bigg[\int\limits_{\omega} |u - u_k|^p + \int\limits_{(L^+\cap M_k^+)\cup(L^-\cap M_k^-)} |\nabla u - \nabla u_k|^p$$

$$+ \int\limits_{L^+\cap M_k^-} |\nabla u|^p \,\mathrm{d}\mathcal{H}_m + \int\limits_{L^-\cap M_k^+} |\nabla u_k|^p \,\mathrm{d}\mathcal{H}_m\Bigg] \le \lim_{k\to\infty}\Bigg[\|u - u_k\|^p_{W^{1,p}(\omega)}$$

$$+ \int\limits_{L^+} \chi_{M_k^-}|\nabla u|^p \,\mathrm{d}\mathcal{H}_m + \left(\|\nabla u - \nabla u_k\|_{L^p(L^-\cap M_k^+)} + \|\nabla u\|_{L^p(L^-\cap M_k^+)}\right)^p\Bigg]$$

$$\le \lim_{k\to\infty}\Bigg[\int\limits_{L^+} \chi_{M_k^-}|\nabla u|^p \,\mathrm{d}\mathcal{H}_m + \int\limits_{L^-} \chi_{M_k^+}|\nabla u|^p \,\mathrm{d}\mathcal{H}_m\Bigg] = 0.$$

Fix k_0 such that $\|u^+ - u_k^+\|_{W^{1,p}(\omega)} < \epsilon/3$ for $k \ge k_0$. According to Theorem 1.21.5, we have for $k \ge k_0$

$$\|u^+ - u_k^+\|_{D^{1,p}(R^m)} \le \|\nabla u\|_{L^p(R^m\setminus\omega)}$$

$$+\|\nabla u_k\|_{L^p(R^m\setminus\omega)} + \|u^+ - u_k^+\|_{W^{1,p}(\omega)} < \epsilon.$$

Let now $u \in \overset{\circ}{D}{}^{1,p}(\Omega)$. Then there exist $\psi_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\psi_n \to u$ in $\overset{\circ}{D}{}^{1,p}(\Omega)$. Fix $\epsilon > 0$. Since $\psi_n^+ \to u^+$ in $D^{1,p}(R^m)$, there is $k \in N$ such that $\|u^+ - \psi_k^+\|_{D^{1,p}(R^m)} < \epsilon/2$. Since ψ_k^+ has a compact support in Ω, there exists $r > 0$ such that $B(x; r) \subset \Omega$ for each $x \in \operatorname{spt}\psi_k^+$. Fix a nonnegative function $\Phi \in \mathcal{C}^\infty(R^m)$ supported in $B(0; 1)$ such that

$$\int\limits_{R^m} \Phi(x)\, d\mathcal{H}_m(x) = 1.$$

For $n \in N$, put $\Phi_n(x) = \Phi(nx)n^m$ and $\varphi_n = \psi_k^+ * \Phi_n$. If $1/n < r$, then $\varphi_n \ge 0$ and $\varphi_n \in \mathcal{C}_c^\infty(\Omega)$. Since $\psi_k^+ * \Phi_n \to \psi_k^+$ in $L^p(R^m)$, $(\partial_j\psi_k^+) * \Phi_n \to \partial_j\psi_k^+$ in $L^p(R^m)$

and $\partial_j(\psi_k^+ * \Phi_n) = (\partial_j \psi_k^+) * \Phi_n$, we infer that $\varphi_n \to \psi_k^+$ in $W^{1,p}(R^m)$ as $n \to \infty$. If we choose $n \in N$ such that $\|\varphi_n - \psi_k^+\|_{W^{1,p}(R^m)} < \epsilon/2$, we have $\|u^+ - \varphi_n\|_{D^{1,p}(R^m)} < \epsilon$.

Since $u, u_k \in \overset{\circ}{D}{}^{1,p}(R^m)$, we have $u^+, u_k^+ \in \overset{\circ}{D}{}^{1,p}(R^m)$. Therefore $u, u_k \in \tilde{W}^{1,p}(\Omega)$.

1.26 Integral Operators

In several chapters, we shall look for a solution of a boundary value problem for the Poisson equation in the special form

$$u(x) = \int_N K(x,y) f(y)\, \mathrm{d}\mu(y).$$

For this reason, we need to know properties of integral operators.

In this section, we shall suppose that Ω is a Borel subset of R^m, ω is a Borel subset of R^n, and $K(x,y)$ and $L(x,y)$ are Borel measurable functions on $\Omega \times \omega$. If μ is a real Radon measure on ω, put

$$K\mu(x) = \int_\omega K(x,y)\, \mathrm{d}\mu(y), \quad L\mu(x) = \int_\omega L(x,y)\, \mathrm{d}\mu(y)$$

whenever it makes sense. If $\mu = f\mathcal{H}_n$, we shall write Kf instead of $K\mu$.

Lemma 1.26.1 *Let $m = n$, μ be a nonnegative Radon measure on ω and $\alpha > 0$. Then*

$$\int_{B(x;t)\setminus B(x;s)} |x-y|^{-\alpha}\, \mathrm{d}\mu(y) = \alpha \int_s^t r^{-\alpha-1} \mu(B(x;r) \setminus B(x;s))\mathrm{d}r$$

$$+t^{-\alpha} \mu(B(x;t) \setminus B(x;s)), \qquad 0 < s < t < \infty.$$

Let $|K(x,y)| \le C_1 |x-y|^{-\alpha}$ for $y \in N$. If $\mu(B(x;r)) \le C_2 r^\beta$ for $0 < r < \rho$ and $\beta > \alpha$, then

$$\left| \int_{B(x;\rho)} K(x,y)\, \mathrm{d}\mu(y) \right| \le \frac{C_1 C_2 \beta}{\beta - \alpha}\, \rho^{\beta-\alpha}.$$

If $\mu(B(x;r)) \le C_2 r^\beta$ for $\rho < r$ and $0 < \beta < \alpha$, then

$$\left| \int_{\omega\setminus B(x;\rho)} K(x,y)\, \mathrm{d}\mu(y) \right| \le \frac{C_1 C_2 \alpha}{|\beta - \alpha|}\, \rho^{\beta-\alpha}.$$

Proof

$$\alpha \int_s^t r^{-\alpha-1}\mu(B(x;r)\setminus B(x;s))\mathrm{d}r = \alpha \int_s^t r^{-\alpha-1} \int_{B(x;r)\setminus B(x;s)} \mathrm{d}\mu(y)\mathrm{d}r$$

$$= \alpha \int_{B(x;t)\setminus B(x;s)} \int_{|x-y|}^t r^{-\alpha-1}\mathrm{d}r\, \mathrm{d}\mu(y)$$

$$= -t^{-\alpha}\mu(B(x;t)\setminus B(x;s)) + \int_{B(x;t)\setminus B(x;s)} |x-y|^{-\alpha}\, \mathrm{d}\mu(y).$$

If $\mu(B(x;r)) \le C_2 r^\beta$ for $0 < r < \rho$, $\beta - \alpha > 0$, then

$$\left| \int_{B(x;\rho)} K(x,y)\, \mathrm{d}\mu(y) \right| \le C_1C_2\alpha \int_0^\rho r^{\beta-\alpha-1}\, \mathrm{d}r + C_1C_2\rho^{\beta-\alpha} = \frac{C_1C_2\beta}{\beta-\alpha}\rho^{\beta-\alpha}.$$

If $\mu(B(x;r)) \le C_2 r^\beta$ for $\rho < r$, $\beta - \alpha < 0$, then

$$\left| \int_{\omega\setminus B(x;\rho)} K(x,y)\, \mathrm{d}\mu(y) \right| \le C_1C_2\alpha \int_\rho^\infty r^{\beta-\alpha-1}\, \mathrm{d}r = \frac{C_1C_2\alpha}{|\beta-\alpha|}\rho^{\beta-\alpha}.$$

Proposition 1.26.2 *Let Ω and ω be compact sets and $K(x,y)$ be a continuous function on $\Omega \times \omega$. If $1 \le p \le \infty$, then K is a compact linear operator from $L^p(\mu)$ into $\mathcal{C}^{0,0}(\Omega)$. If ν is a Radon measure on Ω and $1 \le q \le \infty$, then K is a compact linear operator from $L^p(\mu)$ to $L^q(\nu)$.*

Proof Suppose first that $p = 1$. Since $\Omega \times \omega$ is a compact set (see Example 1.2.8), there is a constant c such that $|K(x,y)| \le c$ (see Lemma 1.2.13). Denote $\mathcal{M} = \{f; \|f\|_{L^1(\mu)} \le 1\}$. If $f \in \mathcal{M}$, then $|Kf(x)| \le c\|f\|_{L^1(\mu)} \le c$.

Fix $\epsilon > 0$. Since K is uniformly continuous by Lemma 1.2.15, there exists $\delta > 0$ such that $|K(x,y) - K(z,y)| < \epsilon$ whenever $|x-z| < \delta$. If $f \in \mathcal{M}$, $|x-z| < \delta$, then

$$|Kf(x) - Kf(y)| \le \int_\omega |K(x,y) - K(z,y)||f(y)|\, \mathrm{d}\mu(y) \le \int_\omega \epsilon|f(y)|\, \mathrm{d}\mu(y) \le \epsilon.$$

Thus $\{Kf; f \in \mathcal{M}\}$ is a relatively compact subset of $\mathcal{C}^{0,0}(\Omega)$ by the Arzela–Ascoli theorem (Theorem 1.16.1), and K is a compact operator.

Let now $p > 1$. Since the identity operator is a bounded operator from $L^p(\mu)$ into $L^1(\mu)$ by Corollary 1.14.2, the operator K is a compact linear operator from $L^p(\mu)$ to $\mathcal{C}^{0,0}(\Omega)$ by Theorem 1.7.4.

Since the identity operator is a bounded operator from $\mathcal{C}^{0,0}(\Omega)$ into $L^q(\nu)$, the operator K is a compact linear operator from $L^p(\mu)$ to $L^q(\nu)$.

Proposition 1.26.3 *Let $m = n$, $\Omega = \omega$ be compact, $K(x, y) \in \mathcal{C}^0(\{[x, y];\, x, y \in \Omega, x \neq y\})$, $\alpha > 0$, $|K(x, y)| \le C|x - y|^{\alpha - m}$. Then K is a compact linear operator from $L^\infty(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$.*

Proof If $\alpha \ge m$ or $C = 0$, then K is a compact linear operator from $L^\infty(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$ by Proposition 1.26.2. Suppose now that $\alpha < m$ and $C > 0$. If $|f| \le 1$, then $|Kf| \le C\mathcal{H}_m(B(0; 1))m(\operatorname{diam}\Omega)^\alpha/\alpha$ by Lemma 1.26.1. Thus K is a bounded linear operator on $L^\infty(\Omega)$. For $n \in N$ put $K_n(x, y) = K(x, y)$ for $|x-y| \ge 2/n$, $K_n(x, y) = 0$ for $|x - y| < 1/n$, $K_n(x, y) = K(x, y)(n|x - y| - 1)$ for $1/n \le |x - y| < 2/n$. Then K_n is a compact linear operator from $L^\infty(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$ by Proposition 1.26.2. If $|f| \le 1$, then

$$|Kf(x) - K_nf(x)| \le \int_\Omega |K(x, y) - K_n(x, y)|\, \mathrm{d}\mathcal{H}_m$$

$$\le \int_{B(x;2/n)} C|x - y|^{\alpha - m}\, \mathrm{d}\mathcal{H}_m(y) \le \frac{C\mathcal{H}_m(B(0; 1))m}{\alpha}\left(\frac{2}{n}\right)^\alpha$$

by Lemma 1.26.1. Since $K_n \to K$ as $n \to \infty$, the operator K is a compact linear operator from $L^\infty(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$ by Theorem 1.7.4.

Proposition 1.26.4 *Let $\Omega = \omega = R^m$, $|K(x, y)| \le c|x - y|^{\alpha - m}$ with $0 < \alpha < m$, $1 \le p < m/\alpha$. If $f \in L^p(\Omega)$, then $Kf(x)$ is finite for almost all $x \in R^m$. If $1 < p$, then there exists a constant C such that*

$$\|Kf\|_{L^q(R^m)} \le C\|f\|_{L^p(R^m)}, \qquad f \in L^p(R^m),$$

where $q = pm/(m - \alpha p)$.

(See [136], Chapter 5, §1, Theorem 1.)

Theorem 1.26.5 *Let $\Omega = \omega$. If $K \in L^2(\mu \times \mu)$ (K is so-called Hilbert–Schmidt operator), then K is a compact linear operator on $L^2(\mu)$.*

(See [149], Chapter X, §2, Example 2.)

Proposition 1.26.6 *Let $m = n$, $\Omega = \omega$ be bounded, $1 \le p \le \infty$, and $\alpha > 0$, $|K(x, y)| \le C|x - y|^{\alpha - m}$. Then K is a bounded linear operator on $L^p(\Omega)$. If $1 < p < \infty$, then K is a compact linear operator on $L^p(\Omega)$.*

Proof Fix $R > 0$ such that $\Omega \subset B(0;R)$. Put $g(z) = C|z|^{\alpha-m}$ for $|z| \leq 2R$, $g = 0$ elsewhere. If $\|f\|_{L^p(\Omega)} \leq 1$ put $f = 0$ outside Ω. Young's inequality (Lemma 1.15.1) yields

$$\|Kf\|_{L^p(\Omega)} \leq \|g * f\|_{L^p(R^m)} \leq \|g\|_{L^1(R^m)} < \infty.$$

So, K is a bounded linear operator on $L^p(\Omega)$. Since $|K(x,y)|^2 \leq C^2|x-y|^{2\alpha-2m}$, we have $K(x,y) \in L^2(\Omega \times \Omega)$. Thus K is a compact linear operator on $L^2(\Omega)$ by Theorem 1.26.5. If $1 < p < \infty$, then K is a compact operator on $L^p(\Omega)$ by Marcinkiewicz's interpolation theorem (Theorem 1.14.9).

Proposition 1.26.7 *Let $m = n$, $\Omega = \omega$ be bounded, $K(x,y) \in \mathcal{C}^0(\{[x,y];\, x,y \in \Omega, x \neq y\})$. Suppose, moreover, that there exist constants $C > 0$, $0 < \alpha < m$ such that $|K(x,y)| \leq C|x-y|^{\alpha-m}$. Let $f \in L^p(\Omega)$ with $m/\alpha < p \leq \infty$. Then $Kf \in \mathcal{C}^{0,0}(\Omega)$. Moreover, there exists a constant $\tilde{C}$ depending only on m, p, and α such that $|Kf| \leq C\tilde{C}(\operatorname{diam}\Omega)^{\alpha-m/p}\|f\|_{L^p(\Omega)}$.*

Proof If $p = \infty$ put $q = 1$. If $p < \infty$ put $q = p/(p-1)$. Remark that $q < m/(m-\alpha)$. Hölder's inequality (Proposition 1.14.1) gives

$$|Kf(x)| \leq C\|f\|_{L^p(\Omega)}\left[\int_{\Omega} |x-y|^{(\alpha-m)q}\,\mathrm{d}\mathcal{H}_m(y)\right]^{1/q}$$

$$\leq C\tilde{C}(\operatorname{diam}\Omega)^{[m+(\alpha-m)q]/q}\|f\|_{L^p(\Omega)} = C\tilde{C}(\operatorname{diam}\Omega)^{\alpha-m/p}\|f\|_{L^p(\Omega)}$$

with $\tilde{C} = \{\mathcal{H}_{m-1}(\partial B(0;1))/[m+(\alpha-m)q]\}^{1/q}$.

Fix $x \in \Omega$, $\epsilon > 0$. Choose $r > 0$ such that

$$C\tilde{C}(\operatorname{diam}\Omega)^{\alpha-m/p}\|f\|_{L^p(\Omega\cap B(x;2r))} < \epsilon/3.$$

Put $f_1 = f$ on $B(x;2r)$, $f_1 = 0$ elsewhere, and $f_2 = f - f_1$. Then $|Kf_1| \leq C\tilde{C}(\operatorname{diam}\Omega)^{\alpha-m/p}\|f_1\|_{L^p(\Omega)} \leq \epsilon/3$. Remark that $f_2 \in L^1(\mathcal{H}_m)$ by Corollary 1.14.2. The function Kf_2 is continuous in x by Theorem on continuity of parametrized integrals (Theorem 1.13.6). So, there exists $\delta > 0$ such that $|Kf_2(y) - Kf_2(x)| < \epsilon/3$ for $|x-y| < \delta$. For $|x-y| < \delta$ we have $|Kf(x) - Kf(y)| \leq |Kf_1(x)| + |Kf_1(y)| + |Kf_2(y) - Kf_2(x)| < \epsilon$.

Lemma 1.26.8 *Let $m = n$, ν be a nonnegative Radon measure on ω with compact support. Let $\beta > 0$, $\alpha \in R^1$, $\alpha + \beta > 0$, $\nu(B(x;r)) \leq C_1 r^\beta$ for $x \in \Omega$ and $r > 0$, $|K(x,y)| \leq C_2|x-y|^\alpha$ for $x \in \Omega$, $y \in \operatorname{spt}\nu$. Suppose that $K(\cdot,y)$ is continuous for each $y \in \operatorname{spt}\nu$. If $f \in L^\infty(\nu)$, $\mu = f\nu$, then $K\mu \in \mathcal{C}^0(\Omega)$.*

Proof If $\alpha \geq 0$, then $K\mu$ is continuous in Ω by the theorem on continuity of parametrized integrals (Theorem 1.13.6). Suppose that $\alpha < 0$. Fix $z \in \Omega$ and $\epsilon > 0$.

According to Lemma 1.26.1, there exists $r > 0$ such that

$$\int\limits_{B(x;2r)} |K(x,y)f(y)| \,\mathrm{d}\nu(y) < \epsilon/2$$

for each $x \in \Omega$. If $|x - z| < r$, then

$$\left|\int\limits_{B(z;r)} K(z,y)\,\mathrm{d}\mu(y)\right| + \left|\int\limits_{B(z;r)} K(x,y)\,\mathrm{d}\mu(y)\right| \leq \epsilon.$$

Since

$$\int\limits_{\omega\setminus B(z;r)} K(x,y)\,\mathrm{d}\mu(y) \to \int\limits_{\omega\setminus B(z;r)} K(z,y)\,\mathrm{d}\mu(y)$$

as $x \to z$ by the theorem on continuity of parametrized integrals (Theorem 1.13.6), we infer that $K\mu$ is continuous.

1.27 Calderón–Zygmund Operators

If $\Omega \subset R^m$ is a bounded open set and K is a measurable function on $\Omega \times \Omega$, then the integral operator

$$Kf(x) = \int\limits_{\Omega} K(x,y)f(y)\,\mathrm{d}\mathcal{H}_m(y)$$

is a continuous operator on $L^p(\Omega)$ provided $|K(x,y)| \leq C|x-y|^{\alpha-m}$ with $\alpha > 0$. In next chapters, we shall need an integral operator whose kernel satisfies only $|K(x,y)| \leq C|x-y|^{-m}$. But if $C_1|x-y|^{-m} \leq |K(x,y)| \leq C_2|x-y|^{-m}$ for positive C_1, C_2, then Kf is well defined only for $f \equiv 0$. We shall see that even for such kernels the integral operator can be defined as a limit well defined integral operators.

Let $K(x,y)$ be a Borel measurable function on $R^m \times R^m$. Suppose that there exists a constant B such that

$$|K(x,y)| \leq B|x-y|^{-m}, \quad x \neq y.$$

For $\epsilon > 0$ denote $K_\epsilon(x,y) = K(x,y)$ for $|x-y| > \epsilon$ and $K_\epsilon(x,y) = 0$ for $|x-y| \leq \epsilon$. If $f \in L^p(R^m)$ and $1 \leq p < \infty$, then $K_\epsilon f(x)$ is well defined for all $x \in R^m$ by Hölder's inequality. Define

$$Kf(x) = \lim_{\epsilon\downarrow 0} K_\epsilon f(x) \tag{1.16}$$

whenever this limit makes a sense,

$$K_*f(x) = \sup_{\epsilon>0} |K_\epsilon f(x)|.$$

Definition 1.27.1 Let $K(x,y)$ be a Borel measurable function on $R^m \times R^m$. We say that K is a Calderón–Zygmund kernel if

- there exists a constant B such that $|K(x,y)| \le B|x-y|^{-m}$ for $x \ne y$;
- K is locally Lipschitz on $\{[x,y]; x \ne y\}$, and there exists a constant C such that $|\nabla K(x,y)| \le C|x-y|^{-m-1}$ for $x \ne y$.
- If $f \in L^2(R^m)$, then there exists the limit $Kf(x)$ given by (1.16) for almost all $x \in R^m$.
- K is a bounded linear operator on $L^2(R^m)$.

Theorem 1.27.2 *Let $\Omega \subset R^m$ be open, $\partial B(0;1) \subset \Omega$, $L \in \mathcal{C}^1(\Omega)$,*

$$\int\limits_{\partial B(0;1)} L \, d\mathcal{H}_{m-1} = 0.$$

Then $K(x,y) = \frac{L((x-y)/|x-y|)}{|x-y|^m}$ is a Calderón–Zygmund kernel.

(See [136], Chapter II, §4.)

Theorem 1.27.3 *Let $m,k \in N$ and suppose that $A : R^m \to R^k$ is a Lipschitz mapping. Let $F : R^k \to R$ be an infinitely differentiable odd function. Then the antisymmetric kernel*

$$K(x,y) = \frac{1}{|x-y|^m} F\left(\frac{A(x)-A(y)}{|x-y|}\right)$$

is a Calderón–Zygmund kernel.

(See [112], Chapter 9, Theorem 11.)

Proposition 1.27.4 *Let K be a Calderón–Zygmund operator. If $f \in L^p(R^m)$, $1 < p < \infty$, then $K_*f \in L^p(R^m)$.*

(See [112], §7.5.)

Proposition 1.27.5 *Let K be a Calderón–Zygmund operator. Then, for $1 \le p < \infty$, there exists a constant C such that for every $f \in L^1_{loc}(R^m)$*

$$\int (K_*f)^p \le C \int (Mf)^p.$$

(See [112], §7.8, Corollary 1.)

Proposition 1.27.6 *Let* $1 < p < \infty$. *Then every Calderón–Zygmund operator can be extended to a continuous linear operator on* $L^p(R^m)$.

(See [112], §7.8, Corollary 2.)

1.28 Nontangential Limits

We shall study the Dirichlet problem for the Laplace equation in Sobolev spaces where a boundary condition is fulfilled in the sense of traces. We shall also study this problem in a weaker sense, where the boundary condition is fulfilled in the sense of a nontangential limit and the regularity of a solution is given by the fact that the nontangential maximal function of a solution is in $L^p(\partial\Omega)$. We now show that such solution lies in the Lebesgue space $L^p(\Omega)$ provided Ω is bounded.

Let $\Omega \subset R^m$ be a domain with Lipschitz boundary. If $x \in \partial\Omega$, $a > 0$, denote the nontangential approach regions of opening a at the point x by

$$\Gamma_a(x) = \{y \in \Omega; |x-y| < (1+a)\operatorname{dist}(y,\partial\Omega)\}.$$

If now v is a function defined in Ω, we denote the nontangential maximal function of v on $\partial\Omega$ by

$$M_a(v)(x) = \sup\{|v(y)|; y \in \Gamma_a(x)\}.$$

Let $\Phi : R^{m-1} \to R$ be a Lipschitz function. Put $\Omega = \{[x,y]; x \in R^{m-1}, y > \Phi(x)\}$. Then Ω is so-called a graph Lipschitz domain. It is well known that for $a, b > 0$ and $1 \le p < \infty$, there exist $c, C > 0$ such that

$$\|M_a v\|_{L^p(\partial\Omega)} \le c\|M_b v\|_{L^p(\partial\Omega)} \le C\|M_a v\|_{L^p(\partial\Omega)}$$

for any measurable function v in Ω. See, e.g., [81] and [137], p. 62. We shall often write $\Gamma(x)$ instead of $\Gamma_a(x)$. Next, define the nontangential limit of v at $x \in \partial\Omega$

$$v(x) = \lim_{\substack{y \to x \\ y \in \Gamma(x)}} v(y)$$

whenever the limit exists.

Lemma 1.28.1 *Let* $G \subset R^m$ *be a bounded open set with Lipscitz boundary,* $\varphi \in C(G)$, *and* $1 < s < \infty$. *If* $M_a(\varphi) \in L^s(\partial G)$, *then* $\varphi \in L^s(G)$.

Proof It is enough to prove that for each $x \in \overline{G}$, there is a neighborhood $U(x)$ of x such that

$$\int_{G\cap U(x)} |\varphi|^s \, d\mathcal{H}_m < \infty.$$

Fix $x \in \partial G$. We may suppose that $x = [0, \dots, 0]$. We can choose a coordinate system, constants $\delta, \epsilon > 0$, and a Lipschitz function f defined on $D \equiv \{y \in R^{m-1}; |y| < 2\delta\}$ such that $G \cap [D \times (-2\epsilon, 2\epsilon)] = \{(y, t); y \in D, -2\epsilon < t < f(y)\}$. Put $U = \{[y_1, \dots, y_{m-1}, t]; |[y_1, \dots, y_{m-1}]| < \delta, |t| < \epsilon\}$. If $a > 0$ is large enough, then $[y, t] \in \Gamma_a([y, f(y)])$ for each $[y, t] \in U \cap G$. If $|f(y)| \le c$ for $|y| \le \delta$, then

$$\int\limits_{G \cap U} |\varphi|^s \, d\mathcal{H}_{m-1} = \int\limits_{\{y \in R^{m-1}; |y| < \delta\}} \int\limits_{-\epsilon}^{f(y)} |\varphi(y, t)|^s \, dt \, d\mathcal{H}_{m-1}(y) \le$$

$$\int\limits_{\{y \in R^{m-1}; |y| < \delta\}} \int\limits_{-\epsilon}^{f(y)} [M_a\varphi(y, f(y))]^s \, dt \, d\mathcal{H}_{m-1}(y)$$

$$\le (c + \epsilon) \int\limits_{\partial G} [M_a\varphi(y, f(y))]^s \, d\mathcal{H}_{m-1}(y).$$

Lemma 1.28.2 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary. If $M_a(\varphi) \in L^s(\partial\Omega)$ with $1 \le s \le \infty$, then there exists $r > 0$ such that φ is bounded on $R^m \setminus B(0; r)$.*

Proof There exists $r > 0$ such that $K := R^m \setminus B(0; r) \subset \Gamma_a(x)$ for all $x \in \partial\Omega$. If φ is not bounded on K, then $M_a(\varphi) \equiv \infty$, what is a contradiction.

Chapter 2
Harmonic Functions

Abstract Chapter 2 studies harmonic functions.

Chapter 2 is devoted to solutions of the Laplace equation. We shall see that a solution of the Laplace equation in the sense of distributions is a classical solution of this equation, the so-called harmonic function. Moreover, this solution is an infinitely differentiable function. We prove classical properties of harmonic functions as the mean value theorem, the maximum principle, and the Harnack inequality. We also discuss under which conditions the limit of sequences of harmonic functions is a harmonic function. As a consequence, we show that the space of functions from $L^p(\Omega)$ harmonic in Ω is a closed subspace of $L^p(\Omega)$.

Applications to boundary value problems on unbounded domains press us to study the behavior of harmonic functions at infinity. First, we study harmonic functions on the whole R^m. A tempered distribution satisfying the Laplace equation in the whole space is a polynomial. We also show that a nonnegative harmonic function in R^m is constant. Using these results, we develop the theory of harmonic functions on exterior domains. The behavior of a harmonic function u at infinity determines a behavior of $\partial^\alpha u$ at infinity and opposite.

A very interesting task is to find a solution of the Dirichlet problem for the ball. This solution is expressed by the Poisson integral corresponding to the boundary condition. The properties of the Poisson integral are discussed.

2.1 Harmonic Functions

In this section, we define harmonic functions as classical solutions of the Laplace equation. We show that a single layer potential and a double layer potential are examples of harmonic functions.

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Let Ω be an open subset of R^m. Denote by Δ the Laplace operator defined for $u \in \mathcal{C}^2(\Omega)$

$$\Delta u = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2}.$$

Definition 2.1.1 A function $u : \Omega \to R$ is called *harmonic* on Ω if $u \in \mathcal{C}^2(\Omega)$ and $\Delta u = 0$ in Ω. Denote by $H(\Omega)$ the set of all harmonic functions on Ω.

Lemma 2.1.2 *Δu does not depend on the choice of coordinates: Let $z \in R^m$, A be a matrix of the type $m \times m$ such that $(Ax) \cdot (Ay) = x \cdot y$ for each $x, y \in R^m$. Put $Tx = Ax + z$. For an open set $\Omega \subset R^m$, $u \in \mathcal{C}^2(T(\Omega))$ put $v(x) = u(Tx)$ for $x \in \Omega$. Then $\Delta v(x) = \Delta u(Tx)$ for $x \in \Omega$.*

Proof Put $w_k = (\delta_{1k}, \delta_{2k}, \ldots, \delta_{mk})$. Then

$$\delta_{jk} = w_k \cdot w_j = (Aw_k) \cdot (Aw_j) = \sum_{n=1}^{m} a_{nk} a_{nj}.$$

This means that $A^T A = I$ and thus $A^{-1} = A^T$ (the transpose matrix of A is the inverse matrix of A). Therefore $I = AA^{-1} = AA^T$, which gives

$$\sum_{n=1}^{m} a_{kn} a_{jn} = \delta_{jk}.$$

Hence

$$\partial_j v(x) = \sum_{k=1}^{m} \partial_k u(Tx) a_{kj},$$

$$\Delta v(x) = \sum_{j=1}^{m} \sum_{n=1}^{m} \sum_{k=1}^{m} \partial_n \partial_k u(Tx) a_{kj} a_{nj} = \sum_{n=1}^{m} \sum_{k=1}^{m} \partial_n \partial_k u(Tx) \sum_{j=1}^{m} a_{kj} a_{nj}$$

$$= \sum_{n=1}^{m} \sum_{k=1}^{m} \partial_n \partial_k u(Tx) \delta_{kn} = \Delta u(Tx).$$

Remark 2.1.3 For $x \in R^m$ denote $h_m(x) = \varphi(|x|)$ where

$$\varphi(t) = \begin{cases} -\frac{1}{2\pi} \ln t, & m = 2, \\ \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial B(0;1))} t^{2-m}, & m > 2. \end{cases} \tag{2.1}$$

Then h_m is a harmonic function in $R^m \setminus \{0\}$.

Proof Since $\partial_j|x| = x_j/|x|$, we have $\partial_j h_m(x) = \varphi'(|x|)x_j/|x|$. Thus $\Delta h_m(x) = \varphi''(|x|) + \varphi'(|x|)(m-1)/|x|]$. Since $\varphi'(t) = c_m t^{1-m}$, $\varphi''(t) = c_m(1-m)^{-1}t^{-m}$ with $c_m = -[\mathcal{H}_{m-1}(\partial B(0;1))]^{-1}$, we get $\Delta h_m = 0$ in $R^m \setminus \{0\}$.

Remark 2.1.4 Let μ be a finite real Borel measure with compact support in R^m. Define the volume potential

$$V\mu(x) = \int\limits_{R^m} h_m(x-y)\, d\mu(y) \tag{2.2}$$

whenever it makes a sense. Then $V\mu$ is a harmonic function in $R^m \setminus \operatorname{spt}\mu$. If $f \in L^1(R^m)$, we can write Vf instead of $V(f\mathcal{H}_m)$.

Proof Suppose first that $\mu \geq 0$. If α is a multiindex, then theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives

$$\partial^\alpha Vf(x) = \int \partial_x^\alpha h_m(x-y)\, d\mu(y) \tag{2.3}$$

for $x \in R^m \setminus \operatorname{spt}\mu$. Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives that $V\mu \in \mathcal{C}^\infty(R^m \setminus \operatorname{spt}\mu)$. Using (2.3) and Remark 2.1.3, we obtain

$$\Delta V\mu(x) = \int\limits_{R^m} \Delta_x h_m(x-y)\, d\mu(y) = 0.$$

Let now μ be arbitrary. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. Then $V\mu = V\mu^+ - V\mu^-$ is a harmonic function in $R^m \setminus \operatorname{spt}\mu$.

Remark 2.1.5 Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $\mu \in \mathcal{C}'(\partial\Omega)$. Define

$$\mathcal{S}^\Omega\mu(x) = \int\limits_{\partial\Omega} h_m(x-y)\, d\mu(y)$$

the single layer potential corresponding to μ and

$$\mathcal{D}^\Omega\mu(x) = -\int\limits_{\partial\Omega} n(y)\cdot\nabla_y h_m(x-y)\, d\mu(y)$$

$$= \int\limits_{\partial\Omega} \frac{n(y)\cdot(y-x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}\, d\mu(y)$$

the double layer potential corresponding to μ whenever these integrals make sense. Here, $n(y)$ denotes the outward unit normal of Ω at y. Then $\mathcal{S}^\Omega \mu and \mathcal{D}^\Omega \mu \in \mathcal{C}^\infty(R^m \setminus \operatorname{spt} \mu) \subset \mathcal{C}^\infty(R^n \setminus \partial\Omega)$ are harmonic functions in $R^m \setminus \operatorname{spt} \mu \subset R^m \setminus \partial\Omega$. If $f \in L^1(\partial\Omega)$, we write $\mathcal{S}^\Omega f$ instead of $\mathcal{S}^\Omega(f\mathcal{H}_{m-1}|\partial\Omega)$ and call it the single layer potential with density f. Similarly, we write $\mathcal{D}^\Omega f$ instead of $\mathcal{D}^\Omega(f\mathcal{H}_{m-1}|\partial\Omega)$ and call it the double layer potential with density f.

Proof According to Remark 2.1.4, it is sufficient to study double layer potentials only. Suppose first that $\mu \geq 0$. If α is a multiindex, then theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives

$$\partial^\alpha \mathcal{D}^\Omega \mu(x) = -\int_{\operatorname{spt} \mu} \sum_{j=1}^m n_j(y) \partial_x^\alpha \frac{\partial h_m(x-y)}{\partial y_j}\, d\mu(y) \tag{2.4}$$

for $x \in R^m \setminus \operatorname{spt} \mu$. The Lebesgue lemma (Lemma 1.13.2) gives that $\mathcal{D}^\Omega \mu \in \mathcal{C}^\infty(R^m \setminus \operatorname{spt} \mu)$. Using (2.4) and Remark 2.1.3, we obtain

$$\Delta \mathcal{D}^\Omega \mu(x) = -\int_{\operatorname{spt} \mu} \sum_{j=1}^m n_j(y) \frac{\partial \Delta_x h_m(x-y)}{\partial y_j}\, d\mu(y) = 0$$

in $R^m \setminus \operatorname{spt} \mu$.

Let now μ be arbitrary. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. Then $\mathcal{D}^\Omega \mu = \mathcal{D}^\Omega \mu^+ - \mathcal{D}^\Omega \mu^-$ is a harmonic function in $R^m \setminus \operatorname{spt} \mu$.

Lemma 2.1.6 *Let $\Omega \subset R^2$ be an open set, $u \in \mathcal{C}^2(\Omega)$. Express u in the polar coordinates, i.e., $u(x_1, x_2) = v(r, \varphi)$ with $x_1 = r\cos\varphi$, $x_2 = r \sin\varphi$. Then*

$$\Delta u = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}.$$

Proof We have

$$\frac{\partial v}{\partial r} = \frac{\partial u}{\partial x_1}\cos\varphi + \frac{\partial u}{\partial x_2}\sin\varphi, \qquad \frac{\partial v}{\partial \varphi} = \frac{\partial u}{\partial x_1}(-r\sin\varphi) + \frac{\partial u}{\partial x_2} r\cos\varphi,$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} = \frac{\partial^2 u}{\partial x_1^2}\cos^2\varphi + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\sin\varphi\cos\varphi + \frac{\partial^2 u}{\partial x_2^2}\sin^2\varphi$$

$$+\frac{1}{r}\frac{\partial u}{\partial x_1}\cos\varphi + \frac{1}{r}\frac{\partial u}{\partial x_2} sin\varphi - \frac{1}{r}\frac{\partial u}{\partial x_1}\cos\varphi - \frac{1}{r}\frac{\partial u}{\partial x_2} sin\varphi + \frac{\partial^2 u}{\partial x_1^2}\sin^2\varphi$$

$$-2\frac{\partial^2 u}{\partial x_1 \partial x_2}\sin\varphi\cos\varphi + \frac{\partial^2 u}{\partial x_2^2}\cos^2\varphi = \Delta u.$$

2.2 Integral Identities

In this section, we prove two useful integral identities—Green identities. Then we express a smooth function u as a sum of three potentials: the volume potential corresponding to $-\Delta u$ plus the single layer potential corresponding to the normal derivative of u plus the double layer potential corresponding to the trace of u. As a consequence, we show that every harmonic function is infinitely differentiable.

Theorem 2.2.1 (Green Identities) *Let Ω be an open subset of R^m, G be a bounded open subset of Ω with Lipschitz boundary such that $\overline{G} \subset \Omega$. If $u, v \in \mathcal{C}^2(\Omega)$ then*

$$\int\limits_G v(-\Delta u)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial G} v\frac{\partial u}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_G \nabla u \cdot \nabla v\,\mathrm{d}\mathcal{H}_m, \tag{2.5}$$

$$\int\limits_{\partial G} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1} = \int\limits_G (u\Delta v - v\Delta u)\, d\mathcal{H}_m. \tag{2.6}$$

Proof The Gauss-Green theorem (Theorem 1.19.1) gives

$$\int\limits_{\partial G} v\frac{\partial u}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_G v\Delta u\,\mathrm{d}\mathcal{H}_m + \int\limits_G \nabla u \cdot \nabla v\,\mathrm{d}\mathcal{H}_m,$$

$$\int\limits_{\partial G} u\frac{\partial v}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_G u\Delta v\,\mathrm{d}\mathcal{H}_m + \int\limits_G \nabla u \cdot \nabla v\,\mathrm{d}\mathcal{H}_m.$$

Subtracting we get (2.6).

Corollary 2.2.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. If $u, v \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ then*

$$\int\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1} = 0.$$

Proof We can suppose that Ω is connected. Let $\Omega(j)$ are domains from Lemma 1.17.13. According to second Green formula (Theorem 2.2.1) and Lebesgue lemma (Lemma 1.13.2), we have

$$\int\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1} = \lim_{j\to\infty} \int\limits_{\partial\Omega(j)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1} = 0.$$

Corollary 2.2.3 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. If $u \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ then*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\mathcal{H}_{m-1} = 0.$$

Proof We use Corollary 2.2.2 for $v \equiv 1$.

Theorem 2.2.4 (Theorem on Three Potentials) *Let Ω be an open subset of R^m and G be a bounded open subset of Ω with Lipschitz boundary such that $\overline{G} \subset \Omega$. If $u \in \mathcal{C}^2(\Omega)$ then*

$$u = V[\chi_G(-\Delta u)] + \mathcal{S}^G \frac{\partial u}{\partial n} + \mathcal{D}^G u$$

in G. Here, n denotes the outward unit normal of G. If $x \in R^m \setminus \overline{G}$, then

$$V[\chi_G(-\Delta u)](x) + \mathcal{S}^G \frac{\partial u}{\partial n}(x) + \mathcal{D}^G u(x) = 0.$$

Proof Fix $x \in R^m \setminus \partial G$. Put $v(y) = h_m(y - x)$. Suppose first that $x \notin \overline{G}$. Since $v \in H(G)$ by Remark 2.1.3, second Green formula (Theorem 2.2.1) gives

$$V[\chi_G(-\Delta u)](x) = \int_G (u\Delta v - v\Delta u)\, d\mathcal{H}_m = \int_{\partial G} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1}$$

$$= -\mathcal{D}^G u(x) - \mathcal{S}^G \frac{\partial u}{\partial n}(x).$$

Suppose now that $x \in G$. Fix $r > 0$ such that $\overline{B(x; r)} \subset G$. Put $G(r) = G \setminus \overline{B(x; r)}$. Then

$$0 = V[\chi_{G(r)}(-\Delta u)](x) + \mathcal{S}^{G(r)} \frac{\partial u}{\partial n}(x) + \mathcal{D}^{G(r)} u(x) = V[\chi_{G(r)}(-\Delta u)](x)$$

$$+\mathcal{D}^G u(x) + \mathcal{S}^G \frac{\partial u}{\partial n}(x) - u(x) - \int_{\partial B(x;r)} \frac{u(y) - u(x)}{\mathcal{H}_{m-1}(\partial B(0; r))}\, d\mathcal{H}_{m-1}(y)$$

$$- \int_{\partial B(x;r)} h_m(y - x) \frac{\partial u(y)}{\partial n}\, d\mathcal{H}_{m-1}(y).$$

Denote

$$M = \sup_{x \in \overline{G}} |\nabla u(y)|.$$

Since

$$|u(y) - u(x)| = \left| \int_0^1 \frac{\partial u(x + t(y-x))}{\partial t} \, dt \right| \le M|x - y|,$$

we get for $r \downarrow 0$

$$0 = V[\chi_G(\Delta u)](x) + \mathcal{D}^G u(x) + \mathcal{S}^G \frac{\partial u}{\partial n}(x) - u(x).$$

Corollary 2.2.5 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. If $u \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, then $u = \mathcal{S}^\Omega(\partial u/\partial n) + \mathcal{D}^\Omega u$ in Ω. If $x \in R^m \setminus \overline{\Omega}$, then $\mathcal{S}^\Omega(\partial u/\partial n)(x) + \mathcal{D}^\Omega u(x) = 0$.*

Proof We can suppose that Ω is connected. Let $\Omega(j)$ are domains from Lemma 1.17.13. Fix $x \in \Omega$.Then there is $k \in N$ such that $x \in \Omega(j)$ for $j \ge k$. According to theorem on three potentials (Theorem 2.2.4) and Lebesgue lemma (Lemma 1.13.2) we have

$$u(x) = \lim_{j \to \infty} [\mathcal{S}^{\Omega(j)}(\partial u/\partial n) + \mathcal{D}^{\Omega(j)} u] = \mathcal{S}^\Omega(\partial u/\partial n) + \mathcal{D}^\Omega u.$$

Let now $x \in R^m \setminus \overline{\Omega}$. Put $v = h_m(y - x)$. Then Corollary 2.2.2 gives $\mathcal{S}^\Omega(\partial u/\partial n)(x) + \mathcal{D}^\Omega u(x) = 0$.

Corollary 2.2.6 *If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary and c is a constant then $\mathcal{D}^\Omega c = c$ in Ω, $\mathcal{D}^\Omega c = 0$ in $R^m \setminus \overline{\Omega}$.*

Proof We use Corollary 2.2.5 for $u \equiv c$.

Corollary 2.2.7 *If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then $\varphi = -V(\Delta\varphi)$ in R^m.*

Proof Fix a bounded domain G with boundary of class C^1 such that $\operatorname{spt}\varphi \subset G$ and use theorem on three potentials (Theorem 2.2.4).

Corollary 2.2.8 *Let Ω be an open subset of R^m. If u is a harmonic function in Ω then $u \in \mathcal{C}^\infty(\Omega)$.*

Proof Let $x \in \Omega$. Fix a bounded open subset G of Ω with boundary of class C^1 such that $x \in G$, $\overline{G} \subset \Omega$. Corollary 2.2.5 gives that $u = \mathcal{S}^G(\partial u/\partial n) + \mathcal{D}^G u$ in G. Remark 2.1.5 shows that $u \in \mathcal{C}^\infty(G)$.

Corollary 2.2.9 *Let Ω be an open subset of R^m. If $u \in H(\Omega)$ then $\partial^\alpha u \in H(\Omega)$ for each multiindex $\alpha = (\alpha_1, \dots, \alpha_m)$.*

Proof Since $u \in \mathcal{C}^\infty(\Omega)$, we have

$$\Delta\partial^\alpha u = \partial^\alpha \Delta u = 0.$$

2.3 The Kelvin Transform

In this section, we study the Kelvin transform which maps functions on a ball to functions on the exterior of a ball and opposite. We show that the Kelvin transform maps harmonic functions to harmonic functions.

Let $x \in R^m$, $r > 0$ be fixed. If $y \in R^m \setminus \{x\}$, then the inverse of y with respect to $\partial B(x;r)$ is the point

$$y^* = \frac{r^2}{|x-y|^2}(y-x) + x.$$

If $y \in \partial B(x;r)$, then $y^* = y$. The inverse of a set E in R^m with respect to $\partial B(x;r)$ is the set $E^* = \{y^*; y \in E \setminus \{x\}\}$. If f is a function defined at least on E, then we define f^* on E^* by

$$f^*(y) = \left(\frac{r}{|x-y|}\right)^{m-2} f(y^*).$$

The mapping $f \mapsto f^*$ is called the Kelvin transform (with respect to $\partial B(x;r)$). Remark that $(f^*)^* = f$.

Theorem 2.3.1 *Let $\Omega \subset R^m$ be an open set, $x \in R^m$, $r > 0$, $\Omega \cap (\partial B(x;r) \cup \{x\}) = \emptyset$. If $f \in C^2(\Omega)$ and f^* is the image of f under the Kelvin transform with respect to $\partial B(x;r)$, then*

$$\Delta f^*(y) = \left(\frac{r}{|x-y|}\right)^{m+2} \Delta f(y^*).$$

If $f \in H(\Omega)$, then $f^* \in H(\Omega^*)$.

Proof Fix j. Then

$$\frac{\partial f(y^*)}{\partial y_j} = \frac{r^2}{|x-y|^2}(\partial_j f)(y^*) - \frac{2r^2(y_j - x_j)}{|x-y|^4}\sum_{k=1}^{m}(y_k - x_k)(\partial_k f)(y^*),$$

$$\frac{\partial^2 f(y^*)}{\partial y_j^2} = -\frac{4r^2(y_j - x_j)}{|x-y|^4}(\partial_j f)(y^*) - \frac{2r^2}{|x-y|^4}\sum_{k=1}^{m}(y_k - x_k)(\partial_k f)(y^*)$$

$$+\frac{8r^2(y_j - x_j)^2}{|x-y|^6}\sum_{k=1}^{m}(y_k - x_k)(\partial_k f)(y^*) + \frac{r^4}{|x-y|^4}(\partial_j \partial_j f)(y^*)$$

$$-\frac{4r^4(y_j-x_j)}{|x-y|^6}\sum_{k=1}^{m}(y_k-x_k)(\partial_k\partial_j f)(y^*)$$

$$+\frac{4r^4(y_j-x_j)^2}{|x-y|^8}\sum_{k=1}^{m}\sum_{n=1}^{m}(y_k-x_k)(y_n-x_n)(\partial_k\partial_n f)(y^*).$$

Since $y \mapsto |x-y|^{2-m}$ is a harmonic function in Ω^* by Remark 2.1.3, we obtain

$$\Delta f^*(y)=\left[\Delta_y\left(\frac{r}{|x-y|}\right)^{m-2}\right]f(y^*)+2\left[\nabla_y\left(\frac{r}{|x-y|}\right)^{m-2}\right]\cdot\nabla_y f(y^*)$$

$$+\left(\frac{r}{|x-y|}\right)^{m-2}\Delta_y f(y^*)=-2(m-2)\sum_{j=1}^{m}\frac{r^m(y_j-x_j)}{|x-y|^m}\left[\frac{1}{|x-y|^2}(\partial_j f)(y^*)\right.$$

$$\left.-\frac{2(y_j-x_j)}{|x-y|^4}\sum_{k=1}^{m}(y_k-x_k)(\partial_k f)(y^*)\right]$$

$$+\left(\frac{r}{|x-y|}\right)^{m-2}\sum_{j=1}^{m}\left[-\frac{4r^2(y_j-x_j)}{|x-y|^4}(\partial_j f)(y^*)\right.$$

$$-\frac{2r^2}{|x-y|^4}\sum_{k=1}^{m}(y_k-x_k)(\partial_k f)(y^*)+\frac{8r^2(y_j-x_j)^2}{|x-y|^6}\sum_{k=1}^{m}(y_k-x_k)(\partial_k f)(y^*)$$

$$+\frac{r^4}{|x-y|^4}(\partial_j\partial_j f)(y^*)-\frac{4r^4(y_j-x_j)}{|x-y|^6}\sum_{k=1}^{m}(y_k-x_k)(\partial_k\partial_j f)(y^*)$$

$$\left.+\frac{4r^4(y_j-x_j)^2}{|x-y|^8}\sum_{k=1}^{m}\sum_{n=1}^{m}(y_k-x_k)(y_n-x_n)(\partial_k\partial_n f)(y^*)\right]$$

$$=\left(\frac{r}{|x-y|}\right)^{m+2}\Delta f(y^*).$$

Lemma 2.3.2 *If z^* is the inverse of $z \in R^m \setminus \{x\}$ with respect of $\partial B(x;r)$ and $y \in \partial B(x;r)$, then $|z^*-y| = |z-y|r/|z-x|$. If $m > 2$, then $h_m(z^*-y) = h_m(z-y)|z-x|^{m-2}r^{2-m}$. If $m = 2$, then $h_2(z^*-y) = h_2(z-y) + [\ln(|z-x|/r)]/2\pi$.*

Proof We can suppose that $x = 0$.

$$|z^*-y|^2=\left|\frac{r^2z}{|z|^2}-y\right|^2=\frac{r^2}{|z|^2}\left|\frac{rz}{|z|}-\frac{|z|\,y}{r}\right|^2=\frac{r^2}{|z|^2}(r^2-2z\cdot y+|z|^2)=\frac{r^2}{|z|^2}|z-y|^2.$$

2.4 Maximum Principle

In this section, we prove the maximum principle: the harmonic function is bounded from above by the supremum of its trace on the boundary and from below by the infimum of its trace on the boundary.

Lemma 2.4.1 *Let $x \in R^m$, $r > 0$, $u \in \mathcal{C}^0(\overline{B(x;r)}) \cap H(B(x;r))$. Then*

$$u(x) = \oint_{\partial B(x;r)} u(y)\, d\mathcal{H}_{m-1}(y) \equiv \int_{\partial B(x;r)} \frac{u}{\mathcal{H}_{m-1}(\partial B(x;r))}\, d\mathcal{H}_{m-1}. \tag{2.7}$$

Proof Let φ be given by (2.1). Fix $0 < \rho < r$. Corollary 2.2.5 and Corollary 2.2.3 give

$$u(x) = \varphi(\rho) \int_{\partial B(x;\rho)} \frac{\partial u}{\partial n}\, d\mathcal{H}_{m-1} - \varphi'(\rho) \int_{\partial B(x;\rho)} u\, d\mathcal{H}_{m-1}$$

$$= \oint_{\partial B(x;\rho)} u(y)\, d\mathcal{H}_{m-1}(y).$$

If $\rho \to r$, we get (2.7).

Theorem 2.4.2 (Maximum Principle) *Let Ω be a bounded open subset of R^m, $u \in \mathcal{C}^0(\overline{\Omega}) \cap H(\Omega)$. Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x), \quad \min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x), \quad \max_{x \in \overline{\Omega}} |u(x)| = \max_{x \in \partial\Omega} |u(x)|.$$

If Ω is connected and u attains its maximum (or minimum) in Ω, then u is constant.

Proof Denote $M = \max u$. Suppose that there is $z \in \Omega$ such that $u(z) = M$. Denote by G the component of Ω containing z. Denote $S = \{x \in G; u(x) = M\}$. Then S is a nonempty relatively closed subset of G. Fix $x \in S$ and $r > 0$ such that $B(x;r) \subset G$. If there is $y \in B(x;r)$ such that $u(y) < M$, then

$$\oint_{\partial B(x;|x-y|)} u(y)\, d\mathcal{H}_{m-1}(y) < M = u(x),$$

what is a contradiction with Lemma 2.4.1. Thus S is an open and relatively closed subset of the metric space G. Since S and $G\setminus S$ are open subsets of G, G is connected, and S is nonempty, we deduce $S = G$.

Using this result for $v = -u$, we get the rest.

Corollary 2.4.3 *Let Ω be an unbounded open subset of R^m, $m \geq 2$, $u \in \mathcal{C}^0(\overline{\Omega}) \cap H(\Omega)$, and $u(x) \to 0$ as $|x| \to \infty$. Then there exists $x_0 \in \partial\Omega$ such that*

$$|u(x_0)| = \max_{x \in \overline{\Omega}} |u(x)|.$$

Proof Denote $M = \sup |u|$. It is enough to suppose that $M > 0$. Choose $R > 0$ such that $|u| < M/2$ on $\Omega \setminus B(0; R)$. According to maximum principle (Theorem 2.4.2), there exists $x_0 \in \partial(\Omega \cap B(0; R))$ such that

$$|u(x_0)| = \max_{x \in \overline{\Omega \cap B(0;R)}} |u(x)| = M.$$

Since $|u| \leq M/2$ on $\partial B(0; R) \cap \overline{\Omega}$, we deduce that $x_0 \in \partial\Omega$.

Corollary 2.4.4 *Let $\Omega \subset R^m$ be an open set, K be a finite set, $u \in H(\Omega)$. Suppose that M is a real constant such that*

$$\limsup_{\Omega \ni y \to z} u(y) \leq M \qquad \forall z \in \partial\Omega \setminus K,$$

$$\limsup_{\Omega \ni y \to z} u(y)/h_m(y - z) \leq 0 \quad \forall z \in K \cap \partial\Omega. \tag{2.8}$$

If Ω is unbounded suppose moreover

$$\limsup_{\substack{x \in \Omega \\ |x| \to \infty}} u(x) \leq M. \tag{2.9}$$

Then $u(x) \leq M$ for all $x \in \Omega$.

Proof First, assume that Ω is bounded. Fix $\epsilon > 0$. If $K \cap \partial\Omega = \emptyset$, put $u_\epsilon = u$. If $K \cap \partial\Omega = \{y_1, \ldots, y_k\}$ put

$$u_\epsilon(x) = u(x) - \epsilon \sum_{j=1}^{k} h_m((x - y_j)/\operatorname{diam} \Omega).$$

Then

$$\limsup_{\Omega \ni y \to z} u_\epsilon(y) \leq M$$

for all $z \in \partial\Omega$. For each $x \in \partial\Omega$ there is $r(x) > 0$ such that $u_\epsilon < M + \epsilon$ for each $y \in \Omega \cap B(x; r(x))$. Since $\partial\Omega$ is compact, we can choose a finite set $\{x_1, \ldots, x_n\} \subset \partial\Omega$ such that

$$\partial\Omega \subset \cup_{j=1}^{n} B(x_j; r(x_j)).$$

Put

$$\Omega_\epsilon = \Omega \setminus \cup_{j=1}^{n} \overline{B(x_j; r(x_j))}.$$

Then $u_\epsilon \in H(\Omega_\epsilon) \cap \mathcal{C}(\overline{\Omega_\epsilon})$, $u_\epsilon \leq M + \epsilon$ on $\partial\Omega_\epsilon$. The Maximum principle (Theorem 2.4.2) gives that $u_\epsilon \leq M + \epsilon$ on Ω_ϵ. Thus $u_\epsilon \leq M + \epsilon$ in Ω. If $x \in \Omega$ then

$$u(x) = \lim_{\epsilon \to 0} u_\epsilon(x) \leq M.$$

Now, consider the case of Ω unbounded. Fix $\epsilon > 0$. Then there is $\rho(\epsilon) > 1/\epsilon$ such that $u(x) \leq M + \epsilon$ for each $x \in \Omega$ with $|x| \geq \rho(\epsilon)$. Put $\Omega_\epsilon = \Omega \cap B(0; \rho(\epsilon))$. Then $u \leq M + \epsilon$ on Ω_ϵ. Then we let $\epsilon \to 0$, and the result follows.

Counterexample 2.4.5 Put $\Omega = R^m \setminus \overline{B(0; 1)}$, $u(x_1, x_2, \ldots, x_m) = x_1^2 - x_2^2$. Then $u \in H(\Omega)$,

$$\limsup_{\Omega \ni y \to z} u(y) \leq 1$$

for all $z \in \partial\Omega$. But

$$\sup_{x \in \Omega} u(x) = \infty,$$

because the condition (2.9) is not satisfied.

Counterexample 2.4.6 Put $\Omega = B(0; r) \setminus \{0\}$, $u(y) = h_m(y)$, $M = h_m(z)$ with $|z| = r$, $K = \{0\}$.Then $u \in H(\Omega)$,

$$\limsup_{\Omega \ni y \to z} u(y) \leq M \quad \forall z \in \partial\Omega \setminus K.$$

But $u > M$ on Ω because the condition (2.8) is not satisfied.

Proposition 2.4.7 *Let $\Omega \subset R^2$ be an unbounded domain, K be a finite set, $\partial\Omega \setminus K \neq \emptyset$. Let u be a harmonic function in Ω and M be a real constant such that*

$$\limsup_{\Omega \ni y \to z} u(y) \leq M \quad \forall z \in \partial\Omega \setminus K,$$

$$\limsup_{\Omega \ni y \to z} u(y)/h_2(|y - z|) \leq 0 \quad \forall z \in K \cap \partial\Omega,$$

$$\limsup_{\substack{|x| \to \infty \\ x \in \Omega}} u(x) < \infty.$$

Then $u(x) \leq M$ for all $x \in \Omega$.

Proof Suppose first $\overline{\Omega} \neq R^2$. Fix $x \in R^2$ and $r > 0$ such that $[\overline{\Omega} \cup K] \cap \overline{B(x;r)} = \emptyset$. We can suppose that $x = 0$. Let u^* be the image of u under the Kelvin transform with respect to $\partial B(0;r)$. Then $u^* \in H(\Omega^*)$ by Theorem 2.3.1, and

$$\limsup_{\Omega^* \ni y \to z} u^*(y) \leq M \quad \forall z \in \partial\Omega^* \setminus K^*.$$

Since $|z^* - y^*| \leq r^2[|z/|z|^2 - y/|z|^2| + |y/|z|^2 - y/|y|^2|] \leq |z-y|r^2[|z|^{-2} + |y|^{-1}|z|^{-2}(|y| + |z|)]$, we obtain

$$\limsup_{\Omega^* \ni y \to z} u^*(y)/h_2(|y-z|) \leq 0 \quad \forall z \in K^* \cap \partial\Omega^*.$$

Corollary 2.4.4 gives $u^* \leq M$ on Ω^* and thus $u \leq M$ on Ω.

Let now $\overline{\Omega} = R^2$. Then fix $r > 0$ large enough such that $K \subset B(0;r)$. Put $\Omega_1 = \Omega \cap B(0;r)$, $\Omega_2 = \Omega \setminus \overline{B(0;r)}$, $S = \Omega \cap \partial B(0;r)$. If $u \leq M$ on S, then we apply Corollary 2.4.4 on Ω_1 and the result of the preceding paragraph on Ω_2.

If it is not the case that $u \leq M$ on S, then u will have a maximum L on S with $L > M$. We have proved that $u \leq L$ on Ω. Fix $z \in S$ such that $u(z) = L$. Theorem 2.4.2 yields that $u \equiv L$ on Ω. Since there is $x \in \partial\Omega \setminus K$, we have

$$L = \limsup_{\Omega \ni y \to x} u(y) \leq M < L,$$

what is a contradiction.

2.5 The Poisson Kernel for the Ball

If Ω is a ball and f is a continuous function on $\partial\Omega$, then there exists a unique classical solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$. This solution is given by the Poisson integral. This section studies the properties of the Poisson integral.

Fix $x \in R^m$, $r > 0$. Denote

$$P(z,y) = \frac{r^2 - |z-x|^2}{\mathcal{H}_{m-1}(\partial B(0;1)) r |z-y|^m}, \quad y \in \partial B(x;r), z \neq y \tag{2.10}$$

the Poisson kernel of $B(x;r)$. For $\nu \in \mathcal{C}'(\partial B(x;r))$ denote

$$P\nu(z) = \int_{\partial B(x;r)} P(z,y)\, d\nu(y), \quad z \in B(x;r) \tag{2.11}$$

the Poisson integral. Put $\mathcal{H} = \mathcal{H}_{m-1}|\partial\Omega$. If $f \in L^1(\mathcal{H})$, we can write Pf instead of $P(f\mathcal{H})$.

Lemma 2.5.1 *Let $x \in R^m$, $m \geq 2$, $r > 0$, $\Omega = B(x; r)$. Denote $\varphi \equiv 1$. If $z \in \overline{\Omega}$ then $\mathcal{S}^\Omega\varphi(z) = r/(m-2)$ for $m > 2$, $\mathcal{S}^\Omega\varphi(z) = r\ln(1/r)$ for $m = 2$. If $z \in R^m \setminus \Omega$ then $\mathcal{S}^\Omega\varphi(z) = h_m(z-x)\mathcal{H}_{m-1}(\partial B(0; 1))r^{m-1}$.*

Proof Easy calculation gives $\mathcal{S}^\Omega\varphi(x) = r/(m-2)$ for $m > 2$, $\mathcal{S}^\Omega\varphi(x) = r\ln(1/r)$ for $m = 2$. Let now $z \in \Omega \setminus \{x\}$. Since $\mathcal{S}^\Omega\varphi \in H(\Omega)$ by Remark 2.1.5, we obtain by Lemma 2.4.1

$$\mathcal{S}^\Omega\varphi(x) = \oint\limits_{\partial B(x;|z-x|)} \mathcal{S}^\Omega\varphi \, d\mathcal{H}_{m-1} = \mathcal{S}^\Omega\varphi(z),$$

because $\mathcal{S}^\Omega\varphi(y)$ depends only on $|y-x|$.

The function $\mathcal{S}^\Omega\varphi$ is continuous in R^m by Lemma 1.26.8. Put $v = \mathcal{S}^\Omega\varphi - h_m\mathcal{H}_{m-1}(\partial B(0; 1))r^{m-1}$. Then $v \in \mathcal{C}^0(R^m \setminus \Omega) \cap H(R^m \setminus \overline{\Omega})$ by Remark 2.1.5, $v = 0$ on $\partial\Omega$. If $m > 2$ then $v(z) \to 0$ as $|z| \to \infty$. So, $v = 0$ in $R^m \setminus \Omega$ by the Maximum principle (Corollary 2.4.3). Let now $m = 2$. Then

$$\lim_{|z|\to\infty} v(z) = \lim_{|z|\to\infty} \int\limits_{\partial B(0;r)} [h_m(z-y) - h_m(z)] \, \mathrm{d}\mathcal{H}_{m-1}(z)$$

$$= -\lim_{|z|\to\infty} \frac{1}{2\pi} \int\limits_{\partial B(0;r)} \ln\frac{|y-z|}{|z|} \, \mathrm{d}\mathcal{H}_{m-1}(z) = 0$$

by theorem on continuity of parametrized integrals (Theorem 1.13.6). So, $v = 0$ in $R^2 \setminus \Omega$ by Maximum principle (Corollary 2.4.3).

Lemma 2.5.2 *For $y \in \partial B(x; r)$, $z \neq y$*

$$\begin{aligned} P(z, y) &= 2n(y) \cdot \nabla_y h_m(z-y) - (m-2)h_m(z-y)/r, && m > 2 \\ &\ 2n(y) \cdot \nabla_y h_m(z-y) - (2\pi r)^{-1}, && m = 2. \end{aligned}$$

where $n(y)$ is the unit outward normal of $B(x; r)$ at y.

Proof We can suppose that $x = 0$. Denote $\sigma_m = \mathcal{H}_{m-1}(\partial B(0; 1))$.

$$n(y) \cdot \nabla_y h_m(z-y) = \frac{y \cdot (y-z)}{r\sigma_m|z-y|^m} = \frac{(r^2 - 2y \cdot z + |z|^2) + (r^2 - |z|^2)}{2r\sigma_m|z-y|^m}.$$

If $m = 2$, then

$$n(y) \cdot \nabla_y h_m(z-y) = \frac{1}{2r\sigma_2} + \frac{1}{2}P(z, y).$$

If $m > 2$, then

$$n(y) \cdot \nabla_y h_m(z-y) = \frac{m-2}{2r} h_m(z-y) + \frac{1}{2} P(z,y).$$

Proposition 2.5.3 *$P1 = 1$ in $B(x;r)$.*

Proof If $m = 2$, then $P1 = 2\mathcal{D}^{B(0;r)}1 - 1 = 2 - 1 = 1$ by Lemma 2.5.2 and Corollary 2.2.6. If $m > 2$, then $P1 = 2\mathcal{D}^{B(0;r)}1 - \mathcal{S}^{B(0;r)}(m-2)/r = 2 - 1 = 1$ by Lemma 2.5.2, Lemma 2.5.1 and Corollary 2.2.6.

Theorem 2.5.4 (Theorem on the Poisson Integral)

1. *If $\nu \in \mathcal{C}'(\partial B(x;r))$ then $P\nu \in H(B(x;r))$.*
2. *If $f \in L^1(\partial B(x;r))$, then*

$$\limsup_{B(x;r)\ni z\to y} Pf(z) \le \limsup_{\partial B(x;r)\ni z\to y} f(z) \quad \forall y \in \partial B(x;r), \tag{2.12}$$

$$\liminf_{B(x;r)\ni z\to y} Pf(z) \ge \liminf_{\partial B(x;r)\ni z\to y} f(z) \quad \forall y \in \partial B(x;r). \tag{2.13}$$

3. *If $f \in L^1(\partial B(x;r))$ is continuous in $y \in \partial B(x;r)$, then*

$$\lim_{B(x;r)\ni z\to y} Pf(z) = f(y). \tag{2.14}$$

4. *If $u \in H(B(x;r)) \cap \mathcal{C}^0(\overline{B(x;r)})$, then $u = Pu$ in $B(x;r)$.*

Proof

1) Since $P\nu = 2\mathcal{D}^{B(x;r)}\nu - (m-2)r^{-1}\mathcal{S}^{B(x;r)}\nu$ for $m > 2$, $P\nu = 2\mathcal{D}^{B(x;r)}\nu - \nu(R^m)/(2\pi r)$ for $m = 2$ by Lemma 2.5.2, the function $P\nu$ is harmonic in $B(x;r)$ by Remark 2.1.4 and Remark 2.1.5.
2) Let

$$\limsup_{\partial B(x;r)\ni z\to y} f(z) < K < \infty.$$

Then there exists $\delta > 0$ such that $f \le K$ on $B(y;\delta) \cap \partial B(x,r)$. Put $f_1 = f$ on $B(y;\delta) \cap \partial B(x,r)$, $f_1 = 0$ elsewhere, $f_2 = f - f_1$. Lebesgue lemma (Lemma 1.13.2) gives

$$\lim_{B(x;r)\ni z\to y} Pf_2(z) = 0.$$

Since $P(z,w)$ is nonnegative for $z \in B(x;r)$ and $w \in \partial B(x;r)$, we have

$$\limsup_{B(x;r)\ni z\to y} Pf_1(z) \le \limsup_{B(x;r)\ni z\to y} PK(z) = K$$

by Proposition 2.5.3. Since K is arbitrary, we conclude (2.12). We obtain (2.13) from (2.12) for $g = -f$.

3) is an easy consequence of 2).

4) Suppose now $u \in H(B(x;r)) \cap \mathcal{C}^0(\overline{B(x;r)})$. Put $v = u - Pu$. Then $v \in H(B(x;r)) \cap \mathcal{C}^0(\overline{B(x;r)})$, $v = 0$ on $\partial B(x;r)$. Maximum principle (Theorem 2.4.2) gives that $v \equiv 0$ in $B(x;r)$ and thus $u = Pu$.

2.6 Mean Value Theorem

In this section, we give several characterization of harmonicity of continuous functions. First, we show that a function u continuous in Ω is harmonic in Ω if and only if the average of u over $B(z;r)$ is equal to $u(z)$ for each ball $B(z;r)$ with $\overline{B(z;r)} \subset \Omega$. Then, we show that a function u continuous in Ω is harmonic in Ω if and only if the average of u over $\partial B(z;r)$ is equal to $u(z)$ for each ball $B(z;r)$ with $\overline{B(z;r)} \subset \Omega$. We also show that a function $u \in \mathcal{C}^1(\Omega)$ is harmonic in Ω if and only if

$$\int\limits_{\partial B(z;r)} \frac{\partial u}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = 0$$

for each ball $B(z;r)$ with $\overline{B(z;r)} \subset \Omega$.

Theorem 2.6.1 (Mean Value Theorem) *Let Ω be an open subset of R^m, $u \in L^1_{loc}(\Omega)$. The following are equivalent:*

1. *u is a harmonic function in Ω.*
2. *$u \in \mathcal{C}^0(\Omega)$ and (2.7) hold true for every $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$.*
3. *$u \in \mathcal{C}^0(\Omega)$ and for every $x \in \Omega$ there is a sequence of positive numbers ρ_k converging to 0 such that (2.7) holds true for each $r = \rho_k$.*
4. *$u \in \mathcal{C}^0(\Omega)$and*

$$u(x) = \oint\limits_{B(x;r)} u\, \mathrm{d}\mathcal{H}_m \equiv \frac{1}{\mathcal{H}_m(B(x;r))} \int\limits_{B(x;r)} u\, \mathrm{d}\mathcal{H}_m \tag{2.15}$$

for every $x \in \Omega$, $0 < r < \operatorname{dist}(x; \partial\Omega)$.

5. *$u \in \mathcal{C}^0(\Omega)$ and for every $x \in \Omega$ there is a sequence of positive numbers ρ_k converging to 0 such that (2.15) holds true for each $r = \rho_k$.*
6. *There is a constant $c \geq 1$ such that (2.15) holds true for every $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)/c$.*

Proof 1) $\Rightarrow$ 2): See Lemma 2.4.1.

3) $\Rightarrow$ 1) : Fix $z \in \Omega$, $0 < \rho < \operatorname{dist}(z, \partial\Omega)$. According to theorem on the Poisson integral (Theorem 2.5.4), there is $v \in \mathcal{C}^0(\overline{B(z;\rho)}) \cap H((B(z;\rho))$ such that $v = u$ on $\partial B(z;\rho)$. Denote by M the supremum of $w \equiv u - v$ over $\overline{B(z;\rho)}$. Suppose that $M > 0$. Then $K = \{x \in \overline{B(z;\rho)}; w(x) = M\}$ is a nonempty compact subset of $\overline{B(z;\rho)}$. Since $w = 0$ on $\partial B(z;\rho)$, we deduce $K \subset B(z;\rho)$. There is $x \in K$ such that $|x - z| = \sup\{|y - z|; y \in K\}$. Fix now $r > 0$ such that (2.7) holds and $\overline{B(x;r)} \subset B(z;\rho)$. We know that $w(y) < M$ for $y \in \partial B(x;r) \setminus \overline{B(z;|z-x|)} \neq \emptyset$. Since $v \in H((B(z;\rho))$, Lemma 2.4.1 gives

$$w(x) = \oint\limits_{\partial B(x;r)} w(y)\, \mathrm{d}\mathcal{H}_{m-1}(y) < M = w(x),$$

what is a contradiction. Thus $w \le 0$. Similarly, $w \ge 0$. Hence, $u = v \in H(B(z;\rho))$.

2) $\Rightarrow$ 4) :

$$\oint\limits_{B(x;r)} u(y)\, \mathrm{d}\mathcal{H}_m(y) = \int\limits_0^r \int\limits_{\partial B(x;\rho)} \frac{u(y)}{\mathcal{H}_m(B(x;r))}\, \mathrm{d}\mathcal{H}_{m-1}(y)\, \mathrm{d}\rho$$

$$= \frac{1}{\mathcal{H}_m(B(x;r))} \int\limits_0^r u(x)\mathcal{H}_{m-1}(\partial B(x;\rho))\, \mathrm{d}\rho = u(x).$$

5) $\Rightarrow$ 1) : Fix $z \in \Omega$, $0 < \rho < \operatorname{dist}(z, \partial\Omega)$. According to theorem on the Poisson integral (Theorem 2.5.4), there is $v \in \mathcal{C}^0(\overline{B(z;\rho)}) \cap H((B(z;\rho))$ such that $v = u$ on $\partial B(z;\rho)$. Denote by M the supremum of $w \equiv u - v$ over $\overline{B(z;\rho)}$. Suppose that $M > 0$. Then $K = \{x \in \overline{B(z;\rho)}; w(x) = M\}$ is a nonempty compact subset of $B(z;\rho)$. Therefore, there is $x \in K$ such that $|x - z| = \sup\{|y - z|; y \in K\}$. Fix now $r > 0$ such that (2.15) holds and $\overline{B(x;r)} \subset B(z;\rho)$. For $v \in H((B(z;\rho))$, we have proved

$$v(x) = \oint\limits_{B(x;r)} v(y)\, \mathrm{d}\mathcal{H}_m(y).$$

We know that $w(y) < M$ for $y \in B(x;r) \setminus \overline{B(z;|z-x|)} \neq \emptyset$. Thus,

$$w(x) = \oint\limits_{B(x;r)} w(y)\, \mathrm{d}\mathcal{H}_m(y) < M = w(x),$$

what is a contradiction. So, $w \le 0$. Similarly, $w \ge 0$. Hence, $u = v \in H(B(z;\rho))$.

6) $\Rightarrow$ 5): Let $x_j \to x \in \Omega$ as $j \to \infty$. Fix $r \in (0, \operatorname{dist}(x, \partial\Omega)/(3c))$. Then, $|u| \in L^1(B(x; 2r))$. We get using Lebesgue lemma (Lemma 1.13.2)

$$\lim_{j\to\infty} u(x_j) = \frac{1}{\mathcal{H}_m(B(x;r))} \lim_{j\to\infty} \int\limits_{B(x;2r)} u\chi_{B(x_j;r)}\, d\mathcal{H}_m$$

$$= \frac{1}{\mathcal{H}_m(B(x;r))} \int\limits_{B(x;2r)} u\chi_{B(x;r)}\, d\mathcal{H}_m = u(x),$$

where χ_M is the characteristic function of a set M. Thus, $u \in \mathcal{C}^0(\Omega)$.

Proposition 2.6.2 *Let $\Omega \subset R^m$, $m \geq 2$, be an open set, $u \in \mathcal{C}^1(\Omega)$. The following are equivalent:*

1. *$u \in H(\Omega)$.*
2. *For each bounded open set G with boundary of class $\mathcal{C}^1$ such that $\overline{G} \subset \Omega$, we have*

$$\int\limits_{\partial G} \frac{\partial u}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{2.16}$$

3. *The relation (2.16) holds true for each ball G with $\overline{G} \subset \Omega$.*

Proof 1) $\Rightarrow$ 2): See Corollary 2.2.3.

3) $\Rightarrow$ 1): Fix $x \in \Omega$. Define

$$f(r) = \oint\limits_{\partial B(x;r)} u\, \mathrm{d}\mathcal{H}_m.$$

Then

$$f'(r) = \frac{d}{dr}\Bigg[\oint\limits_{\partial B(0;1)} u(x+ry)\, \mathrm{d}\mathcal{H}_m(y)\Bigg] = \oint\limits_{\partial B(0;1)} \frac{\partial u(x+ry)}{\partial n}\, \mathrm{d}\mathcal{H}_m(y) = 0.$$

Therefore, $f(r)$ is constant. Since $f(r) \to u(x)$ as $r \searrow 0$, we see

$$\oint\limits_{\partial B(x;r)} u\, \mathrm{d}\mathcal{H}_m = u(x)$$

whenever $\overline{B}(x;r) \subset \Omega$. Mean value theorem (Theorem 2.6.1) gives that $u \in H(\Omega)$.

2.7 Separately Harmonic Function

Theorem 2.7.1 *Let* $\Omega \subset R^m \times R^n$, $u \in L^1_{loc}(\Omega)$. *Suppose for each* $x \in R^m$ *and* $y \in R^n$, *the function* $u(x,\cdot)$ *is harmonic in* $\{z \in R^n; [x,z] \in \Omega\}$ *and* $u(\cdot,y)$ *is a harmonic function in* $\{z \in R^m; [z,y] \in \Omega\}$. *Then* $u \in H(\Omega)$.

Proof Fix $[x,y] \in \Omega$, $x \in R^m$, $y \in R^n$. Fix $0 < r < \operatorname{dist}([x,y],\partial\Omega)$. According to Fubini's theorem and Mean value theorem (Theorem 2.6.1),

$$\int\limits_{B([x,y];r)} u \, \mathrm{d}\mathcal{H}_{m+n} = \int\limits_{B(x;r)} \int\limits_{B(y;\sqrt{r^2-|z-x|^2})} u(z,w)\, \mathrm{d}\mathcal{H}_n(w)\mathrm{d}\mathcal{H}_m(z)$$

$$= \int\limits_{B(x;r)} \mathcal{H}_n(B(y;\sqrt{r^2-|z-x|^2}))u(z,y)\, \mathrm{d}\mathcal{H}_m(z)$$

$$= \int\limits_0^r \int\limits_{\partial B(x;t)} \mathcal{H}_n(B(y;\sqrt{r^2-t^2}))u(t\sigma,y)\, \mathrm{d}\mathcal{H}_{m-1}(\sigma)\mathrm{d}t$$

$$= \int\limits_0^r \mathcal{H}_{m-1}(\partial B(x;t))\mathcal{H}_n(B(y;\sqrt{r^2-t^2}))u(x,y)\, \mathrm{d}t$$

$$= u(x,y)\mathcal{H}_{m+n}(B([x,y];r)).$$

Mean value theorem (Theorem 2.6.1) gives that $u \in H(\Omega)$.

2.8 The Harnack Inequality

In this section we study nonnegative harmonic functions. In particular we show that if u is a nonnegative harmonic function in a domain Ω and $u(x) = 0$ for some $x \in \Omega$ then $u \equiv 0$.

Proposition 2.8.1 (Harnack Inequality) *Let* $0 < \rho < R < \infty$, $z \in R^m$. *If* u *is a nonnegative harmonic function in* $B(z;R)$, *then*

$$\frac{(R-\rho)R^{m-2}}{(R+\rho)^{m-1}}u(z) \le u(x) \le \frac{(R+\rho)R^{m-2}}{(R-\rho)^{m-1}}u(z) \tag{2.17}$$

for each $x \in B(z;\rho)$.

Proof Fix $r \in (\rho, R)$. Let P be the Poisson kernel of $B(z; r)$ (see (2.10)). If $y \in \partial B(z; r)$, then

$$\frac{r-\rho}{r(r+\rho)^{m-1}} \leq \frac{r^2-|z-x|^2}{r|z-y|^m} \leq \frac{r+\rho}{r(r-\rho)^{m-1}}$$

and $(r-\rho)r^{m-2}(r+\rho)^{1-m}P(z, y) \leq P(x, y) \leq (r+\rho)(r-\rho)^{1-m}r^{m-2}P(z, y)$. Theorem on the Poisson integral (Theorem 2.5.4) gives

$$\frac{(r-\rho)r^{m-2}}{(r+\rho)^{m-1}}u(z) \leq u(x) \leq \frac{(r+\rho)r^{m-2}}{(r-\rho)^{m-1}}u(z)$$

If $r \uparrow R$, we get (2.17).

Theorem 2.8.2 *Let K be a compact subset of a domain $\Omega \subset R^m$. Then, there is a positive constant C such that*

$$\sup_{x \in K} u(x) \leq C \inf_{x \in K} u(x)$$

for each nonnegative harmonic function u in Ω.

Proof For each $x \in K$, there is $r(x) > 0$ such that $B(x; 2r(x)) \subset \Omega$. Since $K \subset \cup\{B(x; r(x)); x \in K\}$ and K is compact, there are $x_1, \dots, x_k \in K$ such that

$$K \subset \bigcup_{j=1}^{k} B(x_j; r(x_j)).$$

According to the Harnack inequality (Proposition 2.8.1), there are positive constants $C_1, \dots, C_k$ such that for each nonnegative harmonic function u on Ω and each $x, y \in B(x_j; r(x_j))$ we have $u(x) \leq C_1 u(y)$.

Fix now $j, l \in \{1, \dots, k\}$, $j \neq l$. Since Ω is connected, there is a curve γ in Ω connecting x_j and x_l (see Remark 1.2.19). Since γ is compact, there are open balls $B_1, \dots, B_n$ such that $\gamma \subset B_1 \cup \dots \cup B_n$ and $\overline{B_j} \subset \Omega$ for $j = 1, \dots, n$. We can suppose that $B_j \cap B_{j+1} \neq \emptyset$ for $j = 1, \dots, n-1$ and $x_j \in B_1$, $x_l \in B_n$. According to the Harnack inequality (Proposition 2.8.1), there are positive constants $D_1, \dots, D_n$ such that for each nonnegative harmonic function u on Ω and each $x, y \in B_j$, we have $u(x) \leq D_1 u(y)$. Fix $y_j \in B_j \cap B_{j+1}$. Then,

$$u(x_j) \leq D_1 u(y_1) \leq D_1 D_2 u(y_2) \leq \dots \leq D_1 D_2 \dots D_{n-1} u(y_{n-1})$$

$$\leq D_1 D_2 \dots D_n u(x_l).$$

This means that there are C_{jl} such that $u(x_j) \leq u(x_l)$ for each nonnegative harmonic function u on Ω.

Fix now $x, y \in K$. Then there are j, l such that $x \in B(x_j, r(x_j))$ and $y \in B(x_l, r(x_l))$. If $j = l$, then $u(x) \le C_j u(y)$. If $j \ne l$, then

$$u(x) \le C_j u(x_j) \le C_j C_{jl} u(x_l) \le C_j C_{jl} C_l u(y).$$

2.9 Derivatives of Harmonic Functions

If u is a harmonic function in Ω, then $\partial^\alpha u$ is locally bounded in Ω. In this section, we study behavior of $\partial^\alpha u$ in a neighborhood of $\partial\Omega$. We also derive useful estimates of derivatives of the fundamental solution of the Laplace equation.

Lemma 2.9.1 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, $u \in \mathcal{C}^1(\Omega)$, $\overline{B(x;r)} \subset \Omega$. If $\partial_j u \in H(\Omega)$ (in particular if $u \in H(\Omega)$), then*

$$|\partial_j u(x)| \le \frac{m}{r} \oint_{\partial B(x;r)} |u| \, d\mathcal{H}_{m-1}.$$

Proof Suppose $\partial_j u \in H(\Omega)$. (If $u \in H(\Omega)$, then $\partial_j u \in H(\Omega)$ by Corollary 2.2.9.) According to the Mean value theorem (Theorem 2.6.1) and the Gauss-Green theorem, (Theorem 1.19.1)

$$|\partial_j u(x)| = \left| \int_{B(x;r)} \frac{\partial_j u}{\mathcal{H}_m(B(0;r))} \, d\mathcal{H}_m \right| = \left| \int_{\partial B(x;r)} \frac{n_j u}{\mathcal{H}_m(B(0;r))} \, d\mathcal{H}_{m-1} \right|$$

$$= \frac{m}{r} \left| \oint_{\partial B(0;r)} n_j u \, d\mathcal{H}_{m-1} \right| \le \frac{m}{r} \oint_{\partial B(x;r)} |u| \, d\mathcal{H}_{m-1}.$$

Theorem 2.9.2 *Let $\Omega \subset R^m$ be an open set, $\Omega \ne R^m$. If u is a bounded harmonic function in Ω and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multiindex, then*

$$|\partial^\alpha u(x)| \le \left[\frac{m|\alpha|}{\operatorname{dist}(x, \partial\Omega)} \right]^{|\alpha|} \sup_{y \in \Omega} |u(y)|, \qquad x \in \Omega. \tag{2.18}$$

Proof Suppose first that $|\alpha| = 1$. Let $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$. Lemma 2.9.1 gives

$$|\partial^\alpha u(x)| \le \frac{m}{r} \sup_{y \in \Omega} |u(y)|.$$

If $r \to \operatorname{dist}(x, \partial\Omega)$, we obtain (2.18).

Let now k be a positive integer such that (2.18) holds true for each multiindex α with $|\alpha| = \alpha_1 + \dots + \alpha_m \leq k$. Let now β be a multiindex with $|\beta| = k + 1$. Then, there are $j \in \{1, \dots, m\}$ and a multiindex α with $|\alpha| \leq k$ such that $\partial^\beta = \partial_j \partial^\alpha$. One has $\partial^\alpha u \in H(\Omega)$ by Corollary 2.2.9. Fix $x \in \Omega$. Put $r = \operatorname{dist}(x, \partial\Omega)/(k+1)$. If $z \in \partial B(x; r)$, then

$$|\partial^\alpha u(z)| \leq \left[\frac{m|\alpha|}{\operatorname{dist}(z, \partial\Omega)}\right]^{|\alpha|} \sup_{y\in\Omega} |u(y)| \leq \left[\frac{m(k+1)}{\operatorname{dist}(x, \partial\Omega)}\right]^{k} \sup_{y\in\Omega} |u(y)|.$$

According to Lemma 2.9.1, we have

$$|\partial^\beta u(x)| = |\partial_j \partial^\alpha u(x)| \leq \int\limits_{\partial B(x;r)} \frac{m|\partial^\alpha u(z)|}{\mathcal{H}_{m-1}(\partial B(0;1)) r^m} \, d\mathcal{H}_{m-1}(z)$$

$$\leq \frac{m}{r}\left[\frac{m(k+1)}{\operatorname{dist}(x, \partial\Omega)}\right]^{k} \sup_{y\in\Omega} |u(y)| = \left[\frac{m|\beta|}{\operatorname{dist}(x, \partial\Omega)}\right]^{|\beta|} \sup_{y\in\Omega} |u(y)|.$$

Example 2.9.3 Theorem 2.9.2 shows that for $u \in H(\Omega) \cap \mathcal{C}^{0,0}(\Omega)$, there exists a constant C such that $|\nabla u(x)| \leq C \operatorname{dist}(x, \partial\Omega)^{-1}$. We cannot improve this result as the following example shows. Set $\Omega = \{[x_1, x_2]; 0 < x_1 < 2, 0 < x_2 < 2\}$, $u(x_1, x_2) = \arccos(x_1/|x|)$. Easy calculation yields that $\Delta u = 0$, u is bounded and $|\nabla u(x)| = 1/|x|$. If $0 < x_1 = x_2 < 1$, then $|\nabla u(x)| = 1/|x| = (1/\sqrt{2}) \operatorname{dist}(x, \partial\Omega)^{-1}$.

Theorem 2.9.4 *Let $\Omega \subset R^m$ be an open set, $u \in H(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$, $m \geq 2$, $0 < \alpha \leq 1$. If β is a multiindex with $|\beta| > 0$, then*

$$|\partial^\beta u(x)| \leq (m|\beta|)^{|\beta|} \|u\|_{\mathcal{C}^{0,\alpha}(\Omega)} \operatorname{dist}(x, \partial\Omega)^{\alpha - |\beta|}, \quad x \in \Omega.$$

Proof Let $|u(y) - u(z)| \leq M|y - z|^\alpha$ for $y, z \in \Omega$. Fix $x \in \Omega$. Put $\omega = B(x; \operatorname{dist}(x, \partial\Omega)/2)$. According to Theorem 2.9.2

$$|\partial^\beta u(x)| \leq \left[\frac{m|\beta|}{\operatorname{dist}(x, \partial\omega)}\right]^{|\beta|} \sup_{y\in\omega} |u(y) - u(x)| \leq \frac{(m|\beta|)^{|\beta|} M \operatorname{dist}(x, \partial\Omega)^\alpha}{\operatorname{dist}(x, \partial\Omega)^{|\beta|}}.$$

Theorem 2.9.5 *Let $\Omega \subset R^m$ be an open set, $\Omega \neq R^m$. If u is a nonnegative harmonic function in Ω and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multiindex, then*

$$|\partial^\alpha u(x)| \leq \left[\frac{m|\alpha|}{\operatorname{dist}(x, \partial\Omega)}\right]^{|\alpha|} u(x), \qquad x \in \Omega. \tag{2.19}$$

Proof Suppose first that $|\alpha| = 1$. Let $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$. Lemma 2.9.1 and Mean value theorem (Theorem 2.6.1) give

$$|\partial^\alpha u(x)| \le \frac{m}{r} \oint\limits_{\partial B(x;r)} u \, d\mathcal{H}_{m-1}(y) = \frac{m}{r} u(x).$$

If $r \to \operatorname{dist}(x, \partial\Omega)$, we obtain (2.19).

Let now k be a positive integer such that (2.19) holds true for each multiindex α with $|\alpha| = \alpha_1 + \cdots + \alpha_m \le k$. Let now β be a multiindex with $|\beta| = k + 1$. Then there are $j \in \{1, \dots, m\}$ and a multiindex α with $|\alpha| = k$ such that $\partial^\beta = \partial_j \partial^\alpha$. Corollary 2.2.9 gives $\partial^\alpha u \in H(\Omega)$. Fix $x \in \Omega$. Put $r = \operatorname{dist}(x, \partial\Omega)/(k + 1)$. If $z \in \partial B(x; r)$, then

$$|\partial^\alpha u(z)| \le \left[\frac{m|\alpha|}{\operatorname{dist}(z, \partial\Omega)}\right]^{|\alpha|} u(z) \le \left[\frac{m(k+1)}{\operatorname{dist}(x, \partial\Omega)}\right]^{k} u(z).$$

According to Lemma 2.9.1 and Mean value theorem (Theorem 2.6.1), we have

$$|\partial^\beta u(x)| = |\partial_j \partial^\alpha u(x)| \le \frac{m}{r} \oint\limits_{\partial B(x;r)} |\partial^\alpha u(z)| \, d\mathcal{H}_{m-1}(z)$$

$$\le \frac{m}{r} \oint\limits_{\partial B(x;r)} \left[\frac{m(k+1)}{\operatorname{dist}(x, \partial\Omega)}\right]^{k} u(z) \, d\mathcal{H}_{m-1}(z) = \left[\frac{m|\beta|}{\operatorname{dist}(x, \partial\Omega)}\right]^{|\beta|} u(x).$$

Corollary 2.9.6 *Let $m \ge 2$ and α be a multiindex with $|\alpha| \ge 1$. Then, there exists a constant C depending only on m and α such that $|\partial^\alpha h_m(x)| \le C|x|^{2-m-|\alpha|}$.*

Proof Easy calculation gives that $\partial_j h_m(x) = -x_j |x|^{-m}/\mathcal{H}_m(\partial B(0; 1))$. If α is a multiindex with $|\alpha| > 1$, then $\partial^\alpha h_m = \partial^\beta \partial_j h_m$ for some $j \in \{1, \dots, m\}$ and a multiindex β. For a fixed $x \in R^m \setminus \{0\}$ put $\Omega = \{y \in R^m; |x|/2 < |y|\}$. Then $|\partial_j h_m(y)| \le (|x|/2)^{1-m}/\mathcal{H}_m(\partial B(0; 1))$ in Ω. Theorem 2.9.2 gives

$$|\partial^\alpha h_m(x)| \le \left[\frac{m|\beta|}{\operatorname{dist}(x; \partial\Omega)}\right]^{|\beta|} \sup_{y \in \Omega} |\partial_j h_m(y)|$$

$$\le \left[\frac{m(|\alpha| - 1)}{|x|/2}\right]^{|\alpha|-1} \frac{(|x|/2)^{1-m}}{\mathcal{H}_m(\partial B(0; 1))} \le 2^m (2m|\alpha|)^\alpha |x|^{2-m-|\alpha|}.$$

Lemma 2.9.7 *Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a multiindex, $m \ge 2$. Then, there exists a constant C_α such that for $x, y, z \in R^m$ with $|x - z| > 2|x - y|$ we have $|\partial^\alpha h_m(x - z) - \partial^\alpha h_m(y - z)| \le C_\alpha |x - y||x - z|^{1-m-|\alpha|}$.*

Proof According Corollary 2.9.6, there exists a constant C such that $|\nabla\partial^\alpha h_m(w)| \leq C_\alpha|w|^{1-m-\alpha}$. Put $f(t) = \partial^\alpha h_m((1-t)x + ty - z))$. Then there exists $\theta \in (0;1)$ such that $f(1) - f(0) = f'(\theta)$. Since $|(1-t)x + ty - z| \geq |x-z| - t|y-x| \geq 2|x-y| - |x-y| = |x-y|$, we obtain $|\partial^\alpha h_m(x-z) - \partial^\alpha h_m(y-z)| = |f(1) - f(0)| = |f'(\theta)| = |(x-y)\cdot\nabla\partial^\alpha h_m((1-\theta)x + \theta y - z)| \leq C_\alpha|x-y||x-z|^{1-m-|\alpha|}$.

2.10 Equibounded Harmonic Functions

Proposition 2.10.1 *Let* $\Omega \subset R^m$, $m \geq 2$, *be an open set,* $\mathcal{M} \subset H(\Omega)$. *If* $\mathcal{M}$ *is equibounded on each compact subset of* Ω, *then* $\mathcal{M}$ *is equicontinuous on each compact subset of* Ω.

Proof Fix a compact subset K of Ω. Choose a bounded open set V such that $K \subset V \subset \overline{V} \subset \Omega$. Then, there exists a positive constant L such that $|u| \leq L$ on $\overline{V}$ for each $u \in \mathcal{M}$. According to Remark 1.2.10, there exists a positive constant $r > 0$ such that $B(x;2r) \subset V$ for each $x \in K$. If $x \in K$, $z \in B(x;r)$, $u \in \mathcal{M}$, then $|\nabla u(z)| \leq Lm/r$ by Theorem 2.9.2. Fix $u \in \mathcal{M}$ and $x, y \in K$ with $|x-y| < r$. Put $f(t) = u(x + t(y-x))$. Then, there exist $\theta \in (0,1)$ such that $f(1) - f(0) = f'(\theta)$. Therefore, $|u(x) - u(y)| = |f(1) - f(0)| = |f'(\theta)| = |\nabla u(x + \theta(y-x))\cdot(y-x)| \leq (Lm/r)|x-y|$. Hence, $\mathcal{M}$ is equicontinuous on K.

Proposition 2.10.2 *Let* $\Omega \subset R^m$ *be a domain,* $m \geq 2$, $\mathcal{M} \subset H(\Omega)$. *Suppose that* $\mathcal{M}$ *is equibounded from below on each compact subset of* Ω. *Then either* $\sup\{u; u \in \mathcal{M}\} \equiv \infty$ *or* $\mathcal{M}$ *is equibounded and equicontinuous on each compact subset of* Ω.

Proof Suppose that there exists $z \in \Omega$ such that $L = \sup\{u(z); u \in \mathcal{M}\} < \infty$. Fix a compact set $F \subset \Omega$. Choose a bounded domain ω such that $F \cup \{z\} \subset \omega$, $\overline{\omega} \subset \Omega$. Then, there exists a constant K such that $u \geq K$ on $\overline{\omega}$ for each $u \in \mathcal{M}$. According to Theorem 2.8.2, there exists a positive constant C such that $h(x) \leq Ch(z)$ for each positive $h \in H(\omega)$ and $x \in F$. If $u \in \mathcal{M}$ and $x \in F$, then $u(x) = [u(x) - K] + K \leq C[u(z) - K] + K \leq C(L-K) + K$. So, $\mathcal{M}$ is equibounded on F. Proposition 2.10.1 gives that $\mathcal{M}$ is equicontinuous on F.

2.11 Sequences of Harmonic Functions

In this section, we study a limit of a sequence of harmonic functions. As a consequence, we show that the space of functions from $L^p(\Omega)$ harmonic in Ω is a closed subspace of $L^p(\Omega)$.

Proposition 2.11.1 *Let* Ω *be an open subset of* R^m, u_j *be a sequence of harmonic functions in* Ω *which converges locally uniformly to* u *in* Ω. *Then,* $u \in H(\Omega)$. *If* $\alpha = (\alpha_1, \ldots, \alpha_m)$ *is a multiindex then* $\partial^\alpha u_j$ *converges locally uniformly to* $\partial^\alpha u$ *in* Ω.

Proof $u \in \mathcal{C}^0(\Omega)$ because u_j converges uniformly to u on each compact subset of Ω. If $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, then Mean value theorem (Theorem 2.6.1) and Lebesgue lemma (Lemma 1.13.2) yield

$$u(x) = \lim_{j\to\infty} u_j(x) = \lim_{j\to\infty} \oint\limits_{\partial B(x;r)} u_j \, d\mathcal{H}_{m-1} = \oint\limits_{\partial B(x;r)} u \, d\mathcal{H}_{m-1}.$$

Mean value theorem (Theorem 2.6.1) gives that $u \in H(\Omega)$.

Let K be a compact subset of Ω. Fix a bounded open set G such that $K \subset G \subset \overline{G} \subset \Omega$. According to Remark 1.2.10, there is $r > 0$ such that $B(x; r) \subset G$ for each $x \in K$. Since $u - u_j \in H(G)$, Theorem 2.9.2 gives

$$\sup_{x\in K} |\partial^\alpha u_j(x) - \partial^\alpha u(x)| \le \left(\frac{m|\alpha|}{r}\right)^{|\alpha|} \sup_{y\in\overline{G}} |u_j(y) - u(y)|.$$

Since u_j converges uniformly to u on the compact set $\overline{G}$, the sequence $\partial^\alpha u_j$ converges uniformly to $\partial^\alpha u$ on K.

Proposition 2.11.2 *Let $\Omega \subset R^m$ be a domain, u_j be a sequence of harmonic functions in Ω. Suppose that $\{u_j\}$ is equibounded from below on each compact subset of Ω. Then,*

$$\lim_{j\to\infty} u_j(x) = \infty \qquad \text{on } \Omega \tag{2.20}$$

or there is a subsequence $\{v_k\}$ of $\{u_j\}$ such that v_k converges locally uniformly to $v \in H(\Omega)$.

Proof Suppose that (2.20) does not hold. Then, there exist $z \in \Omega$, $L \in R$ and a subsequence $\tilde{u}_l$ of u_j such that $\tilde{u}_l(z) \le L$. Let Ω_j be a sequence of bounded subdomains of Ω such that $z \in \Omega_j$, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $\cup\Omega_j = \Omega$. Then, $\{\tilde{u}_l\}$ is a set of equicontinuous equibounded functions on $\overline{\Omega_j}$ for each j by Proposition 2.10.2. Arzela-Ascoli theorem (Theorem 1.16.1) gives that $\{\tilde{u}_l\}$ is a relatively compact subset of $\mathcal{C}^0(\overline{\Omega_1})$. Therefore, there exist a subsequence $\{w_{1,k}\}$ of $\{\tilde{u}_l\}$ which converges uniformly in $\overline{\Omega_1}$.

Fix now a positive integer k. Suppose that we have sequences $\{w_{1,j}\}, \dots, \{w_{k,j}\}$ such that $\{w_{l+1,j}\}$ is a subsequence of $\{w_{l,j}\}$ which converges uniformly in $\overline{\Omega_{l+1}}$. By Arzela-Ascoli theorem (Theorem 1.16.1), there exist a subsequence $\{w_{k+1,j}\}$ of $\{w_{k,j}\}$ which converges uniformly in $\overline{\Omega_{k+1}}$. Put $v_k = w_{k,k}$. Then v_k converges uniformly to v on each Ω_j. Proposition 2.11.1 gives that $v \in H(\Omega_j)$.

Proposition 2.11.3 *Let $\Omega \subset R^m$ be a domain, u_j be an increasing sequence of harmonic functions on Ω. Then (2.20) holds true or $\{u_j\}$ converges locally uniformly to $u \in H(\Omega)$.*

Proof If K is a compact subset of Ω, then

$$u_j \geq \inf_{x \in K} u_1(x) > -\infty.$$

The rest is a consequence of Proposition 2.11.2.

Proposition 2.11.4 *Let Ω be an open subset of R^m, $1 \leq p \leq \infty$, $\{f_n\} \subset H(\Omega) \cap L^p(\Omega)$, $f \in L^p(\Omega)$. Suppose that $f_n \to f$ in $L^p(\Omega)$. Then, there is $g \in H(\Omega) \cap L^p(\Omega)$ such that $g(x) = f(x)$ for $\mathcal{H}_m$ almost all $x \in \Omega$ and $f_n(x)$ converges locally uniformly to $g(x)$. Thus $H(\Omega) \cap L^p(\Omega)$ is a closed subspace of $L^p(\Omega)$.*

Proof Let $K \subset \Omega$ be compact. Then, there is $r > 0$ such that $B(x; 2r) \subset \Omega$ for each $x \in K$ (compare Remark 1.2.10). Let $1 \leq q \leq \infty$ be such that $1/p + 1/q = 1$. Fix $\epsilon > 0$. Then there is $n_0 \in N$ such that

$$\|f_n - f\|_{L^p(\Omega)} < \epsilon$$

for each $n > n_0$. If $x \in K$ and $n, k > n_0$ then the Mean value theorem (Theorem 2.6.1) and Hölder' s inequality (Proposition 1.14.1) yield

$$|f_n(x) - f_k(x)| = \left| \oint_{\partial B(x;r)} (f_n - f_k)\, d\mathcal{H}_{m-1} \right| \leq \frac{\epsilon \|\chi_{B(x,r)}\|_{L^q(R^m)}}{\mathcal{H}_{m-1}(\partial B(x;r))}\,.$$

Thus, f_n converges locally uniformly to some function g. Proposition 2.11.2 gives that $g \in H(\Omega)$. If $K \subset \Omega$ is compact, then $f_n \to g$ in $L^p(\mathcal{H}_m|K)$ and $f_n \to f$ in $L^p(\mathcal{H}_m|K)$, which forces $g(x) = f(x)$ for $\mathcal{H}_m$ almost every $x \in K$.

2.12 Growth of Derivatives at Infinity

In this section, we show how behavior of derivatives of a harmonic function u at infinity depends on behavior of u at infinity.

Proposition 2.12.1 *Let $K \subset R^m$ be compact, $u \in H(R^m \setminus K)$, $t \in R^1$. If $u(x) \in O(|x|^t)$ as $|x| \to \infty$ and α is a multiindex, then $\partial^\alpha u(x) \in O(|x|^{t-|\alpha|})$ as $|x| \to \infty$. On the other hand, if $k \in N$, $t > k-1$ and $\partial^\alpha u(x) \in O(|x|^{t-k})$ as $|x| \to \infty$ for each multiindex α with $|\alpha| = k$, then $u(x) \in O(|x|^t)$ as $|x| \to \infty$.*

Proof Suppose first that $|\alpha| = 1$. Fix positive constants r and C such that $|u(x)| \leq C|x|^t$ for$|x| > r$. Fix $x \in R^m \setminus B(0; 2r)$. Put $G = \{y \in R^m; |x|/2 < |y| < 2|x|\}$. Then, $|u(y)| \leq 2^{|t|}C|x|^t$ for $y \in G$. Since $\text{dist}(x; \partial G) = |x|/2$, we obtain by Theorem 2.9.2 that $|\partial_j u(x)| \leq m2^{|t|}C|x|^t/(|x|/2)$.

Suppose that $t > 0$ and $u(x) = O(|x|^{t-1})$ as $|x| \to \infty$. Choose $R > 0$ such that $K \subset B(0; R)$. Let $\varphi \in \mathcal{C}^\infty(R^m)$ be such that $\varphi = 0$ in $B(0; R)$, $\varphi = 1$ on $R^m \setminus B(0; 2R)$. Put $v = u\varphi$. Then, there is a constant C such that $|\nabla v(x)| \leq C|x|^{t-1}$.

Remark $v(0) = 0$. If $|x| \geq 2R$, then

$$|u(x)| = |v(x)| = \left| \int_0^{|x|} \frac{x}{|x|} \cdot \nabla v(sx/|x|)\, \mathrm{d}s \right| \leq \int_0^{|x|} Cs^{t-1}\, \mathrm{d}s = \frac{C|x|^t}{t}.$$

Now, we get the assertion of the proposition by the induction argument.

2.13 Separation of Harmonic Singularities

Proposition 2.13.1 *Let $\Omega \subset R^m$ be a domain, and let F^+, F^- be disjoint closed sets with $F^+ \cup F^- = R^m \setminus \Omega$ and F^+ be bounded, $z \in F^+$. Let $u \in H(\Omega)$. Then, there exist unique $u_1 \in H(\Omega \cup F^+)$, $u_2 \in H(\Omega \cup F^-)$ and a constant c such that $u(x) = u_1(x) + u_2(x) + ch_m(x - z)$ in Ω and $u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$.*

Proof Fix a bounded open set $G \subset \Omega$ with boundary of class C^∞ such that $\overline{G} \subset \Omega$ and $R^m \setminus \overline{G}$ has two components V^+, V^- with $F^+ \subset V^+$, $F^- \subset V^-$. Define

$$w_2(x) = -\mathcal{S}^{V^+}(n^{V^+} \cdot \nabla u) - \mathcal{D}^{V^+} u,$$

$$w_1(x) = -\mathcal{S}^{V^-}(n^{V^-} \cdot \nabla u) - \mathcal{D}^{V^-} u.$$

Then $w_2 \in H(R^m \setminus \overline{V^+})$, $w_1 \in H(R^m \setminus \overline{V^-})$. Corollary 2.2.5 gives that $u = w_1 + w_2$ in G. If we put $u_1 = w_1$ in $R^m \setminus \overline{V^-}$, $u_1 = u - w_2$ in $\Omega \setminus \overline{V^+}$ then u_1 is well defined and harmonic in $\Omega \cup F^+$. Similarly, $v = w_2$ in $R^m \setminus \overline{V^+}$, $v = u - w_1$ in $\Omega \setminus \overline{V^-}$ is well defined and harmonic in $\Omega \cup F^-$. Clearly, $-\mathcal{D}^{V^+} u(x) = O(|x|^{1-m})$ as $|x| \to \infty$. Put

$$c = -\int_{\partial V^+} n^{V^+}(y) \cdot \nabla u(y)\, \mathrm{d}\mathcal{H}_{m-1}(y),$$

$u_2(x) = v(x) - ch_m(x - z)$. With the aid of Lemma 2.9.7

$$u_2(x) = -\mathcal{D}^{V^+} u(x) + \int_{\partial V^+} n^{V^+}(y) \cdot \nabla u(y)[h_m(x - z)$$

$$-h_m(x - y)]\, \mathrm{d}\mathcal{H}_{m-1}(y) = O(|x|^{1-m}) \qquad |x| \to \infty.$$

Let now $u(x) = u_1(x) + u_2(x) + ch_m(x - z) = \tilde{u}_1(x) + \tilde{u}_2(x) + \tilde{c}h_m(x - z)$ in Ω, where $u_1, \tilde{u}_1 \in H(\Omega \cup F^+)$, $u_2, \tilde{u}_2 \in H(\Omega \cup F^-)$ and $u_2(x) = O(|x|^{1-m})$, $\tilde{u}_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$. Then $u_1(x) - \tilde{u}_1(x) = \tilde{u}_2(x) - u_2(x) + (\tilde{c} - c)h_m(x - z)$ in Ω. Put $v(x) = u_1(x) - \tilde{u}_1(x)$ in $\Omega \cup F^+$, $v(x) = \tilde{u}_2(x) - u_2(x) + (\tilde{c} - c)h_m(x - z)$

in $\Omega \cup F^-$. Then, v is well defined and harmonic in R^m. If $j \in \{1, \ldots, m\}$ then $\partial_j v(x) = o(1)$ as $|x| \to \infty$ by Proposition 2.12.1 and Corollary 2.9.6. Maximum modulus principle (Corollary 2.4.4) gives that $\partial_j v \equiv 0$. Therefore, v is constant (compare Lemma 1.20.1). Since $v(x) = \tilde{u}_2(x) - u_2(x) + (\tilde{c} - c)h_m(x - z)$ in $\Omega \cup F^-$, $\tilde{u}_2(x) - u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$, we infer that $v \equiv 0$ and $c = \tilde{c}$, $u_2 = \tilde{u}_2$. Since $v = u_1 - \tilde{u}_1$ in $\Omega \cup F^+$, we obtain $u_1 = \tilde{u}_1$.

2.14 Spherical Harmonics

We want to express harmonic functions locally as a sum of harmonic polynomials. For this reason, define spherical harmonics.

Let $P_k(R^m)$ denote the space of all homogeneous polynomials on R^m of degree k. A spherical harmonic of degree k is the restriction to $\partial B(0; 1)$ of an element of $H(R^m) \cap P_k(R^m)$. Remark that the collection of all spherical harmonics of degree k is a finite-dimensional and therefore closed subspace of $L^2(\partial B(0; 1))$.

Proposition 2.14.1 *If $f \in H(R^m) \cap P_k(R^m)$, $g \in H(R^m) \cap P_l(R^m)$, $l \neq k$, then f and g are orthogonal in $L^2(\partial B(0; 1))$.*

Proof Fix $x \in \partial B(0; 1)$. Put $F(r) = f(rx)$. Since $F(r) = r^k f(x)$, we have $F'(1) = kf(x)$. If n is the outward unit normal of $B(0; 1)$ at x, then

$$\frac{\partial f(x)}{\partial n} = F'(1) = kf(x).$$

Similarly,

$$\frac{\partial g(x)}{\partial n} = lg(x).$$

Using the second Green formula (Theorem 2.2.1), we obtain

$$0 = \int\limits_{B(0;1)} (f \Delta g - g \Delta f)\, d\mathcal{H}_m$$

$$= \int\limits_{\partial B(0;1)} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) d\mathcal{H}_{m-1} = (l - k) \int\limits_{\partial B(0;1)} fg\, d\mathcal{H}_{m-1}.$$

This gives that f and g are orthogonal in $L^2(\partial B(0; 1))$.

Lemma 2.14.2 *Let $p \in P_k(R^m)$, where $k \geq 2$. Then, there exist $h \in H(R^m) \cap P_k(R^m)$ and $q \in P_{k-2}(R^m)$ such that $p(x) = h(x) + |x|^2 q(x)$.*

Proof If

$$u(x) = \sum_{|\alpha|=l} a_\alpha x^\alpha, \qquad v(x) = \sum_{|\alpha|=l} b_\alpha x^\alpha$$

are two elements of $P_l(R^m)$, denote

$$(u, v)_l \equiv u(D)v = \sum_{|\alpha|=l} a_\alpha \partial^\alpha v(0).$$

Easy calculation yields that

$$(u, v)_l = \sum_{|\alpha|=l} \alpha! a_\alpha b_\alpha,$$

so that $(\, , \,)_l$ is an inner product on $P_l(R^m)$. Since the space $P_l(R^m)$ has finite dimension, it is a Hilbert space with this inner product.

Denote $P = \{|x|^2 q(x); q \in P_{k-2}(R^m)\}$. Theorem on orthogonal projection (Theorem 1.9.6) gives that there is $q \in P_{k-2}(R^m)$ and $h \in P^\perp$ with respect to $(\, , \,)_k$ in $P_k(R^m)$ such that $p(x) = h(x) + |x|^2 q(x)$. Since $|x|^2 \Delta h(x) \in P$, we have

$$0 = (|x|^2 \Delta h, h)_k = \Delta h(D)(\Delta h) = (\Delta h, \Delta h)_{k-2}.$$

This forces $\Delta h \equiv 0$.

Corollary 2.14.3 *If p is a polynomial on R^m of degree k, then the restriction of p to* $\partial B(0; 1)$ *is a sum of spherical harmonics of degree at most k.*

Proof If $k = 0$ or $k = 1$, then p is a harmonic, and the proposition of the corollary is true for $k = 0, 1$.

Suppose now that p is a polynomial on R^m of degree $n > 1$ and the proposition of the corollary is true for all $k < n$. We can suppose that $p \in P_n(R^m)$. According to Lemma 2.14.2, there exist $h \in H(R^m) \cap P_n(R^m)$ and $q \in P_{n-2}(R^m)$ such that $p(x) = h(x) + q(x)$ on $\partial B(0; 1)$. Since the restriction of q to $\partial B(0; 1)$ is a sum of spherical harmonics of degree at most $n - 2$, we obtain the proposition of the corollary.

Corollary 2.14.4 *Let $k \in N$, $k \geq 2$. Then the expression of $p \in P_k(R^m)$ in Lemma 2.14.2 is unique. So,* $\dim P_k(R^m) = \dim P_{k-2}(R^m) + \dim[P_k(R^m) \cap H(R^m)]$.

Proof Let $p_k \in [P_k(R^m) \cap H(R^m)]$, $q \in P_{k-2}(R^m)$ be such that $p_k(x) + |x|^2 q(x) \equiv 0$. According to Corollary 2.14.3, there exist $p_j \in H(R^m) \cap P_j(R^m)$ such that

$$q = \sum_{j=0}^{k-1} p_j \quad \text{on } \partial B(0; 1).$$

Since $p_k(x) + |x|^2 q(x) = 0$, we have

$$\sum_{j=0}^{k} p_j = 0 \quad \text{on } \partial B(0;1).$$

If $l \in \{0, \dots, k\}$, then

$$0 = \int\limits_{\partial B(0;1)} p_l \sum_{j=0}^{k} p_j \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial B(0;1)} p_l^2 \, \mathrm{d}\mathcal{H}_{m-1}$$

by Proposition 2.14.1. Since $p_l = 0$ on $\partial B(0;1)$, we have $q = 0$ on $\partial B(0;1)$. Since p_k and q are homogeneous polynomials, we deduce that $p_k \equiv 0$, $q \equiv 0$.

Theorem 2.14.5 *Let $f \in L^2(\partial B(0;1))$. Then there exist unique $p_k \in H(R^m) \cap P_k(R^m)$ for $k = 0, 1, 2, \dots$ such that*

$$f = \sum_{k=0}^{\infty} p_k \tag{2.21}$$

in $L^2(\partial B(0;1))$.

Proof Let $\{\varphi_{kj}; j = 1, \dots, l(k)\}$ be an orthonormal basis of $H(R^m) \cap P_k(R^m)$ as a subspace of $L^2(\partial B(0;1))$. Then $\{\varphi_{kj}; k \in N_0; j = 1, \dots, l(k)\}$ is an orthonormal sequence of $L^2(\partial B(0;1))$ by Proposition 2.14.1. Corollary 2.14.3 gives that the restriction of any polynomial onto $\partial B(0;1)$ is a linear combination of elements from this sequence. The set of all polynomials is dense in $\mathcal{C}^0(\partial B(0;1))$ by the Stone-Weierstrass theorem (Theorem 1.16.2). $\mathcal{C}^0(\partial B(0;1))$ is dense in $L^2(\partial B(0;1))$ by Proposition 1.14.6. Since the space of polynomials is dense in $L^2(\partial B(0;1))$, Theorem 1.9.9 gives that each $f \in L^2(\partial B(0;1))$ can be represented in the form (2.21) with

$$p_k = \sum_{j=1}^{l(k)} (f, \varphi_{kj}) \varphi_{kj}. \tag{2.22}$$

On the other hand, if f is represented in the form (2.21) with $p_k \in H(R^m) \cap P_k(R^m)$, then

$$(f, \varphi_{kj}) = \left(\sum_{l=0}^{\infty} p_l, \varphi_{kj} \right) = (p_k, \varphi_{kj})$$

by Proposition 2.14.1. Since $\{\varphi_{kj}; j = 1, \dots, l(k)\}$ is an orthonormal basis of $H(R^m) \cap P_k(R^m)$ and $p_k \in H(R^m) \cap P_k(R^m)$, we infer that p_k is given by (2.22).

2.15 Analyticity of Harmonic Functions

In this section, we express harmonic functions locally as a sum of harmonic polynomials.

Theorem 2.15.1 *Let* $\Omega \subset R^m$*,* $m \geq 2$*, be a domain,* $u \in H(\Omega)$*,* $z \in \Omega$*,* $0 < r <$ $\operatorname{dist}(z; \partial\Omega)$. *Then there exist* $p_k \in H(R^m) \cap P_k(R^m)$*,* $k = 0, 1, \dots$ *such that*

$$u(x) = \sum_{k=0}^{\infty} p_k(x - z) \tag{2.23}$$

in $B(z, r)$ *and the series converges uniformly. The polynomials* p_k *are unique and*

$$p_k(y) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(z)}{\alpha!} y^\alpha. \tag{2.24}$$

Proof We can suppose that $z = 0$. Suppose first that $r < 1 < \operatorname{dist}(z; \partial\Omega)$. According to Theorem 2.14.5, there are

$$p_k = \sum_{|\beta|=k} c_\beta x^\beta \in H(R^m) \cap P_k(R^m)$$

for $k = 0, 1, 2, \dots$ such that

$$u = \sum_{k=0}^{\infty} p_k$$

in $L^2(\partial B(0; 1))$. Put

$$q_n = \sum_{k=0}^{n} p_k. \tag{2.25}$$

Then $q_n \to u$ in $L^2(\partial B(0; 1))$. Denote by $P(x, y)$ the Poisson kernel of $B(0; 1)$. Since $u - q_n \in H(B(0; 1)) \cap \mathcal{C}^0(B(0; 1))$, we obtain for $x \in B(0; r)$ using theorem on the Poisson integral (Theorem 2.5.4) and Hölder inequality (Proposition 1.14.1)

$$|u(x) - q_n(x)| = \left| \int_{\partial B(0;1)} P(x, y)(u(y) - q_n(y)) \, d\mathcal{H}_{m-1}(y) \right|$$

$$\leq \int_{\partial B(0;1)} \frac{|u(y) - q_n(y)|}{\mathcal{H}_{m-1}(\partial B(0; 1))(1 - r)^m} \, d\mathcal{H}_{m-1}(y)$$

$$\leq \frac{\|u - q_n\|_{L^2(S)}}{\sqrt{\mathcal{H}_{m-1}(\partial B(0; 1))}(1 - r)^m} .$$

Therefore, q_n converges uniformly to u in $B(0; r)$.

Let now $0 < r < \text{dist}(0, \partial\Omega)$ be arbitrary. Fix $\rho \in (r, \text{dist}(0; \partial\Omega))$. If we put $v(x) = u((x)/\rho)$, then there are $\tilde{p}_k \in H(R^m) \cap P_k(R^m)$ such that

$$v(x) = \sum_{k=0}^{\infty} \tilde{p}_k(x)$$

in $B(0, r/\rho)$ and the series converges uniformly. Then

$$u(y) = v(\rho y) = \sum_{k=0}^{\infty} \tilde{p}_k(\rho y).$$

We have proved (2.23). Fix a multiindex α. Let q_n be given by (2.25). Then,

$$q_n(x) = \sum_{|\beta| \le n} a_\beta x^\beta$$

where a_β does not depend on n. Since $q_n \in H(B(z; r))$ converges uniformly to $u \in H(B(z; r))$, Proposition 2.11.1 gives

$$\partial^\alpha u(0) = \lim_{n \to \infty} \partial^\alpha q_n(0) = \alpha!\, a_\alpha\ .$$

So, (2.24) holds.

2.16 Liouville Theorem

In this section, we study harmonic functions on the whole Euclidean space. We show conditions that force that a harmonic function is a polynomial.

Theorem 2.16.1 *Let $u \in H(R^m)$. If u is bounded from below (or above), then u is constant. If $u(x) \to 0$ as $|x| \to \infty$ then $u \equiv 0$.*

Proof Suppose first that $u(x) \ge K$ for each $x \in R^m$. Put $v(x) = u(x) - K$. Then v is a nonnegative harmonic function. Fix $x \in R^m, j \in \{1, \dots, m\}$. If $r > 0$, then $v \in H(B(x; r))$ and Theorem 2.9.5 gives

$$|\partial_j u(x)| = |\partial_j v(x)| \le mv(x)/r.$$

Letting $r \to +\infty$, we get $\partial_j u(x) = 0$. Since the gradient of u vanishes in R^m, the function u is constant.

If u is bounded from above, then $-u$ is bounded from below and therefore constant.

Theorem 2.16.2 *If* $1 \leq p < \infty$, $u \in H(R^m)$, $u \in L^p(R^m)$, $m \geq 2$, *then* $u \equiv 0$.

Proof Fix $x \in R^m$. Put $q = p/(p-1)$, $\mu = \mathcal{H}_m(B(0;1))^{-1}\mathcal{H}_m|B(x;1)$. With the aid of Mean value theorem (Theorem 2.6.1) and Hölder's inequality (Proposition 1.14.1), we obtain

$$|u(x)| = \left|\int u \, d\mu\right| \leq \|u\|_{L^p(\mu)}\|1\|_{L^q(\mu)} \leq \mathcal{H}_m(B(0;1))^{-1/p}\|u\|_{L^p(R^m)}.$$

Since u is bounded, it is constant (see Theorem 2.16.1). Now $u \in L^p(R^m)$ forces $u \equiv 0$.

Theorem 2.16.3 *Let* $u \in H(R^m)$, $\beta \geq 0$. *If* $|u(x)| = O(|x|^\beta)$ *as* $|x| \to \infty$, *then u is a polynomial of degree* $k \leq \beta$.

Proof Let α be a multiindex with $|\alpha| > \beta$. Proposition 2.12.1 gives that $|\partial^\alpha u(x)| = O(|x|^{\beta-|\alpha|})$ as $|x| \to \infty$. Thus, $\partial^\alpha u \in H(R^m)$ (see Corollary 2.2.9) and $\partial^\alpha u(x) \to 0$ as $|x| \to \infty$. Therefore, $\partial^\alpha u \equiv 0$ by Theorem 2.16.1. Since $\partial^\alpha u(x) \equiv 0$ for each multiindex α with $|\alpha| > \beta$, analycity of harmonic functions (Theorem 2.15.1) gives that u is a polynomial of degree $k \leq \beta$.

Corollary 2.16.4 *Let* $u \in H(R^m)$, $m \geq 2$ *and* $k, n \in N_0$ *with* $k \leq n$. *Then the following assertions are equivalent:*

1. *u is a polynomial of degree less than or equal to n.*
2. $\partial^\alpha u(x) = O(|x|^{n-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k$.
3. $\partial^\alpha u(x) = o(|x|^{n+1-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k$.
4. $\partial^\alpha u(x) = o(|x|^{n-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k+1$.
5. *There exists* $1 \leq p < \infty$ *such that* $\partial^\alpha u \in L^p(R^m)$ *for each multiindex* α *with* $|\alpha| = n+1$.

Proof 1) $\Rightarrow$ 2) $\Rightarrow$ 3) is clear.

3) $\Rightarrow$ 4) Fix a multiindex α with $|\alpha| = k$. Theorem 2.16.3 gives that $\partial^\alpha u$ is a polynomial of degree less than or equal to $n+1-k$. The condition $\partial^\alpha u(x) = o(|x|^{n+1-k})$ forces that $\partial^\alpha u$ is a polynomial of degree lees than or equal to $n-k$.

4) $\Rightarrow$ 5) Fix a multiindex α with $|\alpha| = k+1$. Theorem 2.16.3 gives that $\partial^\alpha u$ is a polynomial of degree less than or equal to $n-k$. If $n = k$, then the condition $\partial^\alpha u(x) = o(1)$ forces that $\partial^\alpha u \equiv 0$. Let now $k < n$. Then from the condition $\partial^\alpha u(x) = o(|x|^{n-k})$ as $|x| \to \infty$, it follows that $\partial^\alpha u$ is a polynomial of degree less than or equal to $n-k-1$. Hence, $\partial^\alpha u(x) = O(|x|^{n-k-1})$ as $|x| \to \infty$. If β is a multiindex with $|\beta| = n+1$, then $\partial^\beta u(x) = O(|x|^{-1})$ as $|x| \to \infty$ (see Proposition 2.12.1). Clearly $\partial^\beta u \in L^{2m}(R^m)$.

5) $\Rightarrow$ 1): If α is a multiindex with $|\alpha| = n+1$ then $\partial^\alpha u = 0$ by Theorem 2.16.2. Proposition 2.12.1 forces that $u(x) = O(|x|^{n+1/2})$ as $|x| \to \infty$. Theorem 2.16.3 yields that u is a polynomial of degree less than or equal to n.

2.17 Behaviour at Infinity

This section gives several characterization concerning the grow of u and $\partial^\alpha u$ at infinity for a harmonic function u on an exterior domain.

Proposition 2.17.1 *Let $K \subset R^m$ be compact, $u \in H(R^m \setminus K)$, and $m \geq 2$. Then the following are equivalent:*

1) *There exists $c \in R^1$ such that $u(x) \to c$ as $|x| \to \infty$. Moreover, there exist constants c_1, c_2 such that $u(x) - c_1 - c_2|x|^{2-m} = O(|x|^{1-m})$ as $|x| \to \infty$.*
2) *$u(x) = O(1)$ as $|x| \to \infty$.*
3) *$u(x) = o(\ln|x|)$ as $|x| \to \infty$.*

If $m > 2$, then these assertions are equivalent to

4) *There is $r > 0$ such that u is bounded from below or from above on $R^m \setminus B(0;r)$.*
5) *$u(x) = o(|x|)$ as $|x| \to \infty$.*

Proof Clearly, 1) $\Rightarrow$ 2) $\Rightarrow$ 3) and 1) $\Rightarrow$ 4), 1 $\Rightarrow$ 5).

Put $F^+ = K \cup \{0\}$, $F^- = \emptyset$. According to Proposition 2.13.1, there exist unique $u_1 \in H(R^m)$, $u_2 \in H(R^m \setminus F^+)$ and a constant a such that $u(x) = u_1(x) + u_2(x) + ah_m(x)$ in $R^m \setminus F^+$ and $u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$.

3) $\Rightarrow$ 1): Since $u_1(x) = u(x) - u_2(x) - ah_m(x) = o(|x|)$, there exists a constant c such that $u_1 \equiv c$ (see Corollary 2.16.4). If $m > 2$, then $u(x) \to c$ as $|x| \to \infty$. Let now $m = 2$. Since $u(x) = o(\ln|x|)$ as $|x| \to \infty$, we infer that $a = 0$. Thus, $u(x) \to c$ as $|x| \to \infty$.

Suppose now that $m > 2$:

4) $\Rightarrow$ 1): Since $u_2(x) + ah_m(x)$ is bounded on $R^m \setminus B(0;r)$, the function u_1 is bounded from below or from above. Theorem 2.16.1 gives that there exists a constant c such that $u_1 \equiv c$. Thus, $u(x) \to c$ as $|x| \to \infty$.

5) $\Rightarrow$ 1): Since $u_1(x) = u(x) - u_2(x) - ah_m(x) = o(|x|)$, there exists a constant c such that $u_1 \equiv c$ (see Corollary 2.16.4). Thus $u(x) \to c$ as $|x| \to \infty$.

Example 2.17.2 The function h_2 is a nonnegative harmonic function in $R^2 \setminus B(0;1)$, $h_2(x) = o(|x|)$ as $|x| \to \infty$. But there is not $c \in R$ such that $h_m(x) \to c$ as $|x| \to \infty$.

Proposition 2.17.3 *Let $K \subset R^m$ be compact, $u \in H(R^m \setminus K)$, u be bounded. Let α be a multiindex with $|\alpha| > 0$. If $m > 2$, then $\partial^\alpha u(x) = O(|x|^{2-m-|\alpha|})$ as $|x| \to \infty$. If $m = 2$, then $\partial^\alpha u(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$. If $m > 2$ and $u(x) \to 0$ as $|x| \to \infty$, then $u(x) = O(|x|^{2-m})$ as $|x| \to \infty$. If $m = 2$ and $u(x) \to 0$ as $|x| \to \infty$, then $u(x) = O(|x|^{-1})$ as $|x| \to \infty$.*

Proof According to Proposition 2.17.1, there exists a constant c such that $u(x) \to c$ as $|x| \to 0$. Since $\partial^\alpha u = \partial^\alpha(u - c)$, we can suppose that $u(x) \to 0$ as $|x| \to \infty$.

If $m > 2$, then $u(x) = O(|x|^{2-m})$ as $|x| \to \infty$ by Proposition 2.17.1. Hence, $\partial^\alpha u(x) = O(|x|^{2-m-|\alpha|})$ as $|x| \to \infty$ by Proposition 2.12.1.

If $m = 2$ then $u(x) = O(|x|^{-1})$ as $|x| \to \infty$ by Proposition 2.17.1. Proposition 2.12.1 forces $\partial^\alpha u(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$.

Proposition 2.17.4 *Let* $K \subset R^m$ *be compact,* $u \in H(R^m \setminus K)$*. Then the following are equivalent:*

1. $u(x) \to 0$ *as* $|x| \to \infty$.
2. *There exist* $r >$ *and* $p \in \langle 1, \infty)$ *such that* $u \in L^p(R^m \setminus B(0; r))$.

Proof 1) $\Rightarrow$ 2): Proposition 2.17.3 gives that $u(x) = O(|x|^{-1})$ for $m = 2$ and $u(x) = O(|x|^{2-m})$ for $m > 2$. Thus $u \in L^p(R^m \setminus B(0; r))$ for sufficiently large r and arbitrary $3 < p < \infty$.

2) $\Rightarrow$ 1): Fix $x \in R^m$ such that $|x| > r + 1$. Put $q = p/(p-1)$, $\mu = \mathcal{H}_m(B(0; 1))^{-1}\mathcal{H}_m|B(x; 1)$. With the aid of Mean value theorem (Theorem 2.6.1) and Hölder's inequality (Proposition 1.14.1) we obtain

$$|u(x)| = \left|\int u \, d\mu\right| \le \|u\|_{L^p(\mu)}\|1\|_{L^q(\mu)} \le \mathcal{H}_m(B(0; 1))^{-1/p}\|u\|_{L^p(R^m \setminus B(0;r))}.$$

Since $u(x) = O(1)$ as $|x| \to 1$, Proposition 2.17.1 gives that $u(x) \to c$ as $|x| \to \infty$ for some constant c. Now $u \in L^p(R^m \setminus B(0; r))$ forces $c = 0$.

Proposition 2.17.5 *Let* $K \subset R^m$ *be compact,* $u \in H(R^m \setminus K)$*. Let* $k, n \in N_0$, $k \le n$*. If* $m = 2$ *suppose* $n > 0$*. Then the following are equivalent:*

1. *There exist a harmonic polynomial of* P *degree less than or equal to* n *and a constant* c *such that* $u(x) - P(x) - ch_m(x) = O(|x|^{1-m})$ *as* $|x| \to \infty$.
2. $\partial^\alpha u(x) = O(|x|^{n-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k$.
3. $\partial^\alpha u(x) = o(|x|^{n+1-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k$.
4. $\partial^\alpha u(x) = o(|x|^{n-k})$ *as* $|x| \to \infty$ *for each multiindex* α *with* $|\alpha| = k + 1$.
5. *There exist* $p \in \langle 1, \infty)$ *and* $r > 0$ *such that* $\partial^\alpha u \in L^p(R^m \setminus B(0; r))$ *for each multiindex* α *with* $|\alpha| = n + 1$.
6. *There exists* $r > 0$ *such that* $\partial^\alpha u \in L^p(R^m \setminus B(0; r))$ *for each multiindex* α *with* $|\alpha| = n + 1$ *and* $2 < p < \infty$.

Proof Put $F^+ = K \cup \{0\}$, $F^- = \emptyset$. According to Proposition 2.13.1, there exist unique $P \in H(R^m)$, $u_2 \in H(R^m \setminus F^+)$ and a constant a such that $u(x) = P(x) + u_2(x) + ah_m(x)$ in $R^m \setminus F^+$ and $u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$. If α is a multiindex with $|\alpha| \ge 1$ then Proposition 2.12.1 and Corollary 2.9.6 give $\partial^\alpha[u_2(x) + ah_m(x)] = O(|x|^{2-m-|\alpha|})$ as $|x| \to \infty$. If $|\alpha| = k$, then $\partial^\alpha[u_2(x) + ah_m(x)] = O(|x|^{n-k})$ as $|x| \to \infty$. If $|\alpha| = k + 1$, then $\partial^\alpha[u_2(x) + ah_m(x)] = o(|x|^{n-k})$ as $|x| \to \infty$. If $|\alpha| = n + 1$, then $\partial^\alpha[u_2(x) + ah_m(x)] = O(|x|^{1-m})$ as $|x| \to \infty$ and thus $\partial^\alpha[u_2(x) + ah_m(x)] \in L^p(R^m \setminus B(0; r))$ for sufficiently large r and each $p > 2$. If we use Corollary 2.16.4 for P, we obtain 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4).

Suppose 5). Let α be a multiindex with $|\alpha| = n + 1$. Then $\partial^\alpha u(x) \to 0$ as $|x| \to \infty$ by Proposition 2.17.4. Proposition 2.17.3 gives that $\partial^\alpha u(x) = O(|x|^{-1})$ as $|x| \to \infty$. Thus, $\partial^\alpha u \in L^{m+1}(R^m \setminus B(0; \rho)$ for some $\rho \ge r$. Since $\partial^\alpha[u_2(x) + ah_m(x)] \in L^{m+1}(R^m \setminus B(0; \rho))$, we deduce that $\partial^\alpha P = \partial^\alpha u - \partial^\alpha[u_2(x) + ah_m(x)] \in L^{m+1}(R^m \setminus B(0; \rho))$. Corollary 2.16.4 gives that P is a polynomial of degree at most n.

1) $\Rightarrow$ 6): Let α be a multiindex with $|\alpha| = n+1$. Then $\partial^\alpha[u(x) - P(x) - ch_m(x)] = O(|x|^{-m-n})$ as $|x| \to \infty$ by Proposition 2.12.1. Corollary 2.9.6 gives $\partial^\alpha ch_m(x) = O(|x|^{1-m-n})$ as $|x| \to \infty$. Since $\partial^\alpha P \equiv 0$, we deduce that $\partial^\alpha u(x) = O(|x|^{1-m-n})$ as $|x| \to \infty$. Hence, there exists $r > 0$ such that $\partial^\alpha u \in L^p(R^m \setminus B(0; r))$ for each $p > 2$.

Proposition 2.17.6 *Let $K \subset R^2$ be compact, $u \in H(R^2 \setminus K)$, $z \in K$. Then the following assertions are equivalent:*

1. *$u(x) = o(|x|)$ as $|x| \to \infty$.*
2. *There exist constants c_1, c_2 such that $u(x) - c_1 h_2(x - z) - c_2 = O(|x|^{-1})$ as $|x| \to \infty$.*
3. *$|\nabla u(x)| = O(|x|^{-1})$ as $|x| \to \infty$.*
4. *$|\nabla u(x)| \to 0$ as $|x| \to \infty$.*
5. *There exist $r > 0$ and $p \in \langle 1, \infty)$ such that $|\nabla u| \in L^p(R^2 \setminus B(z; r))$.*
6. *There exists $r > 0$ such that $|\nabla u| \in L^p(R^2 \setminus B(z; r))$ for all $p \in (2, \infty)$.*

Proof We can change a coordinate system in a such way that z is the origin (see Lemma 2.1.2).

1) $\Rightarrow$ 2): According to Proposition 2.17.5, there exists a polynomial P of degree less than or equal to 1 and a constant c_1 such that $u(x) - c_1 h_2(x) - P(x) = O(|x|^{-1})$ as $|x| \to \infty$. Since $P(x) = u(x) - [u(x) - c_1 h_2(x) - P(x)] - c_1 h_2(x) = o(|x|)$, we infer that P is constant.

2) $\Rightarrow$ 3): Since $u(x) - c_1 h_2(x - z) - c_2 = O(|x|^{-1})$ as $|x| \to \infty$, Proposition 2.12.1 gives that $|\nabla[u(x) - c_1 h_2(x - z) - c_2]| = O(|x|^{-2})$ as $|x| \to \infty$. Since $|\nabla[c_1 h_2(x - z) + c_2]| = O(|x|^{-1})$ as $|x| \to \infty$, we obtain the assertion.

3) $\Rightarrow$ 1): Since $|\nabla u(x)| = O(|x|^{-1/2})$ as $|x| \to \infty$, Proposition 2.12.1 gives that $u(x) = O(|x|^{1/2})$ as $|x| \to \infty$. Hence, $u(x) = o(|x|)$ as $|x| \to \infty$.

3) $\Leftrightarrow$ 4) is a consequence of Proposition 2.17.3.

4) $\Leftrightarrow$ 5) is a consequence of Corollary 2.2.9 and Proposition 2.17.4.

3) $\Rightarrow$ 6) is clear.

Proposition 2.17.7 *Let $\Omega \subset R^m$ be open, $R^m \setminus \Omega$ be compact and u be harmonic on a neighborhood of $\overline{\Omega}$. Suppose that $1 < p < \infty$ and α is a multiindex with $|\alpha| > 0$.*

- *If $u \in L^p(\Omega)$, then $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (m/(m-1), \infty)$.*
- *If $u \in L^p(\Omega)$ and $m = 2$, then $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (1, \infty)$.*
- *If $|\nabla u| \in L^p(\Omega)$ and $p \leq m/(m-1)$, then $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (1, \infty)$.*

Proof Suppose first that $u \in L^p(\Omega)$. Then $u(x) \to 0$ as $|x| \to \infty$ by Proposition 2.17.4. Since $\partial^\alpha u(x) = O(|x|^{1-m})$ as $|x| \to \infty$ by Proposition 2.17.3, we deduce $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (m/(m-1), \infty)$. If $m = 2$ then $\partial^\alpha u(x) = O(|x|^{-2})$ as $|x| \to \infty$ by Proposition 2.17.3 and thus $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (1, \infty)$.

Let now $|\nabla u| \in L^p(\Omega)$ and $p \leq m/(m-1)$. According to Proposition 2.17.5 and Proposition 2.17.6, there exist constants c_1, c_2 such that for $v(x) = u(x) - c_1 - c_2 h_m(x)$ we have $v(x) = O(|x|^{1-m})$ as $|x| \to \infty$. Proposition 2.12.1 gives that

$|\nabla v(x)| = O(|x|^{-m})$ as $|x| \to \infty$. Thus, $|\nabla v| \in L^p(\Omega)$. Therefore $|\nabla c_2 h_m(x)| = |\nabla u - \nabla v| \in L^p(\Omega)$. Since $|\nabla h_m(x)| = c_3|x|^{1-m}$ for some positive constant C_3 and $p \leq m/(m-1)$, we deduce that $c_2 = 0$. According to Proposition 2.12.1, we have $\partial^\alpha u(x) = \partial^\alpha v(x) = O(|x|^{-m})$. Therefore, $\partial^\alpha u \in L^q(\Omega)$ for all $q \in (1, \infty)$.

2.18 Laplace Equation in Distributional Sense

Now we prove that every solution of the Laplace equation in distributional sense is a solution of the Laplace equation in the classical sense. We also give another characterization that we use solving boundary value problems in Sobolev spaces. As a consequence, we show a theorem of Liouville type: if u is a tempered distribution that is a solution of the Laplace equation in the whole space, then u is a polynomial.

Definition 2.18.1 We say that a function u defined $\mathcal{H}_m$-a.e. in an open set $\Omega \subset R^m$ is a solution of the equation $\Delta u = 0$ in Ω in the sense of distributions if $u \in L^1_{loc}(\Omega)$ and

$$\int_\Omega u\Delta\varphi \, d\mathcal{H}_m = 0$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Theorem 2.18.2 *Let Ω be an open subset of R^m, u be a distribution in Ω, $1 \leq p \leq \infty$. Then the following assertions are equivalent:*

1. *u is a solution of the equation $\Delta u = 0$ in Ω in the sense of distributions.*
2. *u is representable by a function harmonic in Ω.*
3. *$u \in W^{1,p}_{loc}(\Omega)$ and*

$$\int_\Omega \nabla u \cdot \nabla\varphi \, d\mathcal{H}_m = 0$$

for each $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Proof 1) $\Rightarrow$ 2): Fix $x \in \Omega$ and $r(x) \in (0, \operatorname{dist}(x, \partial\Omega)/4)$. Fix $v \in \mathcal{C}_c^\infty(\Omega)$ such that $v = 1$ on $B(x; 4r(x))$. Put $\tilde{u} = vu$, i.e., $\langle \tilde{u}, \varphi\rangle = \langle u, v\varphi\rangle$ for $\varphi \in \mathcal{C}_c^\infty(R^m)$. Since $\tilde{u} = u$ in $B(x; 4r(x))$, we obtain that $\Delta\tilde{u} = 0$ in $B(x; r(x))$. Fix a nonnegative $\Phi \in \mathcal{C}^\infty(R^1)$ supported in $(0; 1)$ such that

$$\int_{R^m} \Phi(|y|) \, d\mathcal{H}_m(y) = 1.$$

For $0 < \epsilon < r(x)$ put $\psi_\epsilon(y) = \Phi(|y|/\epsilon)\epsilon^{-m}$ and $\tilde{u}_\epsilon = \tilde{u} * \psi_\epsilon$. Then $\tilde{u}_\epsilon \in \mathcal{C}^\infty(R^m)$. If $\varphi \in \mathcal{C}_c^\infty(B(x; 3r(x))$, then $\operatorname{spt}(\psi_\epsilon * \varphi) \subset B(x; 4r(x))$ and

$$\int_{R^m} \varphi \Delta \tilde{u}_\epsilon \, \mathrm{d}\mathcal{H}_m = \langle \Delta(\psi_\epsilon * \tilde{u}), \varphi \rangle = \langle \psi_\epsilon * (\Delta \tilde{u}), \varphi \rangle = \langle \Delta \tilde{u}, \psi_\epsilon * \varphi \rangle = 0.$$

Thus, $\Delta \tilde{u}_\epsilon = 0$ in $B(x; 3r(x))$. Fix $0 < \delta < r(x)$. If $z \in B(x; 2r(x))$ then Mean value theorem (Theorem 2.6.1) gives

$$\tilde{u}_\epsilon * \psi_\delta(z) = \int_{B(0;\delta)} \psi_\delta(y)\tilde{u}_\epsilon(z-y) = \int_0^\delta \psi_\delta(t) \int_{\partial B(z;t)} \tilde{u}_\epsilon(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \, \mathrm{d}t$$

$$= \tilde{u}_\epsilon(z) \int_0^\delta \psi_\delta(t) \int_{\partial B(z;t)} 1 \, \mathrm{d}\mathcal{H}_{m-1}(y) \, \mathrm{d}t = \tilde{u}_\epsilon(z).$$

Thus $\tilde{u}_\epsilon = \tilde{u} * \psi_\epsilon * \psi_\delta = \tilde{u} * \psi_\delta * \psi_\epsilon = \tilde{u}_\delta$ on $B(x; 3r(x))$. If $\varphi \in \mathcal{C}_c^\infty(B(x; r(x))$ then $\partial^\alpha(\varphi * \psi_\delta) = (\partial^\alpha \varphi) * \psi_\delta \to \partial^\alpha \varphi$ uniformly in R^m for each multiindex α. For a fixed $\epsilon \in (0, r(x))$

$$\langle u, \varphi \rangle = \langle \tilde{u}, \varphi \rangle = \lim_{\delta \searrow 0} \langle \tilde{u}, \varphi * \psi_\delta \rangle = \lim_{\delta \searrow 0} \langle \tilde{u}_\delta, \varphi \rangle = \langle \tilde{u}_\epsilon, \varphi \rangle.$$

Therefore, $u = \tilde{u}_\epsilon$ on $B(x; r)$.

We have proved that there is $v_x \in H(B(x; r(x))$ such that $u = v_x$ on $B(x; r(x))$. If $z \in \Omega$ is such that $B(x; r(x)) \cap B(z; r(z)) \neq \emptyset$, then $v_x(y) = v_z(y)$ for almost all $y \in B(x; r(x)) \cap B(z; r(z))$. The continuity gives that $v_x = v_z$ in $B(x; r(x)) \cap B(z; r(z))$. If we put $v(y) = v_x(y)$ for $y \in B(x; r(x))$, then $v \in H(\Omega)$ and $u = v$ in the sense of distributions.

2) ⇒ 3): Since $u \in \mathcal{C}^2(\Omega)$, we have $u \in W^{1,p}_{loc}(\Omega)$. Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Let G be a bounded open subset of Ω with boundary of class C^1 such that $\operatorname{spt} \varphi \subset G$. Since $\Delta u = 0$ in Ω and $\varphi = 0$ on ∂G, the Gauss-Green theorem (Theorem 1.19.1) gives

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \int_G \left[\sum_{j=1}^m \partial_j(\varphi \partial_j u) - \varphi \Delta u \right] = \int_{\partial G} \sum_{j=1}^m n_j(\varphi \partial_j u) \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

3) ⇒ 1): If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then

$$\langle \Delta u, \varphi \rangle = \sum_{j=1}^m \langle u, \partial_j \partial_j \varphi \rangle = -\sum_{j=1}^m \langle \partial_j u, \partial_j \varphi \rangle = -\int_\Omega \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = 0.$$

Corollary 2.18.3 *Let $\Omega \subset R^m$, $m \geq 2$, be an open set, $u \in \mathcal{C}^0(\Omega)$. Then, $\Delta u = 0$ in Ω in the sense of distributions if and only if $u \in H(\Omega)$.*

Proof Suppose that $\Delta u = 0$ in Ω in the sense of distributions. According to Theorem 2.18.2, there exists $v \in H(\Omega)$ such that $u = v$ almost everywhere in Ω. The continuity of u and v gives $u \equiv v$.

Proposition 2.18.4 *Let u be a tempered distribution in R^m and $m \geq 2$. If $k \in N$ and $\Delta^k u = 0$ in R^m in the sense of distributions, then u is a polynomial.*

Proof Denote by $\mathcal{F}w$ the Fourier transformation of w. Then, $0 = |\mathcal{F}(\Delta^k u)(x)| = |x|^{2k}|\mathcal{F}u(x)|$. (See Chap. 1, Sect. 1.20.) If $y \in R^m \setminus \{0\}$ choose $\psi_y \in \mathcal{C}_c^\infty(R^m)$ such that $\psi_y(x) = 1/|x|^{2k}$ on $B(y; |y|/2)$. If $\varphi \in \mathcal{C}_c^\infty(B(y; |y|/2)$, then

$$\langle \mathcal{F}u, \varphi\rangle = \langle |x|^{2k}\mathcal{F}u(x), \psi_y(x)\varphi(x)\rangle = 0.$$

Thus, $\operatorname{spt} \mathcal{F}u \subset \{0\}$. According to Lemma 1.20.4, there exist $n \in N_0$ and constants a_α such that

$$\mathcal{F}u = \sum_{|\alpha| \leq n} a_\alpha \partial^\alpha \delta_0.$$

Put

$$v(x) = \sum_{|\alpha| \leq n} a_\alpha (-ix)^\alpha.$$

Then

$$\mathcal{F}v = \sum_{|\alpha| \leq n} a_\alpha \mathcal{F}[(-ix)^\alpha 1] = \sum_{|\alpha| \leq n} a_\alpha \partial^\alpha \delta_0 = \mathcal{F}u.$$

(See Chap. 1, Sect. 1.20.) Since the Fourier transform is an isomorphism on $\mathcal{S}'(R^m, C)$, we infer that $u = v$.

Corollary 2.18.5 *If $1 < p < \infty$ then $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ is a dense subset of $L^p(R^m)$.*

Proof Denote by M the closure of $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ in $L^p(R^m)$. Suppose that $M \neq L^p(R^m)$. Put $q = p/(p-1)$. According to Proposition 1.5.3 and Theorem 1.14.3, there exists $f \in L^q(R^m)$ such that $\|f\|_{L^q(R^m)} > 0$ and

$$\int_{R^m} f\psi \, \mathrm{d}\mathcal{H}_m = 0 \qquad \forall \psi \in M.$$

Since $\int f\Delta\varphi \, \mathrm{d}\mathcal{H}_m = 0$ for all $\varphi \in \mathcal{C}_c^\infty(R^m)$, Theorem 2.18.2 gives that $f \in H(R^m)$. Since $f \in L^q(R^m)$, we have $f \equiv 0$ (see Theorem 2.16.2), what is a contradiction.

Corollary 2.18.6 *Let* $1 < p < \infty$, $p' = p/(p-1)$, $m \geq 2$. *Denote by* $\mathcal{E}_p(R^m)$ *the closure of* $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ *in* $D^{-1,p}(R^m)$. *If* $p' < m$, *i.e.,* $p > m/(m-1)$, *then* $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$. *If* $p' \geq m$, *i.e.,* $p \leq m/(m-1)$, *then* $\mathcal{E}_p(R^m) = \{\nu \in D^{-1,p}(R^m); \langle \nu, 1\rangle = 0\}$.

Proof Let $\mu \in D^{-1,p}(R^m) \setminus \mathcal{E}_p(R^m)$. The dual space of $D^{-1,p}(R^m)$ is $\overset{\circ}{D}{}^{1,p'}(R^m)$ by Lemma 1.25.3. According to Proposition 1.5.3, there exists a nontrivial $f \in \overset{\circ}{D}{}^{1,p'}(R^m)$ such that $\langle \mu, f\rangle \neq 0$, and

$$0 = \langle \Delta\psi, f\rangle = \int_{R^m} f \Delta\psi \, \mathrm{d}\mathcal{H}_m \qquad \forall \psi \in \mathcal{C}_c^\infty(R^m).$$

Theorem 2.18.2 gives that $f \in H(R^m)$. Since $\partial_j f \in L^{p'}(R^m) \cap H(R^m)$ by Corollary 2.2.9, Theorem 2.16.2 gives that $\partial_j f \equiv 0$. Therefore, f is constant. Lemma 1.5.4 gives that $\mathcal{E}_p(R^m) = \{\nu \in D^{-1,p}(R^m); \langle \nu, f\rangle = 0\}$.

If $p' < m$, then $1 \notin \overset{\circ}{D}{}^{1,p'}(R^m)$ by Proposition 1.25.8. Thus, $f \equiv 0$, what is a contradiction. Thus, $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$.

Suppose now that $m \leq p'$. Then $1 \in \overset{\circ}{D}{}^{1,p'}(R^m)$ by Proposition 1.25.8. If $\varphi \in \mathcal{C}_c^\infty(R^m)$ choose a bounded domain G with smooth boundary such that $\operatorname{spt}\varphi \subset G$. According to the first Green identity (Theorem 2.2.1)

$$\langle \Delta\varphi, 1\rangle = \int_G 1 \cdot \Delta\varphi \, \mathrm{d}\mathcal{H}_m = \int_{\partial G} \frac{\partial\varphi}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Thus $\langle \nu, 1\rangle = 0$ for each $\nu \in \mathcal{E}_p(R^m)$.

2.19 Reflection Principle

Denote $R^m_+ = \{[x_1, \ldots, x_m] \in R^m; x_m > 0\}$.

Theorem 2.19.1 *Let* $\Omega \subset R^m$ *be an open subset of* R^m *such that* $[x_1, \ldots, x_{m-1}, -x_m] \in \Omega$ *for each* $[x_1, \ldots, x_{m-1}, x_m] \in \Omega$. *Let* $u \in \mathcal{C}^0(\Omega) \cap H(\Omega \cap R^m_+)$, $u = 0$ *on* $\Omega \cap \partial R^m_+$, $u(x', -x_m) = -u(x', x_m)$ *with* $x' = [x_1, \ldots, x_{m-1}]$. *Then* $u \in H(\Omega)$.

Proof Easy calculation yields that $u \in H(\Omega \setminus \partial R^m_+)$. Mean value theorem (Theorem 2.6.1) gives that

$$u(x) = \fint_{B(x;r)} u(y) \, \mathrm{d}\mathcal{H}_m \tag{2.26}$$

holds true for each $x \in \Omega \setminus \partial R^m_+$ and $0 < r < \operatorname{dist}(x; \partial(\Omega \setminus \partial R^m_+))$. If $x \in \Omega \cap \partial R^m_+$, then (2.26) holds true for each $0 < r < \operatorname{dist}(x; \partial\Omega)$. Mean value theorem (Theorem 2.6.1) gives that $u \in H(\Omega)$.

Proposition 2.19.2 *Let $\Omega \subset R^m$ be an open subset of R^m such that $[x_1, \dots, x_{m-1}, -x_m] \in \Omega$ for each $[x_1, \dots, x_{m-1}, x_m] \in \Omega$. Put $\Omega_+ = \Omega \cap R^m_+$. Let $u \in \mathcal{C}^0(\Omega) \cap H(\Omega_+)$, $u(x_1, \dots, x_{m-1}, -x_m) = u(x_1, \dots, x_{m-1}, x_m)$. If*

$$\lim_{\Omega_+ \ni x \to y} \partial_m u(x) = 0$$

for each $y \in \Omega \cap \partial R^m_+$, then $u \in H(\Omega)$.

Proof Easy calculation yields that $u \in H(\Omega \setminus \partial R^m_+)$. Fix $\varphi \in \mathcal{C}^\infty_c(\Omega)$. Choose a bounded open set G with smooth boundary such that $\operatorname{spt} \varphi \subset G$, $\overline{G} \subset \Omega$. For $k \in N$ denote $\omega_k = G \cap \{[x_1, \dots, x_m]; x_m > 1/k\}$. Second Green formula (Theorem 2.2.1) gives

$$\int\limits_{\{x \in \Omega; x_m = 1/k\}} [\varphi \partial_m u - u \partial_m \varphi] \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\omega_k} \left[u \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial u}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{\omega_k} [u \Delta\varphi - \varphi \Delta u] \, \mathrm{d}\mathcal{H}_m = \int\limits_{\omega_k} u \Delta\varphi \, \mathrm{d}\mathcal{H}_m.$$

If we put $v = \partial_m u$ on Ω_+, $v = 0$ on $\Omega \cap \partial R^m_+$, then v is a continuous function on the compact set $\overline{R^m_+} \cap \operatorname{spt} \varphi$ and therefore bounded (see Lemma 1.2.13). By virtue of Lebesgue lemma (Lemma 1.13.2)

$$\int\limits_{\Omega_+} u \Delta\varphi \, \mathrm{d}\mathcal{H}_m = \lim_{k \to \infty} \int\limits_{\omega_k} u \Delta\varphi \, \mathrm{d}\mathcal{H}_m$$

$$= \lim_{k \to \infty} \int\limits_{\{x \in \Omega; x_m = 1/k\}} [\varphi v - u \partial_m \varphi] \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega \cap \partial R^m_+} [-u \partial_m \varphi] \, \mathrm{d}\mathcal{H}_{m-1}.$$

By the same way, we prove

$$\int\limits_{\Omega \setminus \Omega_+} u \Delta\varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega \cap \partial R^m_+} u \partial_m \varphi \, \mathrm{d}\mathcal{H}_{m-1}.$$

Adding

$$\int_{\Omega} u\Delta\varphi \, \mathrm{d}\mathcal{H}_m = 0.$$

Therefore, $-\Delta u = 0$ in Ω in the sense of distributions. According to Theorem 2.18.2, there exists $w \in H(\Omega)$ such that $u = w$ almost everywhere in Ω. Continuity of u forces $u \equiv w$.

2.20 Holomorphic Functions

In the last section of this chapter, we study the relations between holomorphic functions and harmonic functions.

Let $\Omega \subset R^2$ be an open set, f_1, f_2 be real functions defined on Ω. We shall identify $[x, y]$ and $z = x + iy$. (Here, i denotes the imaginary unit.) We shall say that $f(x + iy) = f_1(x, y) + if_2(x, y)$ is a holomorphic function in Ω if for each $z_0 \in \Omega$, there exists the complex derivative of f

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Proposition 2.20.1 *Let $\Omega \subset R^2$ be an open set, $f = f_1 + if_2$ be holomorphic in Ω. Then*

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \tag{2.27}$$

Proof

$$\frac{\partial f_1(x_0 + iy_0)}{\partial x} + i\frac{\partial f_2(x_0 + iy_0)}{\partial x} = \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0}$$

$$= f'(x_0 + iy_0) = \lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{iy - iy_0}$$

$$= \frac{1}{i}\frac{\partial f_1(x_0 + iy_0)}{\partial y} + \frac{\partial f_2(x_0 + iy_0)}{\partial y}.$$

This gives the existence of partial derivatives of f_1 and f_2. Comparing the real and imaginary parts of the left side and the right side of this equality, we get (2.27).

Proposition 2.20.2 *If $f = f_1 + if_2 \in \mathcal{C}^1(\Omega, C)$ is holomorphic in Ω, then $f_1, f_2 \in H(\Omega)$.*

Proof Fix a ball $B = B([x, y]; r)$ such that $\overline{B} \subset \Omega$. Put $\Phi(t) = \Phi_1(t) + i\Phi_2(t) = x + iy + re^{it}$ for $0 \leq t \leq 2\pi$. Then $\tau = [\Phi_1'/r, \Phi_2'/r] = [-n_2^B, n_1^B]$ is a tangent vector of B. By virtue of (2.27)

$$\int\limits_{\partial B} \frac{\partial f_1}{\partial n}\, \mathrm{d}\mathcal{H}_m = -\int\limits_{\partial B} \frac{\partial f_2}{\partial \tau}\, \mathrm{d}\mathcal{H}_m = \int\limits_0^{2\pi} (f_2(\Phi_1(t), \Phi_2(t))'\, \mathrm{d}t$$

$$= f_2(\Phi_1(2\pi), \Phi_2(2\pi)) - f_1(\Phi_1(0), \Phi_2(0)) = 0,$$

$$\int\limits_{\partial B} \frac{\partial f_2}{\partial n}\, \mathrm{d}\mathcal{H}_m = \int\limits_{\partial B} \frac{\partial f_1}{\partial \tau}\, \mathrm{d}\mathcal{H}_m = \int\limits_0^{2\pi} (f_1(\Phi_1(t), \Phi_2(t))'\, \mathrm{d}t$$

$$= f_1(\Phi_1(2\pi), \Phi_2(2\pi)) - f_1(\Phi_1(0), \Phi_2(0)) = 0.$$

Proposition 2.6.2 yields that $f_1, f_2 \in H(\Omega)$.

Chapter 3
Solutions of the Poisson Equation

Abstract Chapter 3 studies behaviour of solutions of the Poisson equation.

Chapter 3 studies behavior of solutions of the Poisson equation. This chapter is a small visit to the classical potential theory. But we include only these parts of this theory which will be useful in future. The important notions in the theory of partial differential equations are supersolutions and subsolutions of the equation. Superharmonic and subharmonic functions play an essential role in the potential theory. We show that superharmonic functions (subharmonic functions) are precisely supersolutions (subsolutions) of the Laplace equation, respectively. We shall study the fundamental properties of superharmonic and subharmonic functions. Green's potentials and volume potentials corresponding to nonnegative measures are important examples of superharmonic functions. On the other hand, a superharmonic function is representable by the sum of a harmonic function and a Green potential (or a volume potential) corresponding to a nonnegative measure. A volume potential and Green's potential corresponding to a real measure μ are important solutions of the Poisson equation $-\Delta u = \mu$. So, we discuss their regularity properties and behavior at infinity. As a consequence we obtain regularity results for solutions of the Poisson equation.

In the theory of boundary value problems, we need spaces $D^{k,p}(\Omega)$ and $\tilde{W}^{k,p}(\Omega)$ and behavior of the Laplace operator on these spaces. Using properties of volume potentials, we prove that the Laplace operator maps $\tilde{W}^{k+2,p}(R^m)$ onto $\tilde{W}^{k,p}(R^m)$ and $D^{k+2,p}(R^m)$ onto $D^{k,p}(R^m)$. We characterize the range of the Laplace operator on $\tilde{W}^{1,p}(R^m)$ and on $\tilde{W}^{1,p}(\Omega)$ for a domain with compact Lipschitz boundary. We show that $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega) + \{c; c \in R\}$, in particular $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m) + \{c; c \in R\}$. If $p < m$, then $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m) \oplus \{c; c \in R\}$, and we characterize functions from $\tilde{W}^{1,p}(R^m)$ as functions from $D^{1,p}(R^m) \cap L^{pm/(m-p)}(R^m)$. If $p \geq m$, then $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m)$. We prove for an unbounded domain $\Omega \subset R^m$ with compact Lipschitz boundary that $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega) \oplus \{c; c \in R\}$ for $p < m$ and $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$ for $p \geq m$. Moreover, if $u \in D^{1,p}(\Omega)$ and $p < m$, then there

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exist finite limits

$$\lim_{r\to\infty} \oint\limits_{B(0;r)} u \, \mathrm{d}\mathcal{H}_m, \quad \lim_{r\to\infty} \oint\limits_{\partial B(0;r)} u \, \mathrm{d}\mathcal{H}_{m-1}$$

and these limits coincide. If $u \in \tilde{W}^{1,p}(\Omega)$, then these limits are equal to zero.

3.1 Poisson Equation

This section studies solutions of the Poisson equation in the sense of distributions.

Definition 3.1.1 Let $\Omega \subset R^m$ be an open set, $f \in \mathcal{C}^0(\Omega)$. We say that u is a classical solution of the Poisson equation

$$-\Delta u = f \tag{3.1}$$

in Ω if $u \in \mathcal{C}^2(\Omega)$ and (3.1) holds true in each point of Ω.

Definition 3.1.2 Let $\Omega \subset R^m$ be an open set, and $u, f \in \mathcal{D}'(\Omega)$. We say that u is a solution of the Poisson equation (3.1) in Ω in the sense of distributions if $\langle u, -\Delta\varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$. In particular, if $u \in L^1_{loc}(\Omega)$, then u is a solution of (3.1) if

$$\langle f, \varphi\rangle = \int\limits_\Omega u(-\Delta\varphi) \, d\mathcal{H}_m \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Lemma 3.1.3 *Let u be a classical solution of the Poisson equation (3.1) in an open set $\Omega \subset R^m$. Then u is a solution of the Poisson equation (3.1) in Ω in the sense of distributions.*

Proof Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Choose a bounded open set G with boundary of class C^1 such that $\overline{G} \subset \Omega$ and $\operatorname{spt}\varphi \subset G$. Then second Green formula (Theorem 2.2.1) gives

$$\int\limits_\Omega u\Delta\varphi \, d\mathcal{H}_m = \int\limits_G \varphi\Delta u \, d\mathcal{H}_m + \int\limits_{\partial G} \left(u\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial u}{\partial n}\right) d\mathcal{H}_{m-1} = -\int\limits_\Omega f\varphi \, d\mathcal{H}_m.$$

Definition 3.1.4 Let $u \in \mathcal{D}'(R^m)$. We say that u is a fundamental solution of the Poisson equation (3.1) if $(-\Delta u) = \delta_0$ in the sense of distribution.

Theorem 3.1.5 *If $f \in \mathcal{D}'(R^m)$ with compact support and v is a fundamental solution of the Poisson equation in R^m, then $v * f$ is a solution of the Poisson equation (3.1) in R^m in the sense of distribution. The function $h_m(x)$ is a fundamental solution of the Laplace equation in R^m. If μ is a finite real Borel measure with* *compact support in* R^m*, then* $h_m * \mu = V\mu$*, the volume potential corresponding to* μ*.*

Proof If $f \in \mathcal{D}'(R^m)$ has a compact support and v is a fundamental solution of the Poisson equation, then $-\Delta(v * f) = (-\Delta v) * f = \delta_0 * f = f$.

Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$. Using Corollary 2.2.7 we get

$$\langle \delta_0, \varphi \rangle = \varphi(0) = V(-\Delta\varphi)(0) = \int\limits_{R^m} [h_m(x)](-\Delta\varphi(x)) \, d\mathcal{H}_m(x),$$

what means that $h_m(x)$ is a fundamental solution of the Poisson equation in R^m.

Let μ be a finite real Borel measure with compact support in R^m. If $\varphi \in \mathcal{C}_c^\infty(R^m)$ then Fubini's theorem gives

$$\langle h_m * \mu, \varphi \rangle = \langle h_m, \mu * \varphi \rangle = \int\limits_{R^m} h_m(x) \int\limits_{R^m} \varphi(x+y) \, \mathrm{d}\mu(y) \, \mathrm{d}\mathcal{H}_m(x)$$

$$= \int\limits_{R^m} \int\limits_{R^m} h_m(x)\varphi(x+y) \, \mathrm{d}\mathcal{H}_m(x) \, \mathrm{d}\mu(y) = \int\limits_{R^m} \int\limits_{R^m} h_m(z-y)\varphi(z) \, \mathrm{d}\mathcal{H}_m(z) \, \mathrm{d}\mu(y)$$

$$= \int\limits_{R^m} \int\limits_{R^m} h_m(z-y)\varphi(z) \, \mathrm{d}\mu(y) \, \mathrm{d}\mathcal{H}_m(z) = \langle V\mu, \varphi \rangle.$$

Proposition 3.1.6 *Let* f *be a distribution with compact support in* R^m*. Denote* $Vf = h_m * f$*. Then* $Vf \in \mathcal{C}^\infty(R^m \setminus \operatorname{spt} f)$ *and* $Vf(x) = \langle f, h_m(\cdot - x) \rangle$ *for* $x \notin \operatorname{spt} f$*. Let* $\alpha = (\alpha_1, \ldots, \alpha_m)$ *be a multiindex. If* $|\alpha| > 0$ *or* $m > 2$*, then* $\partial^\alpha Vf(x) = O(|x|^{2-m-|\alpha|})$ *as* $|x| \to \infty$*. If* $m = 2$ *and* $\langle f, 1 \rangle = 0$*, then* $\partial^\alpha V(x) = O(|x|^{-1-|\alpha|})$ *as* $|x| \to \infty$ *for each multiindex* α*.*

Proof $\Delta Vf = 0$ on $R^m \setminus \operatorname{spt} f$ in the sense of distributions by Theorem 3.1.5. Thus $Vf \in H(R^m \setminus \operatorname{spt} f) \subset \mathcal{C}^\infty(R^m \setminus \operatorname{spt} f)$ by Theorem 2.18.2 and Corollary 2.2.8. Fix a bounded open set Ω such that $\operatorname{spt} f \subset \Omega$. According to Lemma 1.20.5, there exist $k \in N$ and $f_\alpha \in L^1(R^m)$ with $f_\alpha = 0$ on $R^m \setminus \Omega$ such that

$$f = \sum_{|\alpha| \le k} \partial^\alpha f_\alpha.$$

Fix a multiindex β. By virtue of Fubini's theorem, we obtain on $R^m \setminus \overline{\Omega}$

$$\partial^\beta Vf(x) = \sum_{|\alpha| \le k} f_\alpha * \partial^{\alpha+\beta} h_m(x) = \sum_{|\alpha| \le k} (-1)^{|\alpha+\beta|} \langle f_\alpha, \partial^{\alpha+\beta} h_m(\cdot - x) \rangle$$

$$= \sum_{|\alpha| \le k} (-1)^{|\beta|} \langle \partial^\alpha f_\alpha, \partial^\beta h_m(\cdot - x) \rangle = (-1)^{|\beta|} \langle f, \partial^\beta h_m(\cdot - x) \rangle.$$

If $|\beta| = 1, m = 2$, or $|\beta| = 0, m > 2$, then

$$\partial^\beta Vf(x) = \sum_{|\alpha|\le k} f_\alpha * \partial^{\alpha+\beta} h_m(x) \to 0, \qquad |x| \to \infty \,.$$

If $|\alpha| > 0$ or $m > 2$, then $\partial^\alpha Vf(x) = O(|x|^{2-m-|\alpha|})$ as $|x| \to \infty$ by Proposition 2.17.3.

Let now $m = 2$ and $\langle f, 1\rangle = 0$. Let ω be a bounded domain such that $\mathrm{spt} f \subset \omega$. Denote $\varphi_x(y) = h_2(y-x) - h_2(x)$. If β is a multiindex, then $\partial^\beta \varphi_x(y) \to 0$ as $|x| \to \infty$ uniformly with respect $y \in \omega$ (compare Corollary 2.9.6 and Lemma 2.9.7). Thus

$$Vf(x) = \langle f, h_2(\cdot - x)\rangle = \langle f, \varphi_x\rangle \to 0, \qquad |x| \to \infty.$$

Since $Vf \in H(R^2 \setminus \mathrm{spt} f)$, Proposition 2.17.3 gives $\partial^\alpha Vf(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for each multiindex α.

Proposition 3.1.7 *Let $\Omega \subset R^m$ be an open set, $u, v, f \in \mathcal{D}'(\Omega)$, and u and v be solutions of the Poisson equation (3.1) in Ω in the sense of distributions, $k \in N_0$, $0 \le \lambda \le 1$, $s \in R$, $1 \le p, q \le \infty$. If $u \in \mathcal{C}^{k,\lambda}_{loc}(\Omega)$, then $v \in \mathcal{C}^{k,\lambda}_{loc}(\Omega)$. If $u \in W^{s,p}_{loc}(\Omega)$, then $v \in W^{s,p}_{loc}(\Omega)$. If u is locally in $B^{p,q}_s(\Omega)$, then v is locally in $B^{p,q}_s(\Omega)$. If $p, q < \infty$ and u is locally in $F^{p,q}_s(\Omega)$, then v is locally in $F^{p,q}_s(\Omega)$.*

Proof Since $v - u$ is a solution of the Laplace equation in the sense of distributions in Ω, we have $v - u \in H(\Omega)$ by Theorem 2.18.2. Corollary 2.2.7 gives that $v - u \in \mathcal{C}^\infty(\Omega)$.

Proposition 3.1.8 *Let f be a distribution on R^m with compact support. If $m = 2$, suppose moreover that $\langle f, 1\rangle = 0$. Let u be a distribution such that $-\Delta u = f$ in R^m. Then $u = Vf$ if and only if $u(x) \to 0$ as $|x| \to \infty$.*

Proof If $u = Vf$, then $u(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6.

Suppose $u(x) \to 0$ as $|x| \to \infty$. Define $v = u - Vf$. Then $\Delta v = 0$ in R^m by Theorem 3.1.5, $v \in \mathcal{C}^\infty(R^m)$ (see Theorem 2.18.2 and Corollary 2.2.8), $v(x) \to 0$ as $|x| \to \infty$ (see Proposition 3.1.6). So, $v \equiv 0$ by Theorem 2.16.1.

3.2 Lower Semicontinuous Functions

In this auxiliary section, we gather properties of lower semicontinuous functions. We need this notion for the definition of superharmonic functions.

Definition 3.2.1 Let $K \subset R^m$. Then $f : K \to \langle -\infty, \infty\rangle$ is called an extended real-valued function on K. We say that an extended real-valued function f on K is lower semicontinuous in K if

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

provided $\{x_n\} \subset K$, $x_n \to x \in K$ as $n \to \infty$. We say that an extended real-valued function f is upper semicontinuous in K if the function $(-f)$ is lower semicontinuous in K.

Lemma 3.2.2 *Let $K \subset R^m$ be compact and f be a lower semicontinuous extended real-valued function on K. Then there is $x_0 \in K$ such that $f(x) \geq f(x_0)$ for each $x \in K$.*

Proof Let $a = \inf\{f(x); x \in K\}$. Choose $\{x_n\} \subset K$ such that $f(x_n) \to a$ as $n \to \infty$. Since K is compact, we can choose subsequence $\{y_k\}$ of $\{x_n\}$ such that $y_k \to x_0$ as $k \to \infty$. Then

$$a \leq f(x_0) \leq \liminf_{k\to\infty} f(y_k) = a.$$

Corollary 3.2.3 *Let $\Omega \subset R^m$ be an open set and f be a lower semicontinuous extended real-valued function on Ω, $f > -\infty$. Then f is a Borel measurable function which is locally bounded from below. If μ is a nonnegative finite Borel measure with compact support in Ω, then $\int f\, d\mu$ makes sense.*

Proof The function f is Borel measurable because $\{x \in \Omega; f(x) > \alpha\}$ is open for each real α. If $K \subset \Omega$ is compact, there is $x \in K$ such that $f \geq f(x) > -\infty$ on K (see Lemma 3.2.2). The rest is clear.

Lemma 3.2.4 *Let $M \subset R^m$, f be an extended real-valued function on M which is bounded from below. The following statements are equivalent.*

1. *f is lower semicontinuous on M.*
2. *There exists an increasing sequence f_k of Lipschitz real functions on M such that $f_k \nearrow f$ as $k \to \infty$.*
3. *There exists an increasing sequence f_k of continuous real functions on M such that $f_k \nearrow f$ as $k \to \infty$.*

Proof 1) ⇒ 2): If $f \equiv \infty$, then we define $f_n \equiv n$. Suppose now that there is $x_0 \in M$ such that $f(x_0) < \infty$. Put

$$\tilde{f}(x) = \begin{cases} f(x), & x \in M, \\ \liminf_{y\to x} f(y), & x \in \partial M \setminus M, \\ +\infty, & x \notin \overline{M}. \end{cases} \tag{3.2}$$

Then $\tilde{f}$ is lower semicontinuous and $\tilde{f} \geq K$ for some $K \in R$. Put

$$f_n(x) = \inf_{y\in R^m} (\tilde{f}(y) + n|x-y|).$$

Then $-\infty < K \leq f_n(x) \leq f(x_0) + n|x-x_0| < \infty$. Clearly, f_n is increasing and $f_n \leq \tilde{f}$ for all n. If $x, z \in R^m$, then

$$f_n(x) \leq \inf_{y\in R^m} (\tilde{f}(y) + n|y-z| + n|x-z|) = f_n(z) + n|x-z|.$$

This gives $|f_n(x) - f_n(z)| \le n|x - z|$ and f_n are Lipschitz.

Fix $x \in R^m$ and $\alpha \in R$ such that $\alpha < \tilde{f}(x)$. Then there is $\delta > 0$ such that $f(x) > \alpha$ in $B(x;\delta)$. Since $f_n(x) \ge \min(\alpha, K + n\delta)$, we deduce that $\lim f_n(x) \ge \alpha$. Thus $f_n \to \tilde{f}$ as $n \to \infty$.

3) ⇒ 1): If $x, x_k \in M$, $x_k \to x$, then

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \liminf_{k\to\infty} f_n(x_k) \\ &\le \lim_{n\to\infty} \liminf_{k\to\infty} f(x_k) = \liminf_{k\to\infty} f(x_k). \end{aligned}$$

3.3 Superharmonic Functions

In this section we define superharmonic functions and study their fundamental properties. We show that superharmonic functions are locally integrable. We prove that a constant function is the only superharmonic function in a domain that attains its minimum.

Definition 3.3.1 Let $\Omega \subset R^m$ be an open set. An extended real-valued function u on Ω is superharmonic on Ω if

- $u > -\infty$ on Ω,
- u is not identically $+\infty$ on any component of Ω,
- u is lower semicontinuous on Ω,
- for every $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, it holds

$$u(x) \ge \oint_{B(x;r)} u \, d\mathcal{H}_m. \tag{3.3}$$

Definition 3.3.2 Let $\Omega \subset R^m$ be an open set. An extended real-valued function u on Ω is called subharmonic on Ω if $(-u)$ is superharmonic on Ω.

Lemma 3.3.3 *Let $\Omega \subset R^m$, $m \ge 2$, be an open set and c be a positive constant. If u and v are superharmonic functions in Ω, then $w = \min(u, v)$, $u + v$, and cu are superharmonic functions in Ω, too. If f and g are subharmonic functions in Ω, then $h = \max(f, g)$, $f + g$, and cf are subharmonic functions in Ω.*

Proof According to Lemma 3.2.4, there are $u_j, v_j \in \mathcal{C}^0(\Omega)$ such that $u_j \nearrow u$, $v_j \nearrow v$. Since $w_j = \min(u_j, v_j) \in \mathcal{C}^0(\Omega)$ and $w_j \nearrow w$, Lemma 3.2.4 gives that w is lower semicontinuous in Ω. If $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, then

$$u(x) \ge \oint_{B(x;r)} u \, d\mathcal{H}_m \ge \oint_{B(x;r)} w \, d\mathcal{H}_m, v(x) \ge \oint_{B(x;r)} v \, d\mathcal{H}_m \ge \oint_{B(x;r)} w \, d\mathcal{H}_m.$$

Thus

$$w(x) \geq \oint\limits_{B(x;r)} w \, d\mathcal{H}_m.$$

Since $-f$ and $-g$ are superharmonic, the function $-h = \min(-f, -g)$ is superharmonic. Therefore h is subharmonic.

The rest is trivial.

Proposition 3.3.4 *Let $\Omega \subset R^m$ be an open set, u be superharmonic in Ω. Then $u \in L^1_{loc}(\Omega)$.*

Proof Corollary 3.2.3 gives that

$$\int\limits_K u \, d\mathcal{H}_m > -\infty$$

is well defined for each compact subset K of Ω. Fix a component G of Ω. Denote $V = \{x \in G; u \in L^1(U)$ for some neighborhood U of $x\}$. Clearly, V is an open set. If $u(x) < \infty$ and $0 < r < \operatorname{dist}(x; \partial\Omega)$, then $B(x; r) \subset V$ by (3.3). Therefore V is nonempty, because u is not identically $+\infty$ in G. Let now $x_n \in V$, $x_n \to x \in G$ as $n \to \infty$. Fix $r \in (0, \operatorname{dist}(x; \partial\Omega)/2)$. Choose x_n such that $|x - x_n| < r$. Since $r < \operatorname{dist}(x_n; \partial\Omega)$, we have $B(x_n; r) \subset V$. Hence $x \in B(x_n; r) \subset V$. The set V is a nonempty open and closed subset of the metric space G. Since G is connected, we infer that $V = G$.

Proposition 3.3.5 *Let $\Omega \subset R^m$ be a domain and $u : \Omega \to (-\infty, \infty\rangle$ be a lower semicontinuous extended real-valued function. Suppose that for every $x \in \Omega$, there is $\rho(x) > 0$ such that (3.3) holds for $0 < r < \rho(x)$. If there is $x_0 \in \Omega$ such that $u(x) \geq u(x_0)$ for each $x \in \Omega$, then u is constant.*

Proof Put $G = \{x \in \Omega; u(x) = u(x_0)\}$. The set G is a closed subset of the metric space Ω, because u is lower semicontinuous.

Fix $x \in G$, $0 < r < \rho(x)$. Then $B(x; r) \setminus G$ is an open set. If $B(x; r) \setminus G = \{y \in B(x; r); u(y)) > u(x_0)\}$ is nonempty, then

$$u(x) = u(x_0) < \oint\limits_{B(x;r)} u \, d\mathcal{H}_m,$$

what contradicts (3.3). Hence $B(x; r) \subset G$ and G is an open set. Since G is an nonempty open and closed subset of the connected metric space Ω, we infer that $G = \Omega$.

Proposition 3.3.6 *Let $\Omega \subset R^m$ be an open set, $A \subset \Omega$, $\mathcal{H}_m(\Omega \setminus A) = 0$, $u : \Omega \to (-\infty, \infty\rangle$ be a lower semicontinuous extended real-valued function, $x \in \Omega$. If there* *is* $\rho > 0$ *such that (3.3) holds for* $0 < r < \rho$*, then*

$$u(x) = \liminf_{A \ni y \to x} u(y).$$

Proof Since u is locally bounded from below (see Lemma 3.2.2), the integral in (3.3) makes sense. If

$$u(x) < \alpha < \liminf_{A \ni y \to x} u(y)$$

then there is $r \in (0, \rho)$ such that $\alpha < u$ on $A \cap B(x; r) \setminus \{x\}$. This contradicts (3.3).

Proposition 3.3.7 *Let* $\Omega \subset R^m$ *be an open set,* $f : \Omega \to (-\infty, \infty\rangle$ *be a lower semicontinuous extended real-valued function not identically equal to* $+\infty$ *on any component of* Ω*. Then the following statements are equivalent.*

1. *u is superharmonic on Ω.*
2. *For every $x \in \Omega$, there is $\rho(x) > 0$ such that (3.3) holds for $0 < r < \rho(x)$.*
3. *For every bounded open set G with $\overline{G} \subset \Omega$ and $h \in H(G) \cap \mathcal{C}^0(\overline{G})$ with $h \le u$ on ∂G, we have $h \le u$ on G.*
4. *If $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, then*

$$u(x) \ge \oint_{\partial B(x;r)} u \, \mathrm{d}\mathcal{H}_{m-1}. \tag{3.4}$$

5. *For every $x \in \Omega$ there is $\rho(x) > 0$ such that (3.4) holds for $0 < r < \rho(x)$.*

Proof 1) ⇒ 2) and 4) ⇒ 5) are clear.

2) ⇒ 3): Put $w(x) = u(x) - h(x)$ in G,

$$w(x) = \liminf_{G \ni y \to x} (u(y) - h(y))$$

for $x \in \partial G$. Then w is a lower semicontinuous extended real-valued function on $\overline{G}$. If $x \in \partial G$ then Proposition 3.3.6 gives

$$0 \le u(x) - h(x) = \liminf_{y \to x} u(y) - h(x) \le \liminf_{G \ni y \to x} [u(y) - h(y)] = w(x).$$

According to Lemma 3.2.2, there is $x \in \overline{G}$ such that $w(y) \ge w(x)$ for each $y \in \overline{G}$. If $x \in \partial G$ then $w(x) \ge 0$ and thus $w \ge 0$. Let now $x \in G$. Fix the component V of G containing x. Proposition 3.3.5 gives that w is constant on V. Since

$$w(x) = \liminf_{V \ni y \to z} w(y) \ge \liminf_{G \ni y \to z} w(y) \ge w(z) \ge 0$$

for each $z \in \partial V$, we see that $w(x) \ge 0$. Thus $w \ge 0$.

3) $\Rightarrow$ 4): Fix $x \in \Omega$, $u(x) < \infty$ and $0 < r < \rho < \operatorname{dist}(x, \partial\Omega)$. Corollary 3.2.3 gives that the function u is bounded from below on $\overline{B(x;\rho)}$. According to Lemma 3.2.4, there exists an increasing sequence f_k of continuous real functions in $B(x;\rho)$ such that $f_k \nearrow u$ as $k \to \infty$. According to Theorem on the Poisson integral (Theorem 2.5.4), there are $u_k \in H(B(x;r)) \cap \mathcal{C}^0(\overline{B(x;r)})$ such that $u_k = f_k$ on $\partial B(x;r)$. Since $u_k = f_k \le u$ on $\partial B(x;r)$ we have $u_k \le u$ on $B(x;r)$. According to Lemma 2.4.1, we have

$$\oint_{\partial B(x;r)} f_k \, \mathrm{d}\mathcal{H}_{m-1} = \oint_{\partial B(x;r)} u_k \, \mathrm{d}\mathcal{H}_{m-1} = u_k(x) \le u(x).$$

Fatou's lemma (Lemma 1.13.5) gives

$$\oint_{\partial B(x;r)} u \, \mathrm{d}\mathcal{H}_{m-1} \le \liminf_{k\to\infty} \oint_{\partial B(x;r)} f_k \, \mathrm{d}\mathcal{H}_{m-1} \le u(x).$$

4) $\Rightarrow$ 1) and 5) $\Rightarrow$ 2): Let $x \in \Omega$ and $\rho > 0$ be such that (3.4) holds for $0 < r < \rho$. If $0 < r < \rho$ we have

$$\oint_{B(x;r)} u \, \mathrm{d}\mathcal{H}_m = \frac{1}{\mathcal{H}_m(B(0;r))} \int_0^r \int_{\partial B(x;s)} u \, \mathrm{d}\mathcal{H}_{m-1} \, \mathrm{d}s$$

$$\le \frac{1}{\mathcal{H}_m(B(0;r))} \int_0^r u(x)\mathcal{H}_{m-1}(\partial B(0;s)) \, \mathrm{d}s = u(x).$$

Corollary 3.3.8 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, and u be an extended real-valued function on Ω. If for each $x \in \Omega$, there is an open set $G(x)$ such that $x \in G(x)$ and u is a superhamonic function on $G(x)$, then u is a superharmonic function on Ω.*

Proof The function u is locally integrable on Ω by Proposition 3.3.4. Proposition 3.3.7 gives that u is a superharmonic function on Ω.

Example 3.3.9 Fix $z \in R^m$, $m \ge 2$. Put $u(z) = \infty$, $u(y) = h_m(z - y)$ for $z \ne y$. The function u is continuous extended real-valued function in R^m. One has $u \in H(R^m \setminus \{z\})$ by Remark 2.1.3. If $x \ne z$ then (3.4) holds for $0 < r < |x - z|$ by Proposition 3.3.7. Clearly, if $x = z$ then (3.4) holds for each $r > 0$. Proposition 3.3.7 gives that u is a superharmonic function in R^m.

Corollary 3.3.10 *Let $\omega \subset \Omega \subset R^m$ be open sets, $m \ge 2$. Let u be a superharmonic function in Ω, v be a superharmonic function in ω, and suppose that*

$$\liminf_{\omega \ni x \to y} v(x) \ge u(y), \qquad y \in \partial\omega \cap \Omega. \tag{3.5}$$

Then the function

$$w(x) = \begin{cases} \min(u(x), v(x)), & x \in \omega, \\ u(x), & x \in \Omega \setminus \omega. \end{cases} \tag{3.6}$$

is superharmonic in Ω.

Proof w is superharmonic in $\Omega \setminus \overline{\omega}$ and in ω (see Lemma 3.3.3). The function w is lower semicontinuous by (3.5). If $x \in \partial\omega \cap \Omega$ and $0 < r < \operatorname{dist}(x; \partial\Omega)$, then

$$w(x) = u(x) \geq \oint\limits_{B(x;r)} u \, \mathrm{d}\mathcal{H}_m \geq \oint\limits_{B(x;r)} w \, \mathrm{d}\mathcal{H}_m.$$

Proposition 3.3.7 gives that w is superharmonic in Ω.

3.4 Minimum Principle

In this section we prove that a superharmonic function is greater than or equal to a subharmonic function provided the same is true for their traces on the boundary. In particular, a superharmonic function is greater than or equal to the infimum of its trace on the boundary.

Proposition 3.4.1 *Let* $\Omega \subset R^m$ *be a domain,* u *be a superharmonic function in* Ω, *and* v *be a subharmonic function in* Ω. *Suppose that*

$$\liminf_{\Omega \ni y \to x} (u(y) - v(y)) \geq 0$$

for each $x \in \partial\Omega$. *If* Ω *is unbounded suppose moreover*

$$\liminf_{|y| \to \infty, y \in \Omega} (u(y) - v(y)) \geq 0. \tag{3.7}$$

Then $u \geq v$ *in* Ω. *If there is* $x \in \Omega$ *such that* $u(x) = v(x)$, *then* $u = v$ *in* Ω.

Proof Put $w(x) = u(x) - v(x)$ in Ω,

$$w(x) = \liminf_{\Omega \ni y \to x} (u(y) - v(y))$$

for $x \in \partial\Omega$. Then w is a lower semicontinuous extended real-valued function on $\overline{\Omega}$, $w = 0$ on $\partial\Omega$. Fix $\alpha < 0$. According to (3.7) there is $r > 0$ such that $w \geq \alpha$ on $\Omega \setminus B(0; r)$. According to Lemma 3.2.2 there is $x \in \overline{\Omega} \cap \overline{B(0; r)}$ such that $w(y) \geq w(x)$ for each $y \in \overline{\Omega} \cap \overline{B(0; r)}$. If $x \in \partial[\overline{\Omega} \cap \overline{B(0; r)}]$ then $w(x) \geq \alpha$. Let now $x \in \Omega \cap B(0; r)$. Fix component G of $\Omega \cap B(0; r)$ containing x. Since w is a superharmonic function in G, Proposition 3.3.5 gives that w is constant on G. So, we can suppose that $x \in \partial[\overline{\Omega} \cap \overline{B(0;r)}]$. Thus $w \geq \alpha$. Since $\alpha < 0$ was arbitrary, we deduce that $w \geq 0$ and thus $u \geq v$ in Ω.

Suppose now that there is $x \in \Omega$ such that $u(x) = v(x)$. Then $w(x) = 0$. Proposition 3.3.5 gives that $w \equiv 0$.

Proposition 3.4.2 *Let $\Omega \subset R^m$ be an open set, u be a superharmonic function in Ω, v be a subharmonic function in Ω, and h be a positive harmonic function in Ω. Suppose that*

$$\liminf_{\Omega \ni y \to x} \frac{u(y) - v(y)}{h(y)} \geq 0$$

for each $x \in \partial\Omega$. If Ω is unbounded, suppose moreover

$$\liminf_{|y| \to \infty, y \in \Omega} \frac{u(y) - v(y)}{h(y)} \geq 0.$$

Then $u \geq v$ in Ω.

Proof Since $u - v$ is superharmonic, we can suppose that $v \equiv 0$. Suppose that $u(z) < 0$ for some $z \in \Omega$. Denote by ω the component of Ω containing z and $\epsilon = -u(z)/(2h(z)) > 0$. Then there exists a compact set $F \subset \omega$ such that $u/h > -\epsilon$ on $\omega - F$. Since $u > -\epsilon h$ on $\omega \setminus F$, Proposition 3.4.1 gives that $u \geq -\epsilon h$ in ω. That is a contradiction with $u(z) = -2\epsilon h(z) < -\epsilon h(z)$.

3.5 Minimum of Superharmonic Functions at a Boundary Point

Proposition 3.5.1 *Let $\Omega \subset R^m$ be a domain, $B \subset \Omega$ be a ball, $x \in \partial\Omega \cap \partial B$, and θ be a unit vector such that $\theta \cdot n^B(x) > 0$. Let $u \in \mathcal{C}^0(\overline{\Omega})$ be superharmonic in Ω. Suppose that there exists $\delta > 0$ such that $u(y) > u(x)$ for all $y \in \Omega \cap B(x;\delta)$. If there exists $\partial u(x)/\partial\theta$, then $\partial u(x)/\partial\theta < 0$.*

Proof Without loss of generality, we can suppose that $x = 0$, $u(0) = 0$, $B = B(z;r) \subset B(0;\delta)$, and $z = [r, 0, 0, \ldots, 0]$. Denote $w = Pu$, the Poisson integral of u with respect to the ball B, $w = u$ in ∂B. Then $w \in \mathcal{C}^0(\overline{B}) \cap H(B)$ by the Theorem on the Poisson integral (Theorem 2.5.4). Moreover, $w(0) = 0$, $w > 0$ on $\overline{B} \setminus \{0\}$ by the maximum principle (Theorem 2.4.2). If $t > 0$, $y = -t\theta \in B$, then the Harnack inequality (Proposition 2.8.1) gives

$$w(y) \geq w(z) \frac{r^{m-2}(r - |z-y|)}{(r + |z-y|)^{m-1}} \geq w(z) \frac{r - |z-y|}{2^{m-1} r},$$

$$\liminf_{t \to 0} \frac{w(-t\theta)}{t} \geq w(z) \lim_{t \to 0} \frac{r - \sqrt{r^2 - 2r|\theta_1|t + t^2}}{2^{m-1} r t} = w(z) \frac{|\theta_1|}{2^{m-1} r} > 0.$$

Since $u = w$ on ∂B, u is superharmonic and w is harmonic, $w \le u$ by Proposition 3.3.7. Thus

$$\frac{\partial u(0)}{\partial(-\theta)} \ge \liminf_{t \to 0} \frac{w(-t\theta)}{t} > 0.$$

Example 3.5.2 The condition that there exists a ball B such that $B \subset \Omega$, $x \in \partial B$ in Proposition 3.5.1 is important. Put $\Omega = \{[x, y]; x > 0, y > 0\}$, $u(x, y) = xy$. Then $\Delta u = 0$, $u(0, 0) = 0 < u(x, y)$ for each $[x, y] \in \Omega$. But $\nabla u(0, 0) = 0$ and $\partial u(0, 0)/\partial\theta = 0$ for each vector θ.

3.6 Harmonic Minorant

In this section we study under which conditions there exists a harmonic minorant of a system of superharmonic functions.

Definition 3.6.1 Let $\Omega \subset R^m$, $m \ge 2$, be an open set and f be an extended real-valued function on Ω. If $u \in H(\Omega)$, $u \le f$ on Ω, then u is called a harmonic minorant of f. The function u is the greatest harmonic minorant of f, if u is a harmonic minorant of f and $v \le u$ whenever v is a harmonic minorant of f. If $u \in H(\Omega)$, $u \ge f$ on Ω, then u is called a harmonic majorant of f. The function u is the least harmonic majorant of f, if u is a harmonic majorant of f and $v \ge u$ whenever v is a harmonic majorant of f.

Proposition 3.6.2 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, u be a superharmonic function on Ω, $z \in \Omega$, and $0 < \rho < \operatorname{dist}(z; \partial\Omega)$. Let $P(x, y)$ be the Poisson kernel corresponding to the ball $B = B(z; \rho)$. Put*

$$u_B(x) = \begin{cases} u(x), & x \in \Omega \setminus B, \\ \displaystyle\int_{\partial B(z;\rho)} P(x, y) u(y) \, \mathrm{d}\mathcal{H}_{m-1}(y), & x \in B. \end{cases} \tag{3.8}$$

Then u_B is a superharmonic function in Ω, harmonic in B, and $u_B \le u$ in Ω.

Proof $u_B \in H(B)$ by Theorem 2.5.4. Fix a bounded open set $V \subset \overline{V} \subset \Omega$ such that $\overline{B} \subset V$. Lemma 3.2.2 gives that u is bounded from below on $\overline{V}$. According to Lemma 3.2.4, there is an increasing sequence $f_j \in \mathcal{C}^0(V)$ such that $f_j \nearrow u$ as $j \to \infty$. Put

$$v_j(x) = \begin{cases} f_j(x), & x \in \partial B, \\ \displaystyle\int_{\partial B(z;\rho)} P(x, y) f_j(y) \, \mathrm{d}\mathcal{H}_{m-1}(y), & x \in B. \end{cases} \tag{3.9}$$

Then $v_j \in H(B) \cap \mathcal{C}^0(\overline{B})$ by Theorem on Poisson integral (Theorem 2.5.4). Since $u \geq v_j$ on ∂B, Proposition 3.3.7 gives that $u \geq v_j$ on B. Since f_j and u are bounded from below on ∂B by the same constant, we obtain by Theorem on monotone convergence of an integral (Theorem 1.13.3) that $v_j \nearrow u_B$ on B as $j \to \infty$. Hence $u \geq u_B$.

If $x \in \partial B$, then Theorem 2.5.4 gives

$$u(x) \leq \liminf_{\partial B \ni y \to x} u(y) \leq \liminf_{B \ni y \to x} u_B(y).$$

u_B is superharmonic by Corollary 3.3.10.

Lemma 3.6.3 *Let $\Omega \subset R^m$ be a domain, $m \geq 2$. Let $\mathcal{A}$ be a nonempty system of superharmonic functions in Ω such that for each $u, v \in \mathcal{A}$ and every ball B with $\overline{B} \subset \Omega$ we have* $\min(u, v) \in \mathcal{A}$, $u_B \in \mathcal{A}$. *Put $h = \inf\{u; u \in \mathcal{A}\}$. Then $h \equiv -\infty$ or $h \in H(\Omega)$.*

Proof Fix $x \in \Omega$, $0 < \rho < \operatorname{dist}(x, \partial\Omega)$. Put $B = B(x; \rho)$. Choose $\{u_n\} \subset \mathcal{A}$ such that $u_n(x) \to h(x)$ as $n \to \infty$. Fix $u \in \mathcal{A}$. Put $v_1 = \min(u_1, u)$, $v_{k+1} = \min(v_k, u_k)$, and $w_n = (v_n)_B$. Then w_k is a decreasing sequence of functions from $\mathcal{A} \cap H(B)$ such that $w_n(x) \to h(x)$ as $n \to \infty$ (see Proposition 3.6.2). Put

$$w^u = \lim_{n \to \infty} w_n.$$

Then $w^u \geq h$, $w^u(x) = h(x)$. Since w_n is a decreasing sequence of harmonic functions on B, Proposition 2.11.3 gives that $w^u = -\infty$ on B or $w^u \in H(B)$. If $w^u = -\infty$ on B, then $h = -\infty$ on B. Suppose now that $w^u \in H(B)$. If $v \in \mathcal{A}$, put $z = \min(v, u)$. Then $w^z \in H(B)$, $w^z(x) = h(x)$. Since $w^z \leq w^u$, the function $w^u - w^z$ is a nonnegative harmonic function in B. Since $w^u(x) - w^z(x) = 0$, Theorem 2.4.2 gives that $w^u - w^z = 0$ in B. Since $v \in \mathcal{A}$ was arbitrary, we deduce $h = w^u \in H(B)$.

We have proved that for each $x \in \Omega$ and $0 < \rho < \operatorname{dist}(x; \partial\Omega)$, we have $h \equiv -\infty$ in $B(x; \rho)$ or $h \in H(B(x; \rho))$. Hence h is continuous. Put $V = \{x \in \Omega; h(x) = -\infty\}$. We have proved that V is an open set. Since h is continuous extended real-valued function, the set V is a closed subset of the metric space Ω. Since Ω is connected, we infer that $V = \Omega$ and $h \equiv -\infty$ or $V = \emptyset$ and $h \in H(\Omega)$.

Proposition 3.6.4 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, $\mathcal{A}$ be a nonempty system of subharmonic functions on Ω, and $\mathcal{B}$ be a nonempty system of superharmonic functions in Ω such that $f \leq g$ for every $f \in \mathcal{A}$ and $g \in \mathcal{B}$. Put $h = \inf\{u; u$ superharmonic in $\Omega, u \geq f\ \forall f \in \mathcal{A}\}$ and $v = \sup\{u; u$ subharmonic in $\Omega, u \leq g\ \forall g \in \mathcal{B}\}$. Then $h, v \in H(\Omega)$, and $f \leq h \leq v \leq g$ for all $f \in \mathcal{A}$ and $g \in \mathcal{B}$.*

Proof Put $V = \{u$ superharmonic in $\Omega; u \geq f\ \forall f \in \mathcal{A}\}$. If $f \in \mathcal{A}$ then $f \leq \inf\{u; u \in V\} = h$. If $u, v \in V$ and B is a ball with $\overline{B} \subset \Omega$, then $\min(u, v) \in V$ (see Lemma 3.3.3) and $u_B \in V$ (see Proposition 3.6.2 and Proposition 3.4.1). Thus $h \equiv -\infty$ or $h \in H(\Omega)$ by Lemma 3.6.3. Since $f \leq h$, we infer $h \in H(\Omega)$.

If $g \in \mathcal{B}$, then $-g$ is subharmonic. Therefore $-v = \inf\{-u \text{ superharmonic }; -u \geq -g\ \forall g \in \mathcal{B}\} \in H(\Omega)$. Clearly $v \leq g$ for all $g \in \mathcal{B}$. Since $h \in H(\Omega)$ and $h \leq g$, we obtain $h \leq v$.

Theorem 3.6.5 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$. If u is a superharmonic function on Ω and u has a harmonic minorant on Ω, then u has a unique greatest harmonic minorant in Ω.*

Proof Put $v = \sup\{f; f \text{ subharmonic in } \Omega, f \leq u\}$. Then $v \in H(\Omega)$ by Proposition 3.6.4. Clearly, v is the greatest harmonic minorant in Ω.

Proposition 3.6.6 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$. If u_j are superharmonic functions on Ω with the greatest harmonic minorants h_j in Ω, $j = 1, 2$, then $h_1 + h_2$ is the greatest harmonic minorant of $u = u_1 + u_2$.*

Proof $h_1 + h_2$ is a harmonic minorant of u. According to Theorem 3.6.5, there exists the greatest harmonic minorant h of u. Clearly, $h_1 + h_2 \leq h \leq u_1 + u_2$. Since $h - u_1 \leq u_2$, $h - u_1$ is subharmonic, and u_2 is superharmonic, $h - u_1 \leq h_2 \leq u_2$ by virtue of Proposition 3.6.4. This forces $h - h_2 \leq u_1$, and thus $h - h_2 \leq h_1$. This gives $h \leq h_1 + h_2$.

3.7 Mean Values of Superharmonic Functions

Proposition 3.7.1 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, u be a superharmonic function on Ω, and $z \in \Omega$. For $0 < r < \operatorname{dist}(z; \partial\Omega)$ denote*

$$A(r) = \oint\limits_{B(z;r)} u \,\mathrm{d}\mathcal{H}_m, \qquad L(r) = \oint\limits_{\partial B(z;r)} u \,\mathrm{d}\mathcal{H}_{m-1}.$$

Then $A(r)$ and $L(r)$ are decreasing functions in the interval $(0, \operatorname{dist}(z; \partial\Omega))$, $A(r) \geq L(r)$ and

$$\lim_{r \searrow 0} A(r) = \lim_{r \searrow 0} L(r) = u(z). \tag{3.10}$$

Proof We can suppose that $z = 0$. Let $0 < r < \rho < \operatorname{dist}(0; \partial\Omega))$. Let u_B be a function from Proposition 3.6.2 with $B = B(0; \rho)$. Mean value theorem (Theorem 2.6.1) gives

$$L(\rho) = u_B(0) = \oint\limits_{\partial B(0;r)} u_B \,\mathrm{d}\mathcal{H}_{m-1} \leq \oint\limits_{\partial B(0;r)} u \,\mathrm{d}\mathcal{H}_{m-1} = L(r).$$

Since L is a decreasing function, we get

$$A(\rho) = \frac{1}{\mathcal{H}_m(B(0;1))\rho^m} \int_0^\rho \int\limits_{\partial B(0;t)} u(x)\, \mathrm{d}\mathcal{H}_{m-1}(x)\, \mathrm{d}t$$

$$= \frac{\mathcal{H}_{m-1}(\partial B(0;1))}{\mathcal{H}_m(B(0;1))\rho^m} \int_0^\rho L(t)t^{m-1}\, \mathrm{d}t = \frac{\mathcal{H}_{m-1}(\partial B(0;1))}{\mathcal{H}_m(B(0;1))} \int_0^1 L(\rho s)s^{m-1}\, \mathrm{d}s$$

$$\le \frac{\mathcal{H}_{m-1}(\partial B(0;1))}{\mathcal{H}_m(B(0;1))} \int_0^1 L(rs)s^{m-1}\, \mathrm{d}s = A(r).$$

Moreover,

$$A(\rho) = \frac{1}{\mathcal{H}_m(B(0;1))\rho^m} \int_0^\rho L(t)\mathcal{H}_{m-1}(\partial B(0;t))\, \mathrm{d}t$$

$$\ge \frac{1}{\mathcal{H}_m(B(0;1))\rho^m} \int_0^\rho L(\rho)\mathcal{H}_{m-1}(\partial B(0;t))\, \mathrm{d}t = L(\rho).$$

Fix $K < u(0)$. Since u is lower semicontinuous, there is $\rho > 0$ such $u \ge K$ on $B(0;\rho)$. Since $L(r) \le A(r) \le u(0)$, we obtain

$$K \le \lim_{r \searrow 0} L(r) \le \lim_{r \searrow 0} A(r) \le u(z).$$

Since $K < u(0)$ is arbitrary, the relation (3.10) holds true.

Corollary 3.7.2 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$. If u and v are superharmonic functions on Ω such that $u = v$ almost everywhere in Ω, then $u = v$ in Ω.*

Proof If $x \in \Omega$, then Proposition 3.7.1 gives

$$u(x) = \lim_{r \searrow 0} \oint\limits_{B(x;r)} u\, \mathrm{d}\mathcal{H}_m = \lim_{r \searrow 0} \oint\limits_{B(x;r)} v\, \mathrm{d}\mathcal{H}_m = v(x).$$

3.8 Regularization and Superharmonic Functions

In this section we prove that a regularization of a superharmonic function is again a superharmonic function.

Lemma 3.8.1 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, and $u, v \in L^1_{loc}(R^m)$ be such that for $\mathcal{H}_m$ almost all $x \in \Omega$*

$$u(x) \le \oint\limits_{B(x;r)} u\, \mathrm{d}\mathcal{H}_m, \quad \oint\limits_{B(x;r)} v\, \mathrm{d}\mathcal{H}_m \le v(x), \qquad 0 < r < \mathrm{dist}(x, \partial\Omega).$$

Suppose that $\varphi \in \mathcal{C}^{\infty}(R^1)$*,* $\varphi \geq 0$*,* $\operatorname{spt}\varphi \subset (0,1)$ *and*

$$\int\limits_{R^m} \varphi(|y|)\, d\mathcal{H}_m(y) = 1.$$

For $\epsilon > 0$ *put* $\varphi_\epsilon(y) = \varphi(|y|/\epsilon)\epsilon^{-m}$ *and* $u_\epsilon = u * \varphi_\epsilon$*,* $v_\epsilon = v * \varphi_\epsilon$*. Then* v_ϵ *is a superharmonic function in* $\Omega_\epsilon = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ *and* u_ϵ *is a subharmonic function in* Ω_ϵ*.*

Proof $v_\epsilon \in \mathcal{C}^{\infty}(R^m)$. Fix $x \in \Omega_\epsilon$ and $0 < r < \operatorname{dist}(x, \partial\Omega_\epsilon)$. If $|z| < \epsilon$, then $B(x - z; r) \subset \Omega$, and Fubini's theorem yields

$$\begin{aligned}
\int\limits_{B(x;r)} \frac{v_\epsilon(y)}{\mathcal{H}_m(B(x;r))}\, \mathrm{d}\mathcal{H}_m(y) &= \int\limits_{B(x;r)} \int\limits_{B(0;\epsilon)} \frac{\varphi_\epsilon(z) v(y-z)}{\mathcal{H}_m(B(x;r))}\, \mathrm{d}\mathcal{H}_m(z)\, d\mathcal{H}_m(y) \\
&= \int\limits_{B(0;\epsilon)} \varphi_\epsilon(z) \int\limits_{B(x;r)} \frac{v(y-z)}{\mathcal{H}_m(B(x;r))}\, \mathrm{d}\mathcal{H}_m(y)\, d\mathcal{H}_m(z) \\
&= \int\limits_{B(0;\epsilon)} \varphi_\epsilon(z) \int\limits_{B(x-z;r)} \frac{v(y)}{\mathcal{H}_m(B(x-z;r))}\, \mathrm{d}\mathcal{H}_m(y)\, d\mathcal{H}_m(z) \\
&\geq \int\limits_{B(0;\epsilon)} \varphi_\epsilon(z) v(x-z)\, d\mathcal{H}_m(z) = v_\epsilon(x).
\end{aligned}$$

Thus v_ϵ is superharmonic in Ω_ϵ. Now we use the proposition for $-u$.

Proposition 3.8.2 *Let* u *be a superharmonic function in* R^m*,* $m \geq 2$*. Suppose that* $\varphi \in \mathcal{C}^{\infty}(R^1)$*,* $\varphi \geq 0$*,* $\operatorname{spt}\varphi \subset (0,1)$*, and*

$$\int\limits_{R^m} \varphi(|y|)\, d\mathcal{H}_m(y) = 1.$$

For $\epsilon > 0$ *put* $\varphi_\epsilon(y) = \varphi(|y|/\epsilon)\epsilon^{-m}$*. Define* $u_\epsilon = u * \varphi_\epsilon$ *on* $\Omega_\epsilon = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \epsilon\}$*. Then* u_ϵ *are superharmonic functions and* $u_\epsilon \uparrow u$ *as* $\epsilon \downarrow 0$*. If* $u \in H(\Omega)$ *then* $u_\epsilon = u$*.*

Proof Fix $x \in R^m$. Denote

$$L(r) = \oint\limits_{\partial B(x;r)} u\, \mathrm{d}\mathcal{H}_{m-1}.$$

According to Proposition 3.7.1,

$$u_\epsilon(x) = \int\limits_{B(0;1)} u(x-\epsilon y)\varphi(|y|) = \int\limits_0^1 \varphi(t)L(t\epsilon)\mathcal{H}_{m-1}(\partial B(0;t))\,\mathrm{d}t \uparrow u(x)$$

as $\epsilon \downarrow 0$. The function u_ϵ is superharmonic by Lemma 3.8.1. If $u \in H(\Omega)$, then $u_\epsilon \le u$, $-u_\epsilon = (-u)_\epsilon \le -u$ and thus $u_\epsilon = u$.

3.9 Green's Function

This section is devoted to the Green function. The Green function and the Green potential play an important role in the study of boundary value problems for the Laplace equation. We characterize open sets for which there exists Green's function and study its properties. We show that Green's function is symmetric and infinitely differentiable outside the diagonal and give estimates of Green's function and its derivatives. We also compare Green's functions corresponding to different open sets.

Definition 3.9.1 Let $\Omega \subset R^m$ be an open set, $m \ge 2$, and g_Ω be a real function on $\Omega \times \Omega$ such that

- $g_\Omega(\cdot, y) \in H(\Omega)$ for each $y \in \Omega$;
- $g_\Omega(x,y) \le h_m(x-y)$;
- $y \in \Omega, u \in H(\Omega), u(x) \le h_m(x-y) \Rightarrow u(x) \le g_\Omega(x,y)$.

Then

$$G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y), \qquad x,y \in \Omega$$

is called the Green function of Ω.

Proposition 3.9.2 *Let $\Omega = R^m$, $m > 2$. Then $G_\Omega(x,y) = h_m(x-y)$ is Green's function corresponding to Ω.*

Proof Put $g_\Omega(x,y) = 0$. Fix $y \in \Omega$. Then $g_\Omega(\cdot,y) \in H(\Omega)$, $g_\Omega(x,y) = 0 \le h_m(x-y)$. Let now $u \in H(R^m)$, $u(x) \le h_m(x-y)$ for all $x \in R^m$. Since

$$\limsup_{|x|\to\infty} u(x) \le \limsup_{|x|\to\infty} h_m(x-y) = 0$$

Corollary 2.4.4 gives that $u \le 0$.

Proposition 3.9.3 *Let $\Omega = R^2$. Then there is not Green's function corresponding to Ω.*

Proof Fix $y \in \Omega$. We show that there is not $u \in H(\Omega)$ such that $u(x) \leq h_2(|x-y|)$ for all $x \in R^2$. Suppose that there is such u. Fix $z \in R^2$. According to Mean value theorem (Theorem 2.6.1),

$$u(z) = \lim_{r\to\infty} \oint\limits_{\partial B(x;r)} u(x)\, \mathrm{d}\mathcal{H}_{m-1} \leq \lim_{r\to\infty} \oint\limits_{\partial B(x;r)} h_2(|x-y|)\, \mathrm{d}\mathcal{H}_{m-1} = -\infty$$

what is a contradiction.

Proposition 3.9.4 *Let* $\Omega = R^m_+ = \{[x', x_m]; x_m > 0\}$. *Let us denote* $\Phi(x', x_m) = [x', -x_m]$. *The Green function* $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ *is given by* $g_\Omega(x,y) = h_m(x-\Phi(y))$.

Proof Fix $y \in \Omega$ and put $v(x) = h_m(x-\Phi(y))$. Clearly, $v \in H(\Omega)$, $v(x) \leq h_m(x-y)$. Let now $u \in H(\Omega)$ and $u(x) \leq h_m(x-y)$. Put $w = u - v$. If $z \in \partial\Omega$, then

$$\limsup_{x\to z} w(x) \leq \limsup_{x\to z}[u(x) - h_m(x-y)] + \limsup_{x\to z}[h_m(x-y) - v(x)] \leq 0,$$

$$\limsup_{|x|\to\infty} w(x) \leq \limsup_{|x|\to\infty}[u(x) - h_m(x-y)] + \limsup_{|x|\to\infty}[h_m(x-y) - v(x)] \leq 0.$$

Corollary 2.4.4 gives that $w \leq 0$. Therefore $g_\Omega(x,y) = v(x)$.

Proposition 3.9.5 *Let* Ω, ω *be open subsets of* R^m, $m \geq 2$, *and* $\omega \subset \Omega$. *If there is the Green function* G_Ω *of* Ω, *then there is the Green function* G_ω *of* ω *and* $G_\omega \leq G_\Omega$ *on* $\omega \times \omega$.

Proof Fix $y \in \omega$. Then $g(x) = h_m(x-y)$ is a superharmonic function in ω (see Example 3.3.9), $f(x) = g_\Omega(x,y)$ is a harmonic function in Ω, and $f \leq g$. Put $v = \sup\{u; u$ subharmonic in $\omega, u \leq g\}$. Then $v \in H(\omega)$ by Proposition 3.6.4. Clearly $g_\omega(x,y) = v(x)$. Since $f \leq g$, we have $f \leq v$ and thus $G_\omega(x,y) = g(x) - v(x) \leq g(x) - f(x) = G_\Omega(x,y)$.

Corollary 3.9.6 *Let* $\Omega \subset R^m$ *be open,* $m > 2$. *Then there is unique Green's function of* Ω.

Proof Since there is the Green function of R^m by Proposition 3.9.2, Proposition 3.9.5 gives the existence of the Green function of Ω. The uniqueness follows from the definition of Green's function.

Proposition 3.9.7 *Let* $\Omega \subset R^m$ *be an open set,* $m \geq 2$. *Then there exists Green's function of* Ω *if and only if there exists Green's function for each component of* Ω. *If* G_Ω *is the Green function of* Ω, ω *is a component of* Ω, *and* $y \in \omega$, *then* $G_\Omega(x,y) = G_\omega(x,y) > 0$ *for* $x \in \omega$, $G_\Omega(x,y) = 0$ *for* $x \in \Omega \setminus \omega$.

Proof If there is Green's function of Ω, then there exists Green's function for each component of Ω by Proposition 3.9.5. Let now ω be a component of Ω, $G_\omega(x,y) = h_m(x-y) - g_\omega(x,y)$ be the Green function of ω. Fix $y \in \omega$. Put $g_\Omega(x,y) = g_\omega(x,y)$

for $x \in \omega$, $g_\Omega(x,y) = h_m(x-y)$ for $x \in \Omega \setminus \omega$. Then $g_\Omega(x,y) \leq h_m(x-y)$. Since $g_\omega(\cdot,y) \in H(\omega)$, $v(x) = h_m(x-y) \in H(\Omega \setminus \omega)$ (see Remark 2.1.3) we infer that $g_\Omega(\cdot,y) \in H(\Omega)$. Let now $u \in H(\Omega)$ be such that $u(x) \leq h_m(x-y)$. Clearly $u(x) \leq g_\Omega(x,y)$ on $\Omega \setminus \omega$. For $x \in \omega$ we have $u(x) \leq g_\omega(x) = g_\Omega(x)$. Thus $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ is the Green function of Ω. By the definition of Green's function, we see that $G_\Omega \geq 0$. Since $G_\Omega(\cdot,y)$ is a superharmonic function in ω which is nonconstant because

$$\lim_{x\to y} G_\Omega(x,y) = \lim_{x\to y}[h_m(x-y) - g_\Omega(x,y)] = \infty,$$

Proposition 3.4.1 gives that $G_\Omega(x,y) > 0$ for $x \in \omega$.

Proposition 3.9.8 *Let $\Omega \subset R^2$ be open. If $R^2 \setminus \partial\Omega$ is not connected, then there is Green's function of Ω. In particular, if $\overline{\Omega} \neq R^2$ then there is Green's function of Ω.*

Proof Suppose that $R^2 \setminus \partial\Omega$ is not connected. Let ω be a component of Ω. Then there is $z \in R^2$ and $r > 0$ such that $B(z;r) \cap \omega = \emptyset$. Fix $y \in \omega$. Since $\partial B(z;r)$ is a compact set (see Example 1.2.8), there is a positive constant c such that $|h_2(|x-y|) - h_2(|x-z|)| \leq c$ for each $x \in \partial B(z;r)$ (see Lemma 1.2.13 and Example 1.2.8). Put $u(x) = -c + h_2(|x-z|) - h_2(|x-y|)$. Then $u \in H(R^2 \setminus \overline{B(z;r)} \setminus \{y\})$. Since $u \leq 0$ on $\partial B(z;r)$,

$$\lim_{x\to y} u(x) = -\infty,$$

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty}\left[-c - \frac{1}{2\pi}\ln\frac{|x-z|}{|x-y|}\right] = -c \leq 0,$$

Corollary 2.4.4 gives that $u \leq 0$ in $R^2 \setminus \overline{B(z;r)} \setminus \{y\}$. Put $v(x) = -c + h_2(|x-z|)$. Then $v \in H(\omega)$, $v(x) \leq h_2(|x-y|)$ in ω. According to Proposition 3.6.4, the function $u_\omega(x,y) = \sup\{h; h \text{ subharmonic in } \omega, h(x) \leq h_2(|x-y|)\}$ has properties from the definition of Green's function of ω. Since there is Green's function for each component ω of Ω, Proposition 3.9.7 gives that there is Green's function of Ω.

Lemma 3.9.9 *Let $\Omega \subset R^m$ be a domain with compact boundary, $m \geq 2$, and $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. Fix $y \in \Omega$. Suppose that $u \in H(\Omega) \cap C^0(\overline{\Omega})$, $u(x) = h_m(x-y)$ for each $x \in \partial\Omega$. If Ω is unbounded, suppose moreover $u(x) - h_m(x-y) = O(|x|^{2-m})$ as $|x| \to \infty$. Then $g_\Omega(x,y) = u(x)$.*

Proof Fix $\delta > 0$ such that $u(x) < h_m(x-y)$ on $\overline{B(y;\delta)}$. Since the function $v(x) = h_m(x-y) \in H(\Omega \setminus \overline{B(y;\delta)})$ by Remark 2.1.3, we have $u \leq v$ in Ω by Corollary 2.4.4 and Proposition 2.4.7. According to the definition of Green's function, we have $u(x) \leq g_\Omega(x,y)$. The function $w(x) = g_\Omega(x,y) - u(x)$ is a nonnegative harmonic function in Ω. If $z \in \partial\Omega$, then

$$0 \leq \liminf_{\Omega\ni x\to z} w(x) \leq \limsup_{\Omega\ni x\to z} w(x) \leq \limsup_{\Omega\ni x\to z}[h_m(x-y) - u(x)] = 0.$$

If Ω is unbounded, we obtain by a similar way that $w(x) \to 0$ as $|x| \to \infty$ for $m > 2$, $w(x) = O(1)$ as $|x| \to \infty$ for $m = 2$. Corollary 2.4.4 and Proposition 2.4.7 give that $w \equiv 0$.

Example 3.9.10 Let $m > 2$, $r > 0$. Denote by $y^* = yr^2|y|^{-2}$. If $\Omega = B(0;r) \subset R^m$, then Lemma 2.3.2 and Lemma 3.9.9 give

$$G_\Omega(x,0) = h_m(x) - \frac{r^{2-m}}{(m-2)\mathcal{H}_{m-1}(\partial B(0;1))},$$

$$G_\Omega(x,y) = h_m(x-y) - h_m(x-y^*)\left(\frac{r}{|y|}\right)^{m-2}, \quad y \neq 0.$$

If $\Omega = R^m \setminus \overline{B(z;r)}$, then Lemma 2.3.2 and Lemma 3.9.9 give

$$G_\Omega(x,y) = h_m(x-y) - h_m(x-y^*)\left(\frac{r}{|y|}\right)^{m-2}.$$

Example 3.9.11 Fix $r > 0$. Denote $y^* = yr^2|y|^{-2}$. If $\Omega = B(0;r) \subset R^2$ then Lemma 2.3.2 and Lemma 3.9.9 give

$$G_\Omega(x,0) = \frac{1}{2\pi}\ln\frac{r}{|x|},$$

$$G_\Omega(x,y) = \frac{1}{2\pi}\left[\ln\frac{1}{|x-y|} - \ln\frac{1}{|x-y^*|} + \ln\frac{|y|}{r}\right], \quad y \neq 0.$$

If $\Omega = R^2 \setminus \overline{B(z;r)}$, then Lemma 2.3.2 and Lemma 3.9.9 give

$$G_\Omega(x,y) = \frac{1}{2\pi}\left[\ln\frac{1}{|x-y|} - \ln\frac{1}{|x-y^*|} + \ln\frac{|y|}{r}\right].$$

Proposition 3.9.12 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, and $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. Then $G_\Omega(x,y) = G_\Omega(y,x)$ for each $x,y \in \Omega$ and $g_\Omega \in H(\Omega \times \Omega)$.*

Proof Fix an open ball B_0 such that $\overline{B_0} \subset \Omega$. Choose a sequence of open balls $\{B_n; n \in N\}$ such that $\overline{B_n} \subset \Omega$, $\overline{B_0} \subset B_1$, $B_j \cap B_0 = \emptyset$ for $j \neq 1$ and

$$\Omega = \bigcup_{j=1}^{\infty} B_j.$$

Let now $\{V(n); n \in N\}$ be a sequence of open balls such that $V(1) = B_1$, for each $n \in N$ there is $k \in N$ such that $V(n) = B_k$; for each $k \in N$ there is an increasing sequence of positive integers n_j such that $V(n_j) = B_k$. Put $v_0(x,y) = h_m(x-y)$ for

$x \in \Omega$, $y \in B_0$. Let $P_j(x,z)$ be the Poisson kernel corresponding to the ball $V(j)$. For $y \in B_0$ and $n \in N$, put

$$v_n(x,y) = \begin{cases} v_{n-1}(x,y), & x \in \Omega \setminus V(n), \\ \int\limits_{\partial V(n)} P_n(x,z)v_{n-1}(z,y)\, \mathrm{d}\mathcal{H}_{m-1}(z), & x \in V(n). \end{cases} \tag{3.11}$$

Then for each $n \in N$:

1. $v_n \in \mathcal{C}(\Omega \times B_0)$.
2. If $y \in B_0$ then $v_n(\cdot,y) \in H(V(n))$.
3. $g_\Omega(x,y) \le v_n(x,y) \le h_m(x-y)$ for each $x \in \Omega$, $y \in B_0$.
4. If $x \in \Omega$ then $v_n(x,\cdot) \in H(B_0)$.
5. If $y \in B_0$ then $v_n(\cdot,y)$ is superharmonic in Ω.
6. $v_{n-1} \ge v_n$.

Suppose first that $n = 1$. Let $y \in B_0$. Since $v_0(\cdot,y)$ is a superharmonic function (see Example 3.3.9), Proposition 3.6.2 gives that $v_1(\cdot,y) \in H(V(1))$, $v_1(\cdot,y)$ is superharmonic in Ω and $v_1(x,y) \le v_0(x,y) = h_m(x-y)$. Clearly, $g_\Omega(x,y) \le h_m(x-y) = v_1(x,y)$ for $x \in \Omega \setminus V(1)$. Since $g_\Omega(\cdot,y) \in H(V(1)) \cap \mathcal{C}(\overline{V(1)})$, $g_\Omega(x,y) \le h_m(x-y) = v_1(x,y)$ for $x \in \partial V(1)$, Proposition 3.3.7 gives $g_\Omega(x,y) \le v_1(x,y)$ for $x \in V(1)$. If $y \in B_0$, $x \in \Omega \setminus V(1)$, then $v_1(x,y) = h_m(x-y)$ and thus $v_1 \in \mathcal{C}((\Omega \setminus V(1)) \times B_0)$. Let now $x_k \in V(1)$, $x_k \to x$, $y_k \in B_0$, $y_k \to y \in B_0$. Fix $\epsilon > 0$. Since $v_1(\cdot,y) \in \mathcal{C}(\overline{V(1)})$ by Theorem on the Poisson integral (Theorem 2.5.4), there is $k_0 \in N$ such that $|v_1(x,y) - v_1(x_k,y)| < \epsilon$ for $k \ge k_0$. Moreover, there is $k_1 \ge k_0$ such that for $k \ge k_1$, we have $|v_0(z,y_k) - v_0(z,y)| < \epsilon$ for all $z \in \partial V(1)$. Since $v_1(\cdot,y_k) - v_1(\cdot,y) \in H(V(1)) \cap \mathcal{C}(\overline{V(1)})$ by Theorem on the Poisson integral (Theorem 2.5.4) and $|v_1(z,y_k) - v_1(z,y)| = |v_0(z,y_k) - v_0(z,y)| \le \epsilon$ on $\partial V(1)$, Maximum principle (Theorem 2.4.2) gives that $|v_1(\cdot,y_k) - v_1(\cdot,y)| \le \epsilon$ on $\overline{V(1)}$. Since

$$|v_1(x_k,y_k) - v_1(x,y)| \le |v_1(x_k,y_k) - v_1(x_k,y)| + |v_1(x_k,y) - v_1(x,y)| < 2\epsilon,$$

we see that $v_1 \in \mathcal{C}(\Omega \times B_0)$. If $x \in \Omega \setminus V(1)$ then $v_1(x,\cdot) = v_0(x,\cdot) \in H(B_0)$. Fix $x \in V(1)$, $y \in B_0$, $0 < r < \operatorname{dist}(y, \partial B_0)$. Since $v_0(z,\cdot) \in H(B_0)$ for each $z \in \partial V(1)$, we get using Fubini's theorem and Mean value theorem (Theorem 2.6.1)

$$\oint\limits_{B(y;r)} v_1(x,w)\, \mathrm{d}\mathcal{H}_m(w) = \oint\limits_{B(y;r)} \int\limits_{\partial V(1)} P_1(x,z)v_0(z,w)\, \mathrm{d}\mathcal{H}_{m-1}(z)\, \mathrm{d}\mathcal{H}_m(w)$$

$$= \int\limits_{\partial V(1)} P_1(x,z) \oint\limits_{B(y;r)} v_0(z,w)\, \mathrm{d}\mathcal{H}_m(w)\, \mathrm{d}\mathcal{H}_{m-1}(z)$$

$$= \int\limits_{\partial V(1)} P_1(x,z) v_0(z,y) \, \mathrm{d}\mathcal{H}_{m-1}(z) = v_1(x,y).$$

Mean value theorem (Theorem 2.6.1) gives that $v_1(x,\cdot) \in H(B_0)$.

Suppose now that (1)–(6) hold for some $n \in N$. Fix $y \in B_0$. Since $v_n(\cdot,y)$ is a superharmonic function, Proposition 3.6.2 gives that $v_{n+1}(\cdot,y)$ is superharmonic in Ω, $v_{n+1}(\cdot,y) \in H(V(n+1))$, $v_{n+1}(x,y) \le v_n(x,y) \le h_m(x-y)$. Clearly, $g_\Omega(x,y) \le v_n(x,y|) = v_{n+1}(x,y)$ for $x \in \Omega \setminus V(n+1)$. Since $g_\Omega(\cdot,y) \in H(V(n+1)) \cap \mathcal{C}(\overline{V(n+1)})$, $g_\Omega(x,y) \le v_n(x,y) = v_{n+1}(x,y)$ for $x \in \partial V(n+1)$, Proposition 3.3.7 gives $g_\Omega(x,y) \le v_{n+1}(x,y)$ for $x \in V(n+1)$. If $y \in B_0$, $x \in \Omega \setminus V(n+1)$ then $v_{n+1}(x,y) = v_n(x,y)$ and thus $v_{n+1} \in \mathcal{C}((\Omega \setminus V(n+1)) \times B_0)$. Let now $x_k \in V(n+1)$, $x_k \to x$, $y_k \in B_0$, $y_k \to y \in B_0$. Fix $\epsilon > 0$. Since $v_{n+1}(\cdot,y) \in \mathcal{C}(\overline{V(n+1)})$ by Theorem on the Poisson integral (Theorem 2.5.4), there is $k_0 \in N$ such that $|v_{n+1}(x,y) - v_{n+1}(x_k,y)| < \epsilon$ for $k \ge k_0$. Since $v_n \in \mathcal{C}(\Omega \times B_0)$, it is uniformly continuous on the compact set $[\partial V(n+1)] \times [\{y\} \cup \{y_k; k \in N\}]$ (see Lemma 1.2.15 and Example 1.2.8), and thus there is $k_1 \ge k_0$ such that for $k \ge k_1$, we have $|v_n(z,y_k) - v_n(z,y)| < \epsilon$ for all $z \in \partial V(n+1)$. Since $v_{n+1}(\cdot,y_k) - v_{n+1}(\cdot,y) \in H(V(n+1)) \cap \mathcal{C}(\overline{V(n+1)})$ by Theorem on the Poisson integral (Theorem 2.5.4) and

$$|v_{n+1}(z,y_k) - v_{n+1}(z,y)| = |v_n(z,y_k) - v_n(z,y)| \le \epsilon$$

for $z \in \partial V(n+1)$, Maximum principle (Theorem 2.4.2) gives that

$$|v_{n+1}(\cdot,y_k) - v_{n+1}(\cdot,y)| \le \epsilon \quad \text{on } \overline{V(n+1)}.$$

Since

$$\begin{aligned}|v_{n+1}(x_k,y_k) - v_{n+1}(x,y)| &\le |v_{n+1}(x_k,y_k) - v_{n+1}(x_k,y)| \\ &\quad + |v_{n+1}(x_k,y) - v_{n+1}(x,y)| < 2\epsilon,\end{aligned}$$

we see that $v_{n+1} \in \mathcal{C}(\Omega \times B_0)$. If $x \in \Omega \setminus V(n+1)$ then $v_{n+1}(x,\cdot) = v_n(x,\cdot) \in H(B_0)$. Fix $x \in V(n+1)$, $y \in B_0$, $0 < r < \operatorname{dist}(y,\partial B_0)$. Since $v_n(z,\cdot) \in H(B_0)$ for each $z \in \partial V(n+1)$, we get using Fubini's theorem and Mean value theorem (Theorem 2.6.1)

$$\oint\limits_{B(y;r)} v_{n+1}(x,w) \, \mathrm{d}\mathcal{H}_m(w)$$

$$= \oint\limits_{B(y;r)} \int\limits_{\partial V(n+1)} P_{n+1}(x,z) v_n(z,w) \, \mathrm{d}\mathcal{H}_{m-1}(z) \, \mathrm{d}\mathcal{H}_m(w)$$

$$= \int\limits_{\partial V(n+1)} P_{n+1}(x,z) \oint\limits_{B(y;r)} v_n(z,w)\, \mathrm{d}\mathcal{H}_m(w)\, \mathrm{d}\mathcal{H}_{m-1}(z)$$

$$= \int\limits_{\partial V(n+1)} P_{n+1}(x,z) v_n(z,y)\, \mathrm{d}\mathcal{H}_{m-1}(z) = v_{n+1}(x,y).$$

Mean value theorem (Theorem 2.6.1) gives that $v_{n+1}(x,\cdot) \in H(B_0)$.

For $x \in \Omega$, $y \in B_0$ denote

$$v(x,y) = \lim_{n\to\infty} v_n(x,y).$$

Then $g_\Omega(x,y) \le v(x,y) \le h_m(x-y)$. Fix $k \in N$. Then there is an increasing sequence of positive integers $n(j)$ such that $V(n(j)) = B_k$. If $x \in \Omega$ then $v_{n(j)}(x,\cdot) \in H(B_0)$. If $y \in B_0$ then $v_{n(j)}(\cdot,y) \in H(B_k)$. Since $v_{n(j)}$ is continuous, Theorem 2.7.1 gives that $v_{n(j)} \in H(B_k \times B_0)$. Since $v_{n(j)}$ is a decreasing sequence of functions and $g_\Omega(x,y) \le v(x,y)$, Proposition 2.11.3 gives that $v \in H(B_k \times B_0)$, $v(x,\cdot) \in H(B_0)$ for each $x \in \Omega$, $v(\cdot,y) \in H(B_k)$ for each $y \in B_0$. Since B_k was arbitrary we infer that $v \in H(\Omega \times B_0)$, $v(\cdot,y) \in H(\Omega)$ for each $y \in B_0$. Since $v(\cdot,y) \in H(\Omega)$, $v(x,y) \le h_m(x-y)$ we obtain from the definition of Green's function that $v(\cdot,y) \le g_\Omega(\cdot,y)$ for each $y \in B_0$. Since $g_\Omega(\cdot,y) \le v(\cdot,y)$ we deduce that $v = g_\Omega$. This forces that $g_\Omega \in H(\Omega \times B_0)$, $g_\Omega(x,\cdot) \in H(B_0)$ for each $x \in \Omega$. Since B_0 was an arbitrary ball in Ω, we have $g_\Omega \in H(\Omega \times \Omega)$, $g_\Omega(x,\cdot) \in H(\Omega)$ for each $x \in \Omega$. Since $g_\Omega(x,\cdot) \in H(\Omega)$, $g_\Omega(x,y) \le h_m(x-y)$ we obtain from the definition of Green's function that $g_\Omega(x,\cdot) \le g_\Omega(\cdot,x)$. Since $g_\Omega(x,y) \le g_\Omega(y,x)$ and $g_\Omega(y,x) \le g_\Omega(x,y)$, we infer that $g_\Omega(x,y) = g_\Omega(y,x)$.

Theorem 3.9.13 *Let $\Omega \subset R^m$, $m \ge 2$, be an open set with the Green function $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$. Let Ω_n be an increasing sequence of open sets with the Green functions $G_n(x,y) = h_m(x-y) - g_n(x,y)$. If $\Omega = \cup\Omega_n$ then G_n converges to G_Ω locally uniformly on $\Omega \times \Omega$.*

Proof According to Proposition 3.9.5, we have $G_n \le G_{n+1} \le G_\Omega$. Thus $g_\Omega(x,y) \le g_{n+1}(x,y) \le g_n(x,y) \le h_m(x-y)$ for each $x, y \in \Omega$. Put

$$v(x,y) = \lim_{n\to\infty} g_n(x,y).$$

Then $g_\Omega(x,y) \le v(x,y) \le h_m(x-y)$. Fix $y \in \Omega$. Since $g_n(\cdot,y) \in H(\Omega_n)$, Proposition 2.11.3 gives that $v(\cdot,y) \in H(\Omega)$. Since $v(\cdot,y) \in H(\Omega)$ and $v(x,y) \le h_m(x-y)$, it follows from the definition of Green's function that $v(x,y) \le g_\Omega(x,y)$. Since $g_\Omega \le v$ we have $v = g_\Omega$. Since $g_n \in H(\Omega_n \times \Omega_n)$, $g_\Omega \in H(\Omega \times \Omega)$ by Proposition 3.9.12, g_n converges locally uniformly to g_Ω on $\Omega \times \Omega$ by Proposition 2.11.3.

Proposition 3.9.14 *Let $\Omega \subset R^m$ be an open set, $m > 2$, and $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. Then $G_\Omega(x,y) \le h_m(x-y)$, $|\partial_x^\alpha G_\Omega(x,y)| \le$*

$(m|\alpha|)^{|\alpha|}h_m(x-y)[\mathrm{dist}(x,\partial\Omega\cup\{y\})]^{-|\alpha|}$ *for all* $x,y\in\Omega$ *and each multiindex* α *with* $|\alpha|>0$.

Proof Since $0\le h_m(x-y)$, we have $0\le g_\Omega(x,y)$ and thus $G_\Omega(x,y)=h_m(x-y)-g_\Omega(x,y)\le h_m(x-y)$. Fix $y\in\Omega$. Since $G_\Omega(\cdot,y)$ is a nonnegative harmonic function on $\Omega\setminus\{y\}$, Theorem 2.9.5 gives $|\partial_x^\alpha G_\Omega(x,y)|\le(m|\alpha|)^{|\alpha|}G_\Omega(x,y)[\mathrm{dist}(x,\partial(\Omega\setminus\{y\}))]^{-|\alpha|}\le(m|\alpha|)^{|\alpha|}h_m(x-y)[\mathrm{dist}(x,\partial\Omega\cup\{y\})]^{-|\alpha|}$.

Proposition 3.9.15 *Let* $\Omega\subset R^2$ *be an open set with the Green function* G_Ω, $y\in\Omega$. *If* Ω *is bounded then* $G_\Omega(x,y)\le(2\pi)^{-1}\ln(\mathrm{diam}\,\Omega/|x-y|)$, $|\partial_x^\alpha G_\Omega(x,y)|\le(2|\alpha|)^{|\alpha|}(2\pi)^{-1}[\ln(\mathrm{dist}(y;\partial\Omega)/|x-y|)][\mathrm{dist}(x,\partial\Omega\cup\{y\})]^{-|\alpha|}$ *for all* $x,y\in\Omega$ *and each multiindex* α *with* $|\alpha|>0$. *If* $|x-y|<\mathrm{dist}(y,\partial\Omega)$ *then* $G_\Omega(x,y)\ge(2\pi)^{-1}\ln[\mathrm{dist}(y,\partial\Omega)/|x-y|]$.

Proof Suppose first that Ω is bounded. Then $\Omega\subset\omega\equiv B(x;\mathrm{diam}\,\Omega)$. So, $G_\Omega(x,y)\le G_\omega(x,y)=(2\pi)^{-1}\ln(\mathrm{diam}\,\Omega/|x-y|)$ by Proposition 3.9.5 and Example 3.9.11. Fix $y\in\Omega$. Since $G_\Omega(\cdot,y)$ is a nonnegative harmonic function on $\Omega\setminus\{y\}$, $|\partial_x^\alpha G_\Omega(x,y)|\le(2|\alpha|)^{|\alpha|}G_\Omega(x,y)[\mathrm{dist}(x,\partial(\Omega\setminus\{y\}))]^{-|\alpha|}\le(2|\alpha|)^{|\alpha|}(2\pi)^{-1}[\ln(\mathrm{dist}(y;\partial\Omega)/|x-y|)][\mathrm{dist}(x,\partial\Omega\cup\{y\})]^{-|\alpha|}$ by Theorem 2.9.5.

Suppose now $|x-y|<\mathrm{dist}(y,\partial\Omega)$. Since $\Omega\ne R^2$ (see Proposition 3.9.3), we have $\partial\Omega\ne\emptyset$. Denote $\omega=B(x;\mathrm{dist}(y;\partial\Omega))$. Then

$$G_\Omega(x,y)\ge G_\omega(x,y)=(2\pi)^{-1}\ln(\mathrm{dist}(y;\partial\Omega)/|x-y|)$$

by Proposition 3.9.5 and Lemma 3.9.9.

Proposition 3.9.16 *Let* $\Omega\subset R^m$ *be an open set with the Green function* G_Ω, $m\ge 2$, *and let* W *be an open neighborhood of* y *such that* $\overline{W}\subset\Omega$. *If* u *is a positive superharmonic function on* Ω *and* $u\ge G_\Omega(\cdot,y)$ *on* ∂W, *then* $u\ge G_\Omega$ *on* $\Omega\setminus W$. *In particular,* $G_\Omega(\cdot,y)$ *is bounded on* $\Omega\setminus W$.

Proof Since $h_m(x-y)-g_\Omega(x,y)=G_\Omega(x,y)\le u(x)$ on ∂W, we have $\max(g_\Omega(x,y),h_m(x-y)-u(x))=g_\Omega(x,y)$ on ∂W. Put

$$s(x)=\begin{cases}\max(g_\Omega(x,y),h_m(x-y)-u(x)), & x\in\Omega\setminus\overline{W},\\ g_\Omega(x,y), & x\in\overline{W},\end{cases}\tag{3.12}$$

Since $g_\Omega(\cdot,y)\in H(\Omega)$, $h_m(\cdot-y)\in H(\Omega\setminus\{y\})$ (see Remark 2.1.3), and $-u$ is a subharmonic function, the function s is a subharmonic function on W and on $\Omega\setminus\overline{W}$ by Lemma 3.3.3. Since $g_\Omega(x,y)$ and $h_m(x-y)-u(x)$ are subharmonic functions on $\Omega\setminus\{y\}$ and $\max(g_\Omega(x,y),h_m(x-y)-u(x))=g_\Omega(x,y)$ on ∂W, the function s is upper semicontinuous on Ω. Since $g_\Omega(x,y)\le h_m(x-y)$ and $u\ge 0$, we infer $g_\Omega(x,y)\le s(x)\le h_m(x-y)$, and thus $s\in L^1_{loc}(\Omega)$. If $x\in\partial W$ and $0<r<\mathrm{dist}(x,\partial\Omega)$, then

$$s(x)=g_\Omega(x,y)=\oint_{B(x;r)}g_\Omega(z,y)\,\mathrm{d}\mathcal{H}_m(y)\le\oint_{B(x;r)}s(z)\,\mathrm{d}\mathcal{H}_m(y)$$

by Mean value theorem (Theorem 2.6.1). So, the function s is subharmonic on Ω by Proposition 3.3.7. According to Proposition 3.6.4, there is $v \in H(\Omega)$ such that $s \leq v \leq h_m(\cdot - y)$. From the definition of the Green function, we see that $v \leq g_\Omega(\cdot, y)$, and thus $s \leq g_\Omega(\cdot, y)$. This forces $h_m(x-y) - u(x) \leq g_\Omega(x, y)$ on $\Omega \setminus W$ and $u \geq G_\Omega$ on $\Omega \setminus W$.

Fix $r > 0$ such that $B(y; r) \subset W$. Since the function $G_\Omega(\cdot, y)$ is continuous on the compact set $\partial B(y; r)$, there is a constant C such that $|G_\Omega(\cdot, y)| \leq C$ on $\partial B(y; r)$ (see Lemma 1.2.13 and Example 1.2.8). Thus $|G_\Omega(\cdot, y)| \leq C$ on $\Omega \setminus B(y; r) \supset \Omega \setminus W$.

3.10 Logarithmic Potentials

Volume potentials and Green's potentials corresponding to μ are important solutions of the Poisson equation $-\Delta u = \mu$. If the dimension of the space is greater than 2, then a volume potential is a Green potential for the whole space. But the Green function for the plane does not exists. So, we study a planar volume potential (so-called logarithmic potential) separately.

Definition 3.10.1 If μ is a real Radon measure on R^2, denote

$$V\mu(x) = \int\limits_{R^2} h_2(|x-y|)\, \mathrm{d}\mu(y)$$

whenever the integral makes sense. The function $V\mu$ is called the logarithmic potential of μ. We also call it the volume potential of μ.

Proposition 3.10.2 *Let μ be a nonnegative Radon measure on R^2, $r > 0$. If*

$$\int\limits_{R^2 \setminus B(0;r)} \ln|y|\, \mathrm{d}\mu(y) = \infty, \tag{3.13}$$

then $V\mu(x) = -\infty$ whenever it makes sense, i.e., $\mathcal{H}_2$—almost everywhere. If

$$\int\limits_{R^2 \setminus B(0;r)} \ln|y|\, \mathrm{d}\mu(y) < \infty, \tag{3.14}$$

then $V\mu$ is a superharmonic function in R^2 which is harmonic in $R^2 \setminus \operatorname{spt}\mu$. In particular, if μ has compact support, then $V\mu$ is a superharmonic function in R^2.

Proof Suppose first that μ has a compact support. Since μ is a Radon measure and $\operatorname{spt}\mu$ is compact, we have $\mu(\operatorname{spt}\mu) < \infty$. Put

$$F_k\mu(x) = \int \min(k, h_2(|x-y|))\, \mathrm{d}\mu(y).$$

Then $F_k\mu$ are continuous functions by theorem on continuity of integrals (Theorem 1.13.6). Since $F_k\mu \uparrow V\mu$, the function $V\mu$ is lower semicontinuous by Lemma 3.2.4. If $0 < r$, then Fubini's theorem gives

$$\oint\limits_{B(z;r)} V\mu(x)\,\mathrm{d}\mathcal{H}_m(x) = \int\limits_{R^2}\oint\limits_{B(z;r)} h_2(|x-y|)\,\mathrm{d}\mathcal{H}_m(x)\,\mathrm{d}\mu(y)$$

$$\leq \int\limits_{R^2} h_2(|z-y|)\,\mathrm{d}\mu(y) = V\mu(z).$$

Since $V\mu \in H(R^2 \setminus \operatorname{spt}\mu)$ by Theorem on differentiability of parametrized integrals (Theorem 1.13.7), we see that $V\mu$ is superharmonic on R^2.

Let now μ be arbitrary. Fix $\rho \in (r,\infty)$. Denote $\mu_0(A) = \mu(A \cap B(0;2\rho))$ and $\mu_k(A) = \mu(A \cap B(0;2k\rho) \setminus B(0;2\rho))$. We have proved that $V\mu_0$ is superharmonic and $V\mu_k \in H(B(0;\rho))$ for $k \in N$. Proposition 3.3.4 gives that $V\mu_0$ is finite a.e. in R^2. Clearly $V\mu_k \downarrow V(\mu-\mu_0)$ as $k \to \infty$. If (3.13) holds then $V(\mu-\mu_0)(0) = -\infty$ and thus $V(\mu-\mu_0) = -\infty$ in $B(0;\rho)$ by Proposition 2.11.3. If (3.14) holds then $V(\mu-\mu_0)(0)$ is finite and thus $V(\mu-\mu_0) \in H(B(0;\rho))$ by Proposition 2.11.3. So, $V\mu = V\mu_0 + V(\mu-\mu_0)$ is superharmonic in $B(0;\rho)$ by Lemma 3.3.3. Since ρ was arbitrary, $V\mu$ is superharmonic in R^2 by Corollary 3.3.8.

Theorem 3.10.3 *Let μ be a real Radon measure on R^2 such that $V\mu(z)$ is finite for some $z \in R^2$. Then $-\Delta V\mu = \mu$ in the sense of distributions.*

Proof Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Choose bounded open sets $U, W \subset R^2$ with smooth boundary such that $\operatorname{spt}\varphi \subset W \subset \overline{W} \subset U$. Put $\mu_1 = \mu|\overline{U}$, $\mu_2 = \mu - \mu_1$. Proposition 3.10.2 gives that $V\mu_2 \in H(U)$. The second Green formula (Theorem 2.2.1) gives

$$\int\limits_{R^2}(-\Delta\varphi)V\mu_2 = \int\limits_{W}[\varphi\Delta V\mu_2 - (\Delta\varphi)V\mu_2] - \int\limits_{\partial W}\left[\varphi\frac{\partial V\mu_2}{\partial n} - (V\mu_2)\frac{\partial\varphi}{\partial n}\right] = 0.$$

Since $-\Delta V\mu_1 = \mu_1$ in the sense of distributions, by Theorem 3.1.5, one has

$$\int\limits_{R^2}(-\Delta\varphi)V\mu\,\mathrm{d}\mathcal{H}_2 = \int\limits_{R^2}(-\Delta\varphi)V\mu_1\,\mathrm{d}\mathcal{H}_2 = \int\limits_{R^2}\varphi\,\mathrm{d}\mu_1 = \int\limits_{R^2}\varphi\,\mathrm{d}\mu.$$

3.11 Green's Potentials

In this section we study properties of the Green potential $G_\Omega\mu$ for a nonnegative Radon measure μ. We prove that $G_\Omega\mu$ is a superharmonic function, $-\Delta G_\Omega\mu = \mu$, and that the zero function is the greatest harmonic minorant of $G_\Omega\mu$.

Definition 3.11.1 Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \geq 2$. If μ is a nonnegative Radon measure on Ω, denote

$$G_\Omega \mu(x) = \int\limits_\Omega G_\Omega(x, y)\, \mathrm{d}\mu(y), \quad x \in \Omega,$$

the Green potential of μ. If $\mu = \mu^+ - \mu^-$ is a real Radon measure, we write $G\mu = G\mu^+ - G\mu^-$ whenever it makes sense. If $\Omega = R^m$ with $m > 2$, we also call it the Newton potential of μ (or the volume potential of μ), and we can write $V\mu$ instead of $G_\Omega \mu$. Remark that

$$V\mu(x) = \int\limits_{R^m} h_m(x - y)\, \mathrm{d}\mu(y).$$

If $f \in L^1_{loc}(\Omega)$ we can write $G_\Omega f$ instead of $G_\Omega(f\mathcal{H}_m|\Omega)$.

Proposition 3.11.2 *Let $\Omega \subset R^m$ be a domain with the Green function G_Ω, $m \geq 2$. If μ is a nonnegative Radon measure on Ω, then $G_\Omega \mu \equiv +\infty$ or $G_\Omega \mu$ is a superharmonic function on Ω which is harmonic on $\Omega \setminus \operatorname{spt} \mu$.*

Proof Since $G_\Omega(x, y) = h_m(x - y) - g_\Omega(x, y)$ and $g_\Omega \in H(\Omega \times \Omega)$ by Proposition 3.9.12, Fatou's lemma (Lemma 1.13.5) gives that G_μ is lower semicontinuous in Ω. Fix $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$. Since $G_\Omega(\cdot, y)$ is a superharmonic nonnegative function in Ω for each $y \in \Omega$ (compare Example 3.3.9), Fubini's theorem gives

$$\oint\limits_{B(x;r)} G_\Omega(z)\, \mathrm{d}\mathcal{H}_m(z) = \int\limits_\Omega \oint\limits_{B(x;r)} G_\Omega(z, y)\, \mathrm{d}\mathcal{H}_m(z)\mathrm{d}\mu(y)$$

$$\leq \int\limits_\Omega G_\Omega(x, y)\, \mathrm{d}\mu(y) = G_\Omega \mu(x).$$

Suppose now that $G_\mu \not\equiv +\infty$. Then G_μ is a superharmonic function by the definition of the superharmonic function. Choose an increasing sequence K_n of compact subsets of Ω such that $\Omega = \cup K_n$. For a Borel set A, denote by $\mu_n(A) = \mu(A \cap K_n)$. Then $G_\Omega \mu_n \in H(\Omega \setminus \operatorname{spt} \mu)$ by the Theorem on differentiability of parametrized integrals (Theorem 1.13.7). Clearly, $G_\Omega \mu_n \uparrow G_\Omega \mu$ as $n \to \infty$. Since $G_\Omega \mu \in L^1_{loc}(\Omega)$ by Proposition 3.3.4, Proposition 2.11.3 gives that $G_\Omega \mu \in H(\Omega \setminus \operatorname{spt} \mu)$.

Proposition 3.11.3 *Let $\Omega \subset R^m$ be a domain with the Green function G_Ω, $m \geq 2$, and μ be a nonnegative Radon measure on Ω. If $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, then $G_\Omega \mu$ is a superharmonic function in Ω if and only if*

$$\int\limits_{\Omega \setminus B(x;r)} G_\Omega(x, y)\, \mathrm{d}\mu(y) < \infty. \tag{3.15}$$

In particular, if $\mu(\Omega) < \infty$, then $G_\Omega \mu$ is a superharmonic function in Ω.

Proof For a Borel set A, put $\nu(A) = \mu(A \setminus B(x;r))$. Suppose first that $G_\Omega\mu$ is a superharmonic function in Ω. Since $G_\Omega\nu \le G_\Omega\mu$, the function $G_\nu \in H(B(x;r))$ by Proposition 3.11.2, and thus (3.15) holds true.

Suppose now that (3.15) holds true. Since $G_\Omega\nu(x) < \infty$, Proposition 3.11.2 gives that $G_\Omega\nu$ is a superharmonic function in Ω. Since $r < \operatorname{dist}(x;\partial\Omega)$, there is $z \in \Omega \setminus \overline{B(x;r)}$. Since μ is a Radon measure, we have $\mu(B(x;r)) < \infty$. According to Proposition 3.9.12, the function $G_\Omega(z,\cdot) = G_\Omega(\cdot,z)$ is a continuous finite function on $\overline{B(x;r)}$. Thus $G_\Omega(\mu-\nu)(z) < \infty$. Proposition 3.11.2 gives that $G_\Omega(\mu-\nu)$ is a superharmonic function in Ω. Since $G_\Omega(\mu-\nu)$ and $G_\Omega\nu$ are superharmonic functions in Ω the function $G_\Omega\mu$ is a superharmonic function in Ω.

Suppose now that $\mu(\Omega) < \infty$. According to Proposition 3.9.12 and Proposition 3.9.16, the function $G_\Omega(x,\cdot) = G_\Omega(\cdot,x)$ is bounded on $\Omega \setminus B(x;r)$. Hence (3.15) holds true.

Lemma 3.11.4 *Let $\Omega \subset R^m$ be an open set with the Green function $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$. Let μ be a nonnegative Radon measure with $\mu(R^m) < \infty$, $\mu(R^m \setminus \Omega) = 0$. If $m = 2$ suppose moreover that μ has a compact support. Put*

$$g_\Omega\mu(x) = \int_\Omega g_\Omega(x,y)\, \mathrm{d}\mu(y).$$

Then $g_\Omega\mu \in H(\Omega)$.

Proof Fix $z \in \Omega$. Choose $r > 0$ such that $B(z;3r) \subset \Omega$. Put $\mu_1 = \mu|B(z;2r)$, $\mu_2 = \mu - \mu_1$. According to Proposition 3.11.2, Proposition 3.11.3, and Proposition 3.10.2, we have $g_\Omega\mu_2 = G_\Omega\mu_2 - V\mu_2 \in H(B(z;r))$. Proposition 3.9.12 gives $g_\Omega \in H(\Omega\times\Omega)$. If ω is a bounded open set with $\overline{\omega} \subset \Omega$, then $\partial_x^\alpha g_\Omega(x,y)$ is bounded on $\overline{\omega} \times (\operatorname{spt}\mu_1)$ for each multiindex α. Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives that $g_\Omega\mu_1 \in \mathcal{C}^2(\omega)$ and

$$\Delta g_\Omega\mu_1(x) = \int_\Omega \Delta_x g_\Omega(x,y)\, \mathrm{d}\mu_1(y) = 0.$$

Theorem 3.11.5 *Let $\Omega \subset R^m$ be an open set with the Green function $G_\Omega(x,y)$, $m \ge 2$, and μ be a nonnegative Radon measure on Ω such that $G_\Omega\mu$ is finite almost everywhere in Ω. Then $-\Delta G_\Omega\mu = \mu$ in the sense of distributions.*

Proof If ω is a component of Ω, then $G_\Omega(x,y) = G_\omega(x,y)$ on $\omega\times\omega$ and $G_\Omega(x,y) = 0$ on $\omega \times (\Omega \setminus \omega)$ (see Proposition 3.9.7). Therefore we can suppose that Ω is connected.

Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Choose bounded open sets U, W with smooth boundary such that $\operatorname{spt}\varphi \subset W \subset \overline{W} \subset U \subset \overline{U} \subset \Omega$. Put $\mu_1 = \mu|\overline{U}$ (the restriction of μ onto $\overline{U}$), $\mu_2 = \mu - \mu_1$. Proposition 3.11.2 gives that $G_\Omega\mu_1$, $G_\Omega\mu_2$ are superharmonic functions in Ω and $G_\Omega\mu_2 \in H(U)$. Put

$$u(x) = \int_\Omega g_\Omega(x,y)\,\mathrm{d}\mu_1(y).$$

Then $u \in H(\Omega)$ by Lemma 3.11.4. Since $-\Delta V\mu_1 = \mu_1$ in the sense of distributions (see Theorem 3.1.5), second Green formula (Theorem 2.2.1) gives

$$\int_\Omega (-\Delta\varphi)G_\Omega\mu\,\mathrm{d}\mathcal{H}_m = \int_W (-\Delta\varphi)(G_\Omega\mu_2 + u)\,\mathrm{d}\mathcal{H}_m + \int_{R^m} (-\Delta\varphi)V\mu_1\,\mathrm{d}\mathcal{H}_m$$
$$= \int_W \varphi(-\Delta G_\Omega\mu_2 - \Delta u)\,\mathrm{d}\mathcal{H}_m + \int_{R^m} \varphi\,\mathrm{d}\mu_1 = \int_{R^m} \varphi\,\mathrm{d}\mu.$$

Proposition 3.11.6 *Let $\Omega \subset R^m$ be an open set with the Green function $G_\Omega(x,y)$, $m \ge 2$, and μ be a nonnegative Radon measure on Ω such that $G_\Omega\mu$ is finite almost everywhere in Ω. Then the greatest harmonic minorant of $G_\Omega\mu$ is the zero function.*

Proof We can suppose that Ω is connected (see Proposition 3.9.7). Since $G_\Omega\mu$ is a superharmonic function by Proposition 3.11.2 and 0 is a harmonic minorant of $G_\Omega\mu$, there exists the greatest harmonic minorant h of $G_\Omega\mu$ (see Theorem 3.6.5). Clearly, $0 \le h$.

Suppose first that $K = \operatorname{spt}\mu$ is a compact subset of Ω. Choose a bounded domain U such that $\operatorname{spt}\mu \subset U \subset \overline{U} \subset \Omega$. Fix $z \in K$. Then there is a positive constant C such that $u(y) \le Cu(z)$ for each nonnegative $u \in H(U)$ and $y \in K$ (see Theorem 2.8.2). If $x \in \Omega \setminus U$ then $G_\Omega(x,\cdot)$ is a nonnegative harmonic function in U (see Proposition 3.9.12). Therefore $G_\Omega(x,y) \le CG_\Omega(x,z)$ for each $y \in K$. Since μ is a Radon measure and K is compact, we have $\mu(\Omega) = \mu(K) < \infty$. Thus

$$h(x) \le G_\Omega\mu(x) = \int_K G_\Omega(x,y)\,\mathrm{d}\mu(y) \le CG_\Omega(x,z)\mu(\Omega), \quad x \in \Omega \setminus U.$$

Since $G_\Omega(\cdot,z)$ is superharmonic in Ω, $h \in H(\Omega)$, $h \le CG(\cdot,z)$ on ∂U, Proposition 3.3.7 forces that $h \le CG(\cdot,z)$ on U. Hence $h(x)/C$ is a harmonic minorant of $G_\Omega(x,z)$. Thus $h \le 0$ by virtue of the definition of Green's function.

Let now μ be general. Let K_n be an increasing sequence of compact subsets of Ω such that $\Omega = \cup K_n$. Put $\mu_n = \mu|K_n$, $\nu_n = \mu - \mu_n$. Since 0 is the greatest harmonic minorant of $G_\Omega\mu_n$, Proposition 3.6.6 gives that h is the greatest harmonic minorant of $G_\Omega\nu_n$. Fix $x \in \Omega$. Choose $n \in N$ such that $x \notin K_n$. Then $G_\Omega\nu_n(x) < \infty$ by Proposition 3.11.2 and

$$0 \le h(x) \le \lim_{k\to\infty} G_\Omega\nu_k(x) = 0.$$

3.12 Behavior of Volume Potentials at Infinity

Let μ be a real Radon measure in R^m such that the volume potential $V\mu$ is finite for some $x \in R^m$. We study the limit

$$\lim_{r\to\infty} \oint_{\partial B(0,r)} V\mu.$$

We show that this limit is equal to 0 for $m > 2$. In a planar case, we suppose moreover that μ has compact support. Then the limit is 0, ∞, or $-\infty$ dependent on $\mu(R^2)$.

Proposition 3.12.1 *Let μ be a real Radon measure on R^m, $m > 2$. If $V\mu(z)$ is well defined and finite for some $z \in R^m$, then*

$$\lim_{r\to\infty} \oint_{\partial B(0;r)} |V\mu| \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{3.16}$$

Proof Fix $\epsilon > 0$. Since

$$\int_{R^m \setminus B(0;1)} h_m \, \mathrm{d}|\mu| < \infty$$

by Proposition 3.11.3, there exists $\rho > 0$ such that

$$\int_{R^m \setminus B(0;\rho)} h_m \, \mathrm{d}|\mu| < \epsilon/2.$$

Put $\mu_1 = \mu|B(0;\rho)$, $\mu_2 = \mu - \mu_1$. Since $V|\mu_2|$ is a superharmonic function by Proposition 3.11.2, Proposition 3.3.7 gives

$$\oint_{\partial B(0;r)} V|\mu_2| \, \mathrm{d}\mathcal{H}_{m-1} \le V|\mu_2|(0) < \epsilon/2, \qquad \forall r > 0.$$

Since $V|\mu_1|(x) \to 0$ as $|x| \to \infty$, there exists $r_0 > 0$ such that

$$\oint_{\partial B(0;r)} V|\mu_1| \, \mathrm{d}\mathcal{H}_{m-1} \le \epsilon/2, \qquad \forall r > r_0.$$

If $r > r_0$, then

$$\oint_{\partial B(0;r)} |V\mu| \, \mathrm{d}\mathcal{H}_{m-1} \le \oint_{\partial B(0;r)} V|\mu_1| \, \mathrm{d}\mathcal{H}_{m-1} + \oint_{\partial B(0;r)} V|\mu_2| \, \mathrm{d}\mathcal{H}_{m-1} \le \epsilon.$$

Lemma 3.12.2 *Let μ be a real Radon measure in R^2 with compact support. If $\mu(R^2) = 0$ and α is a multiindex, then $\partial^\alpha V\mu(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$. If $\mu(R^2) > 0$ then $V\mu(x) \to -\infty$ as $|x| \to \infty$. If $\mu(R^2) < 0$ then $V\mu(x) \to \infty$ as $|x| \to \infty$.*

Proof Suppose first that $\mu(R^2) = 0$. According to Lemma 2.9.7, there is a constant C such that $|h_2(x-y) - h_2(x)| \le C|y||x|^{-1}$ whenever $|x| > 2|y|$. Choose $r > 0$ such that $\operatorname{spt}\mu \subset B(0;r)$. If $|x| > 2r$, then

$$|V\mu(x)| = \left| \int_{B(0;r)} [h_2(x-y) - h_2(x)] \, \mathrm{d}\mu(y) \right| \le \int_{B(0;r)} C|y||x|^{-1} \, \mathrm{d}|\mu|(y) \to 0.$$

Since $V\mu \in H(R^2 \setminus \operatorname{spt}\mu)$ by Proposition 3.10.2, we have $\partial^\alpha V\mu(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ (see Proposition 2.17.3).

Let now $c = \mu(R^2) \neq 0$. Put $\nu = \mu - c\delta_0$. Then $V\nu(x) = V\mu(x) - ch_2(x) \to 0$ as $|x| \to \infty$. If $c > 0$ then $V\mu \to -\infty$; if $c < 0$ then $V\mu \to \infty$.

Proposition 3.12.3 *Let μ be a real Radon measure on R^2. If $V\mu(z)$ is well defined and finite for some $z \in R^2$, then*

$$\lim_{r\to\infty} \oint_{\partial B(0;r)} V\mu \, \mathrm{d}\mathcal{H}_1 = \begin{array}{r} 0 \textit{ for } \mu(R^2) = 0, \\ -\infty \textit{ for } \mu(R^2) > 0, \\ \infty \textit{ for } \mu(R^2) < 0. \end{array} \tag{3.17}$$

Proof Since $V\mu(z)$ is finite, Proposition 3.10.2 gives

$$\int_{R^2 \setminus B(0;r)} |h_2| \, \mathrm{d}|\mu| < \infty \qquad \forall r > 0. \tag{3.18}$$

In particular $|\mu|(R^2) < \infty$.

Suppose first that $\mu(R^2) = 0$ and $0 \notin \operatorname{spt}\mu$. Let $\mu = \mu^+ - \mu^-$. Then $V\mu^+$ is a superharmonic function in R^2 which is finite in 0 (see Proposition 3.10.2). Fix $r > 1$. Then $V\mu^+$ is bounded from below on $\partial B(0;r)$ by Lemma 3.2.2 and

$$-\infty < \oint_{\partial B(0;r)} V\mu^+ \, \mathrm{d}\mathcal{H}_1 \le V\mu^+(0) < \infty$$

(see Proposition 3.3.7). Denote $\sigma_r = \mathcal{H}_1|\partial B(0;r)$. Put $\nu = \mu^+|B(0;2r)$. If $x \in \partial B(0;r)$ and $y \in B(0;2r)$ then $h_2(x-y) \geq [\ln(3r)]/(2\pi)$. According to Fubini's theorem and Lemma 2.5.1,

$$\int_{\partial B(0;r)} V\nu \,\mathrm{d}\mathcal{H}_1 = \int_{B(0;2r)} V\sigma_r \,\mathrm{d}\nu = \int_{B(0;2r)} r\ln(1/r)\,\mathrm{d}\mu^+.$$

If $|x| = r$ and $|y| \geq 2r$ then $|y-x| \leq 2|y|$ and $|h_2(x-y)| \leq |h_2(2y)| \leq |h_2(y)| + 1$. Since $|\mu|(R^2) < \infty$ and (3.18) holds we can use Fubini's theorem. By virtue of Lemma 2.5.1,

$$\int_{\partial B(0;r)} V(\mu^+ - \nu)\,\mathrm{d}\mathcal{H}_1 = \int_{R^2\setminus B(0;2r)} V\sigma_r \,\mathrm{d}\mu^+ = \int_{R^2\setminus B(0;2r)} c_r h_2 \,\mathrm{d}\mu^+$$

with $c_r = \mathcal{H}_1(\partial B(0;r))$. Similarly for μ^-. Thus

$$\oint_{\partial B(0;r)} V\mu \,\mathrm{d}\mathcal{H}_1 = \int_{B(0;2r)} \frac{\ln(1/r)}{2\pi}\,\mathrm{d}\mu + \int_{R^2\setminus B(0;2r)} h_2 \,\mathrm{d}\mu.$$

Since $\mu(R^2) = 0$, we get by virtue of (3.18)

$$\lim_{r\to\infty}\left|\oint_{\partial B(0;r)} V\mu \,\mathrm{d}\mathcal{H}_1\right| = \lim_{r\to\infty}\left|-\int_{R^2\setminus B(0;2r)} \frac{\ln(1/r)}{2\pi}\,\mathrm{d}\mu + \int_{R^2\setminus B(0;2r)} h_2 \,\mathrm{d}\mu\right|$$
$$\leq \lim_{r\to\infty} 2\int_{R^2\setminus B(0;2r)} |h_2|\,\mathrm{d}|\mu| = 0.$$

Let now $\mu(R^2) = 0$. Denote $c = \mu(B(0;1))$,

$$\nu = \mu|B(0;1) - (c/2\pi)\mathcal{H}_1|\partial B(0;1).$$

Then $\nu(R^2) = (\mu - \nu)(R^2) = 0$. Lemma 3.12.2 gives that $V\nu(x) \to 0$ as $|x| \to 0$. We have proved (3.17) for $\mu - \nu$. So, (3.17) holds also for μ.

Let now $c = \mu(R^2) \neq 0$. Define $\nu = (c/2\pi)\mathcal{H}_1|\partial B(0;1)$. If $|x| > 0$ then $V\nu(x) = c\ln(1/|x|)$ by Lemma 2.5.1. So, (3.17) holds for ν. Since $(\mu-\nu)(R^2) = 0$, we have

$$\lim_{r\to\infty} \oint_{\partial B(0;r)} V(\mu - \nu)\,\mathrm{d}\mathcal{H}_1 = 0.$$

Therefore (3.17)) holds for μ.

3.13 Regularity of Volume Potentials

If $f \in L^1(R^m)$ has compact support, then the volume potential

$$Vf(x) = \int\limits_{R^m} h_m(y-x) f(y)\, d\mathcal{H}_m(y)$$

is a solution of the Poisson equation $-\Delta u = f$. This solution is a classical solution on an open set $\Omega \subset R^m$ if and only if $Vf \in \mathcal{C}^2(\Omega)$. In this section we shall study how smooth the volume potential is with dependence on the smoothness of f.

Proposition 3.13.1 *Let $m \geq 2$, μ be a nonnegative Radon measure in R^m such that $V\mu(z)$ is finite for some z. Then $V\mu$ is absolutely continuous on almost all line segments parallel to the coordinate axes and*

$$\partial_j V\mu(x) = \int\limits_{R^m} \frac{\partial h_m(x-y)}{\partial x_j}\, \mathrm{d}\mu(y) \tag{3.19}$$

for almost all x and $j \in \{1, \ldots, m\}$. Moreover, $V\mu \in W^{1,p}_{loc}(R^m)$ for $1 \leq p < m/(m-1)$.

Proof Without loss of generality, we can suppose that $j = m$. Fix $r > 0$ and put $\mu_1(A) = \mu(A \cap B(0;r))$, $\mu_2 = \mu - \mu_1$. There exists a constant c_1 such that $|\nabla h_m(x)| \leq c_1 |x|^{1-m}$. According to Proposition 3.10.2, Proposition 3.11.2, and Proposition 3.11.3,

$$\int\limits_{R^m \setminus B(0;r)} |h_m(y)|\, \mathrm{d}\mu(y) < \infty$$

Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives that $V\mu_2 \in \mathcal{C}^1(B(0;r))$ and (3.19) holds for μ_2.

Since μ_1 is a Radon measure with compact support, there exists a constant c_2 such that $\mu_1(R^m) \leq c_2$. Set

$$F(x) = \int\limits_{R^m} \frac{\partial h_m(x-y)}{\partial x_m}\, \mathrm{d}\mu_1(y).$$

Let $1 < p < m/(m-1)$. Since $p(m-1) < m$, we have by Hölder's inequality (Proposition 1.14.1) and Fubini's theorem

$$\begin{aligned}\int\limits_{B(0;r)} |F|^p \,\mathrm{d}\mathcal{H}_m &\le \int\limits_{B(0;r)} \mu_1(R^m)^{p-1} \int\limits_{B(0;r)} (c_1|x-y|^{1-m})^p \,\mathrm{d}\mu_1(y)\,\mathrm{d}\mathcal{H}_m(x)\\ &\le \int\limits_{B(0;r)} \int\limits_{B(0;r)} c_1^p c_2^{p-1} |x-y|^{p(1-m)} \,\mathrm{d}\mathcal{H}_m(x)\,\mathrm{d}\mu_1(y)\\ &\le \int\limits_{B(0;r)} \int\limits_{B(0;2r)} c_1^p c_2^{p-1} |x|^{p(1-m)} \,\mathrm{d}\mathcal{H}_m(x)\,\mathrm{d}\mu_1(y)\\ &\le \int\limits_{B(0;2r)} c_1^p c_2^p |x|^{p(1-m)} \,\mathrm{d}\mathcal{H}_m(x) < \infty.\end{aligned}$$

Therefore $F \in L^p(B(0;r)) \subset L^1(B(0;r))$ and for almost all $x' \in R^{m-1}$ with $|x'| < r$, we have integrability of the function $F(x',\cdot)$ on $M(x') = \{x_m; |x_m| < \sqrt{r^2 - |x'|^2}\}$. Fix such x'. If $t, s \in M(x')$, $t > s$, then Fubini's theorem yields

$$\int\limits_s^t F(x',\tau)\,\mathrm{d}\tau = \int\limits_{R^m} \int\limits_s^t \partial_m h_m([x',\tau]-y)\,\mathrm{d}\tau\,\mathrm{d}\mu_1(y)$$

$$= \int\limits_{R^m} (h_m([x',t]-y) - h_m([x',s]-y))\,\mathrm{d}\mu_1(y) = V\mu_1(x',t) - V\mu_1(x',s).$$

Hence $V\mu_1(x',\cdot)$ is absolutely continuous on $M(x')$ and $\partial_m V\mu_1(x) = F(x)$.

Since $V\mu$ is absolutely continuous on almost all line segments parallel to the coordinate axes, (3.19) holds and $|\nabla V\mu| \in L^1_{loc}(R^m)$, Theorem 1.21.1 gives that $V\mu \in W^{1,1}_{loc}(R^m)$, $V\mu \in W^{1,p}_{loc}(R^m)$.

Proposition 3.13.2 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$, $1 \le p \le \infty$. If $f \in L^p(R^m)$, $f = 0$ on $R^m \setminus \Omega$ then $Vf \in W^{1,p}(\Omega)$,*

$$\|Vf\|_{W^{1,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

with a constant C depending only on Ω.

Proof Fix $R > 0$ such that $\Omega \subset B(0;R)$. Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $B(0;2R)$. Put $g(y) = \varphi(y)h_m(y)$. Then $Vf = g * f$ on $B(0;R)$. Since $g, \partial_j g \in L^1(R^m)$, Young's inequality (Lemma 1.15.1) gives $Vf = g*f \in L^p(B(0;R))$, $\partial_j Vf = \partial_j(g*f) = (\partial_j g)*f \in L^p(B(0;R))$, $\|Vf\|_{L^p(\Omega)} \le \|g\|_{L^1(R^m)}\|f\|_{L^p(\Omega)}$, $\|\partial_j Vf\|_{L^p(\Omega)} \le \|\partial_j g\|_{L^1(R^m)}\|f\|_{L^p(\Omega)}$.

Theorem 3.13.3 *Let $\Omega \subset R^m$ be a bounded open set with the Green function G_Ω, $m \geq 2$, $m/2 < p \leq \infty$. For $f \in L^p(\Omega)$ put $f = 0$ on $R^m \setminus \Omega$. Then $V : f \mapsto Vf$, $G_\Omega : f \mapsto G_\Omega f$ are bounded linear operators from $L^p(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$.*

Proof Since $m/2 < p$, we have $m - m/p > m - 2$. Choose $\gamma \in (m-2, m-m/p)$. Then $m/(m-\gamma) < m/2 < p$. Since there exists a constant C such that $|h_m(x-y)| + |G_\Omega(x,y)| \leq C|x-y|^{-\gamma}$ for $x, y \in \Omega$ (see Propositions 3.9.14 and 3.9.15), Proposition 1.26.7 gives that V, G_Ω are a bounded linear operators from $L^p(\Omega)$ to $\mathcal{C}^{0,0}(\Omega)$.

Theorem 3.13.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, $f \in L^\infty(\Omega)$. Put $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^1(R^m)$.*

Proof $Vf \in \mathcal{C}^0(R^m)$ by Theorem 3.13.3. Since $\partial_j Vf = f * \partial_j h_m$ by Proposition 3.13.1, Proposition 1.26.3 gives that $\partial_j Vf \in \mathcal{C}^0(R^m)$.

Proposition 3.13.5 *Let $f \in L^p(R^m)$, $m > 2$, $1 < p < m/2$. Then $Vf \in L^{pm/(m-2p)}(R^m)$, $\partial_j Vf \in L^{pm/(m-p)}(R^m)$ is given by*

$$\partial_j Vf(x) = f * (\partial_j h_m)(x) = \int_{R^m} \frac{f(y)(y_j - x_j)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} \, \mathrm{d}\mathcal{H}_m(y).$$

for $j = 1, \dots, m$. Moreover, there exists a constant C depending only on m and p such that

$$\|Vf\|_{L^{pm/(m-2p)}(R^m)} \leq C\|f\|_{L^p(R^m)}, \quad \|\nabla Vf\|_{L^{pm/(m-p)}(R^m)} \leq C\|f\|_{L^p(R^m)}.$$

Proof $V : f \mapsto Vf$ is a bounded linear operator from $L^p(R^m)$ to $L^{pm/(m-2p)}(R^m)$ by Proposition 1.26.4. For a fixed $j \in \{1, \dots, m\}$, put

$$K(x,y) = \frac{y_j - x_j}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}.$$

Then

$$Kg(x) = \int_{R^m} K(x,y) g(y) \, \mathrm{d}\mathcal{H}_m(y)$$

is a bounded linear operator from $L^p(R^m)$ to $L^{pm/(m-p)}(R^m)$ by Proposition 1.26.4. Put $f_n(x) = f(x)$ for $|x| < n$, $f_n(x) = 0$ elsewhere. Since $Vf_n \to Vf$ in $L^{pm/(m-2p)}(R^m)$, $Vf_n \to Vf$ in the sense of distributions. Therefore $\partial_j Vf_n \to \partial_j Vf$ in the sense of distributions. But $\partial_j Vf_n = Kf_n$ by Proposition 3.13.1. The continuity of the operator K gives $\partial_j Vf = Kf$.

Example 3.13.6 Put $f(x) = (1 + |x|)^{-2}$ for $x \in R^m$, $m > 2$. Then $f \in L^p(R^m)$ for each $p > m/2$. But $Vf \equiv \infty$.

3.14 Second Derivatives of Volume Potentials

In this section we show that the volume potential $Vf \in D^{2,p}(R^m)$ provided $f \in L^p(R^m)$.

Proposition 3.14.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $z \in \Omega$. Let f be a measurable function on R^m, $f = 0$ in $R^m \setminus \Omega$. If there are positive constants C, λ such that $|f(x) - f(z)| \leq C|x-z|^\lambda$ for each $x \in R^m$, then there exists $\partial_j \partial_k Vf(z)$ for all $j, k \in \{1, \dots, m\}$ and*

$$\partial_j \partial_k Vf(z) = \int_\Omega \partial_j \partial_k h_m(y-z)[f(y) - f(z)] \, \mathrm{d}\mathcal{H}_m(y)$$

$$+ f(z) \int_{\partial\Omega} n_k^\Omega(y) \partial_j h_m(y-z) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Proof We can suppose that $\lambda < 1$. Put $f_1 = f(z)\chi_\Omega$, $f_2 = f - f_1$. According to Proposition 3.13.1 and the Gauss-Green theorem (Theorem 1.19.1),

$$\partial_k Vf_1(x) = f(z) \int_\Omega \partial_k h_m(x-y) = -f(z) \int_{\partial\Omega} n_k^\Omega(y) h_m(x-y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives

$$\partial_j \partial_k Vf_1(x) = -f(z) \int_{\partial\Omega} n_k^\Omega(y) \partial_j h_m(x-y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Since $f_2 \in L^\infty(R^m)$, Theorem 3.13.4 and Proposition 3.13.1 give that $Vf_2 \in \mathcal{C}^1(R^m)$ and $\partial_k Vf_2 = f_2 * (\partial_k h_m)$. According to Corollary 2.9.6 and Lemma 2.9.7, there exists a positive constant C_1 such that

$$|\partial^\alpha h_m(x)| \leq C_1 |x|^{2-m-|\alpha|}$$

for $x \in R^m \setminus \{0\}$ and every multiindex α with $1 \leq |\alpha| \leq 3$.

Fix $x, y \in R^m$ such that $|z-y| > 2|z-x|$. Put $g(t) = \partial_k h_m(t(y-x) + (1-t)(y-z))$. Then there exists $0 < \theta < 1$ such that $g(1) - g(0) - g'(0) = g''(\theta)/2$. Since $|\theta(z-x) + y - z| \geq |y-z| - \theta|z-x| > |z-y|/2$, we have

$$|\partial_k h_m(y-x) - \partial_k h_m(y-z) - (x-z) \cdot \nabla \partial_k h_m(y-z)| = |g(1) - g(0) - g'(0)|$$

$$= |g''(\theta)/2| \leq |x-z|^2 |\nabla^2 \partial_k h_m(\theta(z-x) + y - z)|/2$$

$$\leq C_1 m^2 |x-z|^2 \left(\frac{|z-y|}{2} \right)^{-m-1}.$$

Put $e = [e_1, \dots, e_m]$ with $e_n = \delta_{nj}$. If $x = z + he$, then

$$\left|\frac{\partial_k Vf_2(z+he) - \partial_k Vf_2(z)}{h} - \int\limits_{\Omega} \partial_j\partial_k h_m(y-z) f_2(y)\, \mathrm{d}\mathcal{H}_m(y)\right|$$

$$\leq \int\limits_{B(z;2|x-z|)} |f_2(y)| \left[\frac{|\partial_k h_m(y-x)| + |\partial_k h_m(y-z)|}{h} + |\partial_j\partial_k h_m(y-z)|\right]$$

$$+ \int\limits_{\Omega\setminus B(z;2|x-z|)} \frac{|f_2(y)|}{h} |\partial_k h_m(y-x) - \partial_k h_m(y-z)$$

$$-(x-z)\cdot\nabla\partial_k h_m(y-z)|\, \mathrm{d}\mathcal{H}_m(y) \leq \frac{C(2|h|)|^{\lambda}}{|h|}\Bigg[\int\limits_{B(x;3|h|)} C_1|y-x|^{1-m}\, \mathrm{d}\mathcal{H}_m(y)$$

$$+ \int\limits_{B(z;2|h|)} C_1|y-z|^{1-m}\, \mathrm{d}\mathcal{H}_m(y)\Bigg] + \int\limits_{B(z;2|h|)} C|y-z|^{\lambda} C_1|y-z|^{-m}\, \mathrm{d}\mathcal{H}_m(y)$$

$$+ \int\limits_{R^m\setminus B(z;2|h|)} |h|^{-1} C|y-z|^{\lambda} C_1 m^2 |x-z|^2 \left(\frac{|z-y|}{2}\right)^{-m-1} \mathrm{d}\mathcal{H}_m(y)$$

$$= CC_1\mathcal{H}_{m-1}(\partial B(0;1))\left[2^{\lambda}|h|^{\lambda}3 + 2^{\lambda+1}|h|^{\lambda} + \frac{(2|h|)^{\lambda}}{\lambda} + \frac{m^2|h|^{\lambda}2^{m+1}}{1-\lambda}\right].$$

Therefore

$$\partial_j\partial_k Vf_2(z) = \lim_{h\to 0}\frac{\partial_k Vf_2(z+he) - \partial_k Vf_2(z)}{h} = \int\limits_{\Omega} \partial_j\partial_k h_m(y-z) f_2(y).$$

Theorem 3.14.2 *Let $\Omega, \omega \subset R^m$ be bounded open sets, $\overline{\omega} \subset \Omega$, $m \geq 2$, $0 < \lambda < 1$. If $f \in \mathcal{C}^{0,\lambda}(\Omega)$, $f = 0$ on $R^m \setminus \Omega$, then $Vf \in \mathcal{C}^{2,\lambda}(\omega)$.*

Proof We can suppose that Ω has boundary of class $\mathcal{C}^1$. Theorem 3.13.4 gives $Vf \in \mathcal{C}^1(R^m)$. According to Proposition 3.14.1, one has $\partial_j\partial_k Vf(z) = I_1(z) + f(z)I_2(z)$ in Ω, where

$$I_1(z) = \int\limits_{\Omega} \partial_j\partial_k h_m(y-z)[f(y) - f(z)]\, \mathrm{d}\mathcal{H}_m(y),$$

$$I_2(z) = \int\limits_{\partial\Omega} n_k^{\Omega}(y)\partial_j h_m(y-z)\, \mathrm{d}\mathcal{H}_{m-1}(y).$$

$I_2 \in \mathcal{C}^1(\Omega)$ (and therefore $I_2 \in \mathcal{C}^{0,\lambda}(\omega)$) by Theorem on differentiability of parametrized integrals (Theorem 1.13.7). Lemma 1.16.8 gives that $f(z)I_2(z) \in \mathcal{C}^{0,\lambda}(\omega)$.

Fix a constant C such that $|f| \leq C$ and $|f(x) - f(y)| \leq C|x-y|^\lambda$ for $x, y \in \Omega$. According to Corollary 2.9.6 and Lemma 2.9.7, there exists a constant C_1 such that $|\nabla h_m(x)| \leq C_1|x|^{1-m}$, $|\nabla^2 h_m(x)| \leq C_1|x|^{-m}$ and $|\partial_j\partial_k h_m(x-y) - \partial_j\partial_k h_m(z-y)| \leq C_1|x-z||x-y|^{-m-1}$ for $x, y, z \in R^m$, $|x-y| > 2|x-z|$. Fix $x, z \in \omega$. Then $|I_1(x) - I_1(z)| \leq J_1(x,z) + J_2(x,z) + J_3(x,z) + J_4(x,z)$, where

$$J_1(x,z) = \int\limits_{\Omega\cap B(z;2|x-z|)} |\partial_j\partial_k h_m(y-z)||f(y) - f(z)| \,\mathrm{d}\mathcal{H}_m(y),$$

$$J_2(x,z) = \int\limits_{\Omega\cap B(z;2|x-z|)} |\partial_j\partial_k h_m(y-x)||f(y) - f(x)| \,\mathrm{d}\mathcal{H}_m(y),$$

$$J_3(x,z) = \left|\int\limits_{\Omega\setminus B(z;2|x-z|)} \partial_j\partial_k h_m(y-z)[f(x) - f(z)] \,\mathrm{d}\mathcal{H}_m(y)\right|,$$

$$J_4(x,z) = \int\limits_{\Omega\setminus B(z;2|x-z|)} |\partial_j\partial_k h_m(y-z) - \partial_j\partial_k h_m(y-x)||f(y) - f(x)| \,\mathrm{d}\mathcal{H}_m(y).$$

The Gauss-Green theorem (Theorem 1.19.1) gives

$$J_3(x,z) = \left|\int\limits_{\partial[\Omega\setminus B(z;2|x-z|)]} n_j(y)\partial_k h_m(y-z)[f(x) - f(z)] \,\mathrm{d}\mathcal{H}_{m-1}(y)\right|$$

$$\leq C|x-z|^\lambda\left[\int\limits_{\partial\Omega} C_1 \operatorname{dist}(\omega, \partial\Omega)^{1-m} \,\mathrm{d}\mathcal{H}_{m-1}(y) + 1\right].$$

We obtain by virtue of Lemma 1.26.1

$$J_4(x,z) \leq \int\limits_{\Omega\setminus B(z;2|z-x|)} C_1|x-z||y-z|^{-m-1}C|x-y|^\lambda \,\mathrm{d}\mathcal{H}_m(y)$$

$$\leq CC_1|x-z| \int\limits_{\Omega\setminus B(z;2|z-x|)} |y-z|^{-m-1}(2|z-y|)^\lambda \,\mathrm{d}\mathcal{H}_m(y)$$

$$\leq 2CC_1|x-z|\mathcal{H}_m(B(0;1))(2|x-z|)^{\lambda-1}\frac{m+1-\lambda}{1-\lambda} \leq \tilde{C}|x-z|^\lambda,$$

$$J_1(x,z) \le \int\limits_{B(z;|2|x-z|)} C_1 C|y-z|^{\lambda-m}\,\mathrm{d}\mathcal{H}_m(y) \le \frac{2CC_1\mathcal{H}_m(B(0;1))m}{\lambda}|x-z|^{\lambda},$$

$$J_2(x,z) \le \int\limits_{B(x;|3|x-z|)} C_1 C|y-x|^{\lambda-m}\,\mathrm{d}\mathcal{H}_m(y) \le \frac{3CC_1\mathcal{H}_m(B(0;1))m}{\lambda}|x-z|^{\lambda}.$$

Lemma 3.14.3 *Let $j,k \in \{1,\dots,m\}$. Then*

$$\int\limits_{\partial B(0;1)} \partial_j\partial_k h_m\,\mathrm{d}\mathcal{H}_{m-1} = 0. \tag{3.20}$$

Proof Easy calculation yields that

$$\partial_j\partial_k h_m(x) = \frac{m x_j x_k}{\mathcal{H}_{m-1}(\partial B(0;1))|x|^{m+2}} - \frac{\delta_{kj}}{\mathcal{H}_{m-1}(\partial B(0;1))|x|^{m}}.$$

If $j \ne k$ then (3.20) is clear. Let now $j = k$. Then

$$\begin{aligned}\int\limits_{\partial B(0;1)} \partial_j\partial_j h_m\,\mathrm{d}\mathcal{H}_{m-1} &= \oint\limits_{\partial B(0;1)} m x_j^2\,\mathrm{d}\mathcal{H}_{m-1} - \oint\limits_{\partial B(0;1)} 1\,\mathrm{d}\mathcal{H}_{m-1}\\ &= \oint\limits_{\partial B(0;1)} x_1^2\,\mathrm{d}\mathcal{H}_{m-1} + \dots + \oint\limits_{\partial B(0;1)} x_m^2\,\mathrm{d}\mathcal{H}_{m-1} - 1\\ &= \oint\limits_{\partial B(0;1)} [x_1^2 + x_2^2 + \dots + x_m^2]\,\mathrm{d}\mathcal{H}_{m-1} - 1 = 0.\end{aligned}$$

Proposition 3.14.4 *Let $j,k \in \{1,\dots,m\}$, $1 < p < \infty$. For $f \in L^p(R^m)$, $\epsilon > 0$, define*

$$T_{jk}^{\epsilon} f(x) = \int\limits_{|y|\ge\epsilon} \partial_j\partial_k h_m(y) f(x-y)\,\mathrm{d}\mathcal{H}_m(y).$$

Then there exists a constant A such that

- *If $\epsilon > 0$ then T_{jk}^{ϵ} is a bounded linear operator on $L^p(R^m)$ and $\|T_{jk}^{\epsilon}\| \le A$.*
- *There exists a bounded linear operator T_{jk} on $L^p(R^m)$ with $\|T_{jk}\| \le A$ such that $T_{jk}^{\epsilon} f \to T_{jk} f$ in $L^p(R^m)$ as $\epsilon \to 0_+$.*
- *If $f \in L^p(R^m)$, then there exists*

$$\lim_{\epsilon \searrow 0} T_{jk}^{\epsilon} f(x)$$

for almost all $x \in R^m$.

- *If $f \in L^p(R^m)$, $x \in R^m$, denote*

$$T^*_{jk}f(x) = \sup_{\epsilon>0} |T^\epsilon_{jk}f(x)|.$$

Then

$$\|T^*_{jk}f\|_{L^p(R^m)} \le A\|f\|_{L^p(R^m)}.$$

Proof Fix j, k. Easy calculation yields $\partial_j\partial_k h_m(x) = L(x/|x|)|x|^{-m}$ where $L(x) = (mx_jx_k - \delta_{jk})/\mathcal{H}_{m-1}(\partial B(0;1))$. If $x, y \in \partial B(0;1)$ then $|L(x)| \le m+1$,

$$|L(x) - L(y)| \le m[|x_j(x_k - y_k)| + |y_k(x_j - y_j)|] \le 2m|x-y|.$$

Lemma 3.14.3 gives

$$\int_{\partial B(0;1)} L \,\mathrm{d}\mathcal{H}_m = 0.$$

$\partial_j\partial_k h_m(x-y)$ is a Calderón-Zygmund kernel by Theorem 1.27.2. Now we use properties of Calderón-Zygmund operators (Propositions 1.27.4–1.27.6).

Theorem 3.14.5 *Let $f \in L^p(R^m)$, $1 < p < \infty$, $m \ge 2$. Suppose moreover that $Vf(z)$ is finite for some $z \in R^m$. Then $Vf \in D^{2,p}(R^m)$. If $j, k \in \{1, \dots, m\}$, then*

$$\partial_j\partial_k Vf = T_{jk}f - \frac{\delta_{jk}f}{m},$$

where T_{jk} is a linear operator from Proposition 3.14.4.

Proof Suppose first that $f \in \mathcal{C}^\infty_c(R^m)$. Then $Vf \in \mathcal{C}^2(R^m)$ by Theorem 3.14.2. Fix $\rho > 0$ such that $\operatorname{spt} f \subset B(z;\rho)$. According to Proposition 3.14.1,

$$\partial_j\partial_k Vf(z) = \int_{B(z;\rho)} \partial_j\partial_k h_m(y-z)[f(y) - f(z)] \,\mathrm{d}\mathcal{H}_m(y)$$
$$+ f(z) \int_{\partial B(z;\rho)} n_k(y)\partial_j h_m(y-z) \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

Lemma 3.14.3 gives

$$\partial_j\partial_k Vf(z) = T_{jk}f(z) + f(z) \int_{\partial B(z;\rho)} n_k(y)\partial_j h_m(y-z) \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

Since $n_k(y)\partial_j h_m(y-z) = -(y_j - z_j)(y_k - z_k)/\rho^{m+1}\mathcal{H}_{m-1}(\partial B(0;1))$, we have $\partial_j\partial_k Vf(z) = T_{jk}f(z)$ for $j \neq k$. Let now $j = k$. Then

$$\int\limits_{\partial B(z;\rho)} n_k(y)\partial_j h_m(y-z)\,\mathrm{d}\mathcal{H}_{m-1}(y) = -\oint\limits_{\partial B(0;\rho)} \frac{y_j^2}{|y|^2}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$= \frac{-1}{m}\left[\,\oint\limits_{\partial B(0;\rho)} \frac{y_1^2}{|y|^2}\,\mathrm{d}\mathcal{H}_{m-1}(y) + \cdots + \oint\limits_{\partial B(0;\rho)} \frac{y_m^2}{|y|^2}\,\mathrm{d}\mathcal{H}_{m-1}(y)\right] = \frac{-1}{m}.$$

Let now f be general. Then Vf is well defined in R^m by Proposition 3.10.2 and Proposition 3.11.2. Fix $\rho > 0$. Put $f_1 = f\chi_{B(0;2\rho)}$, $f_2 = f - f_1$. Then $Vf_2 \in H(B(0;2\rho))$ by Propositions 3.10.2, 3.11.2, and 3.9.2. Hence $Vf_2 \in W^{2,p}(B(0;\rho))$. Theorem on differenciability of parametrized integrals (Theorem 1.13.7) gives $\partial_j\partial_k Vf_2 = T_{jk}f_2 = T_{jk}f_2 - \delta_{jk}f_2/m$ in $B(0;\rho)$. Proposition 3.13.2 forces $Vf_1 \in W^{1,p}(B(0;\rho))$. Choose $\varphi_n \in \mathcal{C}_c^\infty(B(0;3\rho))$ such that $\varphi_n \to f_1$ in $L^p(R^m)$. (See Proposition 1.14.7.) Then $V\varphi_n \to Vf_1$ in $W^{1,p}(B(0;\rho))$ by Proposition 3.13.2. Moreover, $V\varphi_n \in W^{2,p}(B(0;\rho))$ and $\partial_j\partial_k V\varphi_n = T_{jk}\varphi_n - \varphi_n\delta_{jk}/m \to T_{jk}f_1 - f_1\delta_{jk}/m$ in $L^p(B(0;\rho))$ (see Proposition 3.14.4). Thus $V\varphi_n$ is a Cauchy sequence in $W^{2,p}(B(0;\rho))$. Since $W^{2,p}(B(0;\rho))$ is a Banach space, we deduce that $Vf_1 \in W^{2,p}(B(0;\rho))$ and $\partial_j\partial_k Vf_1 = T_{jk}f_1 - f_1\delta_{jk}/m$. Proposition 3.14.4 gives that $\partial_j\partial_k Vf \in L^p(R^m)$, and therefore $Vf \in D^{2,p}(R^m)$.

3.15 Hölder Continuous Volume Potentials

In this section we study a volume potential as a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{0,\lambda}(\Omega)$ or from $L^p(\Omega)$ to $\mathcal{C}^{1,\lambda}(\Omega)$.

Proposition 3.15.1 *Let $\Omega \subset R^m$ be a bounded open set, $2 \le m < p < \infty$. For $f \in L^p(\Omega)$, put $f = 0$ on $R^m \setminus \Omega$. If $\lambda = 1 - m/p$, then $V : f \mapsto Vf$ is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ and $\partial_j V : f \mapsto \partial_j Vf$ is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{0,\lambda}(R^m)$ for $j = 1,\dots,m$.*

Proof Without loss of generality we can suppose that Ω has Lipschitz boundary. Let $f \in L^p(\Omega)$. Then $Vf \in W^{2,p}_{loc}(R^m)$ by Theorem 3.14.5. Theorem 1.21.13 forces that $Vf \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. If $f_n \to f$ in $L^p(\Omega)$ and $Vf_n \to g$ in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ then $Vf = g$ by Theorem 3.13.3. Closed-graph theorem (Theorem 1.4.15) gives that V is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$.

$\partial_j Vf, \partial_j\partial_k Vf \in \mathcal{C}^\infty(R^m \setminus \overline{\Omega})$ and tends to 0 at infinity. Thus $\partial_j Vf \in \mathcal{C}^{0,0}(R^m)$. Fix $r > 1$ such that $\overline{\Omega} \subset B(0;r)$. Then there exist constants M_1, M_2, and M_3 such that $|\partial_j Vf| \le M_1$, $|\partial_j Vf(x) - \partial_j Vf(y)| \le M_2|x-y|^\lambda$ for $|x|, |y| \le 2r$, $|\nabla^2 Vf(x)| \le M_3$ for $|x| \ge r$. If $|x-y| \ge 1$ then $|\partial_j Vf(x) - \partial_j Vf(y)| \le 2M_1 \le 2M_1|x-y|^\lambda$. If $|x| \ge 2r$, $|x-y| < 1$ then $|y| > r$ and $|\partial_j Vf(x) - \partial_j Vf(y)| \le M_3|x-y| \le M_3|x-y|^\lambda$. Thus

$\partial_j Vf \in \mathcal{C}^{0,\lambda}(R^m)$. Let $f_n \to f$ in $L^p(\Omega)$ and $\partial_j Vf_n \to g$ in $\mathcal{C}^{0,\lambda}(R^m)$. We have proved that $\partial_j Vf = g$. Closed-graph theorem (Theorem 1.4.15) gives that $\partial_j V$ is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{0,\lambda}(R^m)$.

Proposition 3.15.2 *Let $\Omega \subset R^m$ be a bounded open set, $1 \le m/2 < p \le m$. For $f \in L^p(\Omega)$ put $f = 0$ on $R^m \setminus \Omega$. If $0 < \lambda < 1$, $\lambda \le 2 - m/p$ then $V : f \mapsto Vf$ is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$.*

Proof Without loss of generality, we can suppose that Ω has Lipschitz boundary. Suppose first that $p < m$ and $\lambda = 2 - m/p$. Let $f \in L^p(\Omega)$. Then $Vf \in W^{2,p}(\Omega)$ by Theorem 3.14.5. Theorem 1.21.13 forces that $Vf \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$. If $f_n \to f$ in $L^p(\Omega)$ and $Vf_n \to g$ in $\mathcal{C}^{0,\lambda}(\overline{\Omega})$ then $Vf = g$ by Theorem 3.13.3. Closed-graph theorem (Theorem 1.4.15) gives that V is a bounded linear operator from $L^p(\Omega)$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$.

Let now $m/2 < q < p \le m$. Then $L^p(\Omega)$ is continuously embedded into $L^q(\Omega)$ by Corollary 1.14.2. Thus V is a bounded linear operator from $L^p(\Omega)$ into $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ with $\alpha = 2 - m/q$.

3.16 Modified Volume Potentials

We want to prove that $D^{2,p}(R^m) = \mathring{D}^{2,p}(R^m) + P_0(R^m) + P_1(R^m)$, where $P_k(R^m)$ is the space of homogeneous polynomials of order k. We are also interested about the Laplace operator as a mapping between homogeneous Sobolev spaces. For this reason, we define modified volume potentials and study their properties.

Let $f \in L^p(R^m)$, $1 < p < \infty$, $m \ge 2$. Denote $B := B(0;1)$. We define the modified volume potentials $V^{m,p}f$:

$$V^{m,p}f = Vf, \qquad 1 < p < m/2,$$

$$V^{m,p}f(x) = V(f\chi_B)(x) + \int\limits_{R^m\setminus B} [h_m(x-y) - h_m(-y)] f(y)\, \mathrm{d}\mathcal{H}_m(y)$$

for $m/2 \le p < m$, and

$$V^{m,p}f(x) = \int\limits_{R^m} [h_m(x-y) - (1-\chi_B(y))(h_m(-y) + x \cdot \nabla h_m(-y))] f(y)\, \mathrm{d}\mathcal{H}_m(y)$$

for $m \le p < \infty$.

Theorem 3.16.1 *If $1 < p < \infty$, $m \ge 2$, then $V^{m,p} : f \mapsto V^{m,p}f$ is a bounded linear operator from $L^p(R^m)$ to $D^{2,p}(R^m)$. Let $f \in L^p(R^m)$, T_{jk} be an operator from Proposition 3.14.4. Then $-\Delta V^{m,p}f = f$,*

$$\partial_j \partial_k V^{m,p} f = T_{jk} f - \frac{\delta_{jk} f}{m}, \tag{3.21}$$

$$\partial_j V^{m,p} f(x) = \int\limits_{R^m} \partial_j h_m(x-y) f(y) \, \mathrm{d}\mathcal{H}_m(y), \qquad p < m, \tag{3.22}$$

$$\partial_j V^{m,p} f(x) = \int\limits_{R^m} [\partial_j h_m(x-y) - (1-\chi_B(y))\partial_j h_m(-y)] f(y) \, \mathrm{d}\mathcal{H}_m(y) \tag{3.23}$$

for $p \geq m$. If $\omega \subset R^m$ is an open set and $f \in L^q(\omega)$ with $q > m/2$ then $V^{m,p} f \in \mathcal{C}^0(\omega)$.

Proof Suppose first $p < m/2$. If $f \in L^p(R^m)$, then $V^{m,p} f \in D^{2,p}(R^m)$ by Proposition 3.13.5 and Theorem 3.14.5. Moreover, $-\Delta V^{m,p} f = f$ by Theorem 3.11.5 and Theorem 3.10.3. Proposition 3.13.1 gives (3.22), Theorem 3.14.5 gives (3.21). According to Propositions 3.14.4 and 3.13.5 and Corollary 1.14.2, there exists a constant C_1 such that

$$\|V^{m,p} f\|_{L^p(B(0;1))} + \|\nabla^2 V^{m,p} f\|_{L^p}(R^m) \leq C_1 \|f\|_{L^p}(R^m).$$

So, $V^{m,p}$ is a bounded linear operator from $L^p(R^m)$ to $D^{2,p}(R^m)$. Let now $f \in L^p(R^m) \cap L^q(\omega)$, where $q > m/2$ and ω is a bounded open set. Define $g = f$ on ω, $g = 0$ otherwise. Then $V^{m,p} g = Vg \in \mathcal{C}^0(R^m)$ by Theorem 3.13.3. Propositions 3.10.2 and 3.11.2 give that $V^{m,p}(f-g) = V(f-g) \in H(\omega) \subset \mathcal{C}^0(\omega)$.

Let now $p \geq m/2$. For $r > 0$ denote $f_r(x) = f(x)$ for $|x| < r$, $f_r(x) = 0$ elsewhere. Suppose first $m/2 \leq p < m$. Then $Vf_r \in D^{2,p}(R^m)$ by Propositions 3.10.2 and 3.11.3 and Theorem 3.14.5. Since $V^{m,p} f_r$ and Vf_r differ by a constant, we deduce that $V^{m,p} f_r \in D^{2,p}(R^m)$. Proposition 3.13.1 gives (3.22) for f_r. Moreover, $-\Delta V^{m,p} f_r = f_r$ by Theorem 3.11.5 and Theorem 3.10.3. According to Lemma 2.9.7 there exists a constant c_0 such that for $|y| \geq 2|x|$ we have $|h_m(x-y) - h_m(-y)| \leq c_1 |x| |y|^{1-m}$. Since $p \geq m/2$ we have $(m-1)p/(p-1) \geq (m-1)(m/2)/[(m/2)-1] = m(m-1)/(m-2) > m$. If $|x| < r/2$, then Hölder's inequality (Proposition 1.14.1) gives

$$|V^{m,p}(f-f_r)(x)| \leq \int\limits_{R^m \setminus B(0;r)} |f(y)| \frac{c_1 r}{|y|^{m-1}} \, \mathrm{d}\mathcal{H}_m(y)$$

$$\leq c_1 r \|f\|_{L^p(R^m)} \left[\int\limits_{R^m \setminus B(0;r)} |y|^{(1-m)p/(p-1)} \, \mathrm{d}\mathcal{H}_m(y) \right]^{(p-1)/p} < \infty.$$

Proposition 3.13.2 and Hölder's inequality (Proposition 1.14.1) force that $\|V^{m,p} f\|_{L^p(B(0;r/2))} \leq C_2 \|f\|_{L^p(R^m)}$ with C_2 independent on f. According to Corollary 2.9.6, there exists a constant c_2 such that $|\nabla h_m(x)| \leq c_2 |x|^{1-m}$. If $|x| \leq r/2$, $|y| \geq r$ then $\partial[h_m(x-y) - h_m(y)]/\partial x_j| \leq c_2 |x-y|^{1-m} \leq c_2 (|y|/2)^{1-m}$. We have proved that $|f(y) - f_r(y)||y|^{1-m}$ is integrable. By virtue of Theorem on differentiability of parametrized integrals (Theorem 1.13.7),

$$\partial_j V^{m,p}(f - f_r)(x) = \int_{R^m} \partial_j h_m(x-y)[f(y) - f_r(y)] \, \mathrm{d}\mathcal{H}_m(y).$$

So, (3.22) holds true. Theorem 3.14.5 gives (3.21) for f_r. Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives (3.21) for $f - f_r$ in $B(0;r/2)$. So, (3.21) holds true. Since T_{jk} is a bounded linear operator on $L^p(R^m)$ by Proposition 3.14.4, $V^{m,p}$ is a bounded linear operator from $L^p(R^m)$ to $D^{2,p}(R^m)$. Thus

$$-\Delta V^{m,p} f = \lim_{r \to \infty} [-\Delta V^{m,p} f_r] = \lim_{r\to\infty} f_r = f. \tag{3.24}$$

Let now $f \in L^p(R^m) \cap L^q(\omega)$, where ω is a bounded open set and $q > m/2$. Fix $r > 0$ such that $\omega \subset B(0;r/2)$. Put $g = f$ on ω, $g = 0$ elsewhere. Then $V^{m,p} g \in \mathcal{C}^0(\omega)$ by Theorem 3.13.3. Propositions 3.10.2 and 3.11.2 give that $V^{m,p}(f_r - g) \in H(\omega) \subset \mathcal{C}^0(\omega)$. Theorem on continuity of parametrized integrals (Theorem 1.13.6) gives that $V^{m,p}(f - f_r) \in \mathcal{C}^0(\omega)$.

Let now $p \geq m$. Fix $r > 0$. Then $Vf_r \in D^{2,p}(R^m)$ by Propositions 3.10.2 and 3.11.3 and Theorem 3.14.5. Since $V^{m,p} f_r$ and Vf_r differ by a polynomial of degree less than or equal to 1, we deduce that $V^{m,p} f_r \in D^{2,p}(R^m)$. Moreover, $-\Delta V^{m,p} f_r = f_r$ by Theorems 3.11.5 and 3.10.3. By virtue of Proposition 3.13.1, we obtain (3.23) for f_r. According to Lemma 2.9.7 there exists a constant c_3 such that $|\nabla h_m(z - y) - \nabla h_m(-y)| \leq c_3 |z||y|^{-m}$ for $|z| < 2|y|$. If $|x| < r/2$, $|y| > r$ then there exists $\theta \in (0,1)$ such that $h_m(x-y) - h_m(-y) = x \cdot \nabla(-y + \theta x)$. Thus $|h_m(x-y) - h_m(-y) - x \cdot \nabla h_m(-y)| = |x \cdot \nabla h_m(-y + \theta x) - x \cdot \nabla h_m(-y)| \leq c_3 |x||\theta x||y|^{-m} \leq c_3 r^2 |y|^{-m}$. Hölder's inequality (Proposition 1.14.1) gives

$$|V^{m,p}(f - f_r)(x)| \leq \int_{R^m \setminus B(0;r)} |f(y)| \frac{c_3 r^2}{|y|^m} \, \mathrm{d}\mathcal{H}_m(y)$$

$$\leq c_3 r^2 \|f\|_{L^p(R^m)} \left[\int_{R^m \setminus B(0;r)} |y|^{-mp/(p-1)} \, \mathrm{d}\mathcal{H}_m(y) \right]^{(p-1)/p} < \infty.$$

Proposition 3.13.2 and Hölder's inequality (Proposition 1.14.1) forces that $\|V^{m,p} f\|_{L^p(B(0;r/2))} \leq C_3 \|f\|_{L^p(R^m)}$ with C_3 independent on f. If $|x| < r/2$, $r < |y|$ then $|\partial[h_m(x-y) - h_m(-y) - x \cdot \nabla h_m(-y)]/\partial x_j| = |\partial_j h_m(x-y) - \partial_j h_m(-y)| \leq c_3 |x||y|^{-m} \leq c_3 r |y|^{-m}$. We have proved that $|f(y) - f_r(y)| c_3 r |y|^{-m} \in L^1(R^m)$. By virtue of Theorem on differentiability of parametrized integrals (Theorem 1.13.7),

$$\partial_j V^{m,p}(f - f_r)(x) = \int_{R^m} [\partial_j h_m(x-y) - \partial_j h_m(-y)](f(y) - f_r(y)) \, \mathrm{d}\mathcal{H}_m(y)$$

for $r > 1$. So, (3.23) holds true. Theorem on differentiability of integrals (Theorem 1.13.7) gives (3.21) for $f - f_r$ in $B(0; r/2)$. So, (3.21) holds true. Since T_{jk} is a bounded linear operator on $L^p(R^m)$ by Proposition 3.14.4, $V^{m,p}$ is a bounded linear operator from $L^p(R^m)$ to $D^{2,p}(R^m)$. Thus (3.24) holds. Since $V^{m,p}f \in D^{2,p}(R^m)$ we have $V^{m,p}f \in \mathcal{C}^0(R^m)$ by Proposition 1.25.2 and Theorem 1.21.13.

Corollary 3.16.2 *Let $f \in L^p(R^m)$, $1 < p < \infty$, $m \geq 2$. Then there exist $u_1, \ldots, u_m \in D^{1,p}(R^m)$ such that for $u = (u_1, \ldots, u_m)$, we have $\nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \cdots + \partial_m u_m = f$.*

Proof Put $u_j = \partial_j V^{m,p}(-f)$. Since $V^{m,p}f \in D^{2,p}(R^m)$ by Theorem 3.16.1, we have $u_j \in D^{1,p}(R^m)$. Moreover, $\nabla \cdot u = \Delta V^{m,p}(-f) = f$.

Corollary 3.16.3 *Let $u \in D^{2,p}(R^m)$, $f \in L^p(R^m)$, $1 < p < \infty$, $m \geq 2$. Then $-\Delta u = f$ in the sense of distributions if and only if $v = u - V^{m,p}f$ is a polynomial of degree less than or equal to* 1.

Proof If v is a polynomial of degree less than or equal to 1, then $-\Delta u = f$ by Theorem 3.16.1. Suppose now that $-\Delta u = f$. Then $\Delta v = f + \Delta V^{m,p}f = 0$ by Theorem 3.16.1. Theorem 2.18.2 gives that $v \in H(R^m)$. Since $u \in D^{2,p}(R^m)$ and $V^{m,p}f \in D^{2,p}(R^m)$ by Theorem 3.16.1, we deduce that $v \in D^{2,p}(R^m)$. Corollary 2.16.4 forces that v is a polynomial of degree less than or equal to 1.

Corollary 3.16.4 *Let $1 < p < \infty$, $m \geq 2$. Then $D^{2,p}(R^m) = \{V^{m,p}f; f \in L^p(R^m)\} \oplus P_0(R^m) \oplus P_1(R^m)$. (Here $P_k(R^m)$ denotes the space of all homogeneous polynomials on R^m of degree k.)*

Proof Let $u \in D^{2,p}(R^m)$. Then $f = -\Delta u \in L^p(R^m)$. According to Corollary 3.16.3, we have $u - V^{m,p}f \in P_0(R^m) \oplus P_1(R^m)$.

Let $f \in L^p(R^m)$. If $V^{m,p}f \in P_0(R^m) \oplus P_1(R^m)$ then $f = -\Delta V^{m,p}f = 0$ by Theorem 3.16.1.

Proposition 3.16.5 *Let $1 < p < \infty$, $m \geq 2$. If $p < m/2$ then $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m) \oplus P_0(R^m) \oplus P_1(R^m)$. If $m/2 \leq p < m$ then $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m) \oplus P_1(R^m)$. If $p \geq m$ then $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m)$.*

Proof Clearly $\overset{\circ}{D}{}^{2,p}(R^m) + P_0(R^m) + P_1(R^m) \subset D^{2,p}(R^m)$.

Let $f \in L^p(R^m)$. According to Corollary 2.18.5, there exists a sequence $\varphi_n \in \mathcal{C}_c^\infty(R^m)$ such that $\Delta \varphi_n \to f$ in $L^p(R^m)$. Corollary 2.2.7 gives $V(-\Delta \varphi_n) = \varphi_n \in \overset{\circ}{D}{}^{2,p}(R^m)$. Since $V^{m,p}\varphi_n - V\varphi_n \in P_0(R^m) + P_1(R^m)$ we obtain $V^{m,p}\varphi_n \in \overset{\circ}{D}{}^{2,p}(R^m) + P_0(R^m) + P_1(R^m)$. Since $\overset{\circ}{D}{}^{2,p}(R^m) + P_0(R^m) + P_1(R^m)$ is a closed subspace of $D^{2,p}(R^m)$ by Proposition 1.3.7, Theorem 1.3.7 gives that $V^{m,p}f \in \overset{\circ}{D}{}^{2,p}(R^m) + P_0(R^m) + P_1(R^m)$. Corollary 3.16.4 forces $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m) + P_0(R^m) + P_1(R^m)$.

If $p < m/2$, then $\overset{\circ}{D}{}^{2,p}(R^m) \cap [P_0(R^m) \oplus P_1(R^m)] = \emptyset$ by Proposition 1.25.8 and thus $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m) \oplus P_0(R^m) \oplus P_1(R^m)$.

If $p \geq m$, then $[P_0(R^m) \oplus P_1(R^m)] \subset \overset{\circ}{D}{}^{2,p}(R^m)$ by Proposition 1.25.8 and thus $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m)$.

Let now $m/2 \le p < m$. Then $P_0(R^m) \subset \overset{\circ}{D}{}^{2,p}(R^m)$ by Proposition 1.25.8. Let now $u \in P_1(R^m) \cap \overset{\circ}{D}{}^{2,p}(R^m)$. Then $\partial_j u \in \overset{\circ}{D}{}^{1,p}(R^m) \cap P_0(R^m)$. Proposition 1.25.8 gives that $\partial_j u = 0$. Since u is a homogeneous polynomial of degree 1 and $\nabla u = 0$, we deduce that $u \equiv 0$. Thus $D^{2,p}(R^m) = \overset{\circ}{D}{}^{2,p}(R^m) \oplus P_1(R^m)$.

3.17 The Laplace Operator on $\tilde{W}^{k+2,p}(R^m)$

In this section we prove that the Laplace operator maps $\tilde{W}^{k+2,p}(R^m)$ onto $\tilde{W}^{k,p}(R^m)$.

Lemma 3.17.1 *Denote by $P_l(R^m)$ the space of all homogeneous polynomials on R^m of degree l. If $k \in N_0$ and $P \in P_k(R^m)$ then there exists $Q \in P_{k+2}(R^m)$ such that $\Delta Q = P$.*

Proof The operator $T : u \mapsto \Delta u$ is a linear operator from $P_{k+2}(R^m)$ to $P_k(R^m)$. Since $\operatorname{Ker} T = P_{k+2}(R^m) \cap H(R^m)$ we have $\dim T(P_{k+2}(R^m)) = \dim P_{k+2}(R^m) - \dim[P_{k+2}(R^m) \cap H(R^m)]$. Corollary 2.14.4 gives that $\dim T(P_{k+2}(R^m)) = \dim P_k(R^m))$. Thus $T(P_{k+2}(R^m)) = P_k(R^m)$.

Theorem 3.17.2 *Let $1 < p < \infty$ and $k \in N_0$. Then the Laplace operator Δ is a bounded linear operator from $D^{k+2,p}(R^m)$ into $D^{k,p}(R^m)$, and from $\tilde{W}^{k+2,p}(R^m)$ onto $\tilde{W}^{k,p}(R^m)$.*

Proof Define on $D^{n,p}(R^m)$ the norm

$$\|u\|_{D^{n,p}(R^m)} = \left[\int_{R^m} |\nabla^n u|^p \, d\mathcal{H}_m + \int_{B(0;1)} |u|^p \, d\mathcal{H}_m\right]^{1/p}.$$

If $u \in D^{2+k,p}(R^m)$, then

$$\int_{R^m} |\nabla^k(\Delta u)|^p \, d\mathcal{H}_m \le \int_{R^m} |\nabla^{k+2} u|^p \, d\mathcal{H}_m.$$

According to Proposition 1.25.2, there is a constant C_1 that does not depend on u such that

$$\|u\|_{W^{k+2,p}(B(0;1))} \le C_1 \left[\int_{B(0;1)} |\nabla^{k+2} u|^p \, d\mathcal{H}_m + \int_{B(0;1)} |u|^p \, d\mathcal{H}_m\right]^{1/p}.$$

Hence $\|\Delta u\|_{L^p(B(0;1))} \le C_1 \|u\|_{D^{2+k,p}(R^m)}$, $\Delta : D^{k+2,p}(R^m) \to D^{k,p}(R^m)$ is bounded. If $u \in \mathcal{C}_c^\infty(R^m)$ then $\Delta u \in \mathcal{C}_c^\infty(R^m)$. Thus $\Delta u \in \tilde{W}^{k,p}(R^m)$ for all $u \in \tilde{W}^{k+2,p}(R^m)$.

Fix $f \in \tilde{W}^{k,p}(R^m)$. We show that there exists $u \in \tilde{W}^{k+2,p}(R^m)$ such that $\Delta u = f$. Suppose first that $k = 0$. According to Theorem 3.16.1, there exists $v \in D^{2,p}(R^m)$ such that $\Delta v = f$. According to Proposition 3.16.5, there exist $u \in \tilde{W}^{2,p}(R^m)$ and a polynomial P of degree at most 1 such that $v = u + P$. Clearly, $\Delta u = f$.

In the rest of the proof, we suppose that $k > 0$. Suppose now that $f \in \mathcal{C}_c^\infty(R^m)$. Set $u = -Vf(= -f * h_m)$. Then $u \in \mathcal{C}^\infty(R^m)$ by § 1.15, and $\Delta u = f$ by Theorem 3.1.5. Fix an index α with $|\alpha| = k$. Then $\partial^\alpha u = h_m * \partial^\alpha f = V(\partial^\alpha f)$. (See § 1.15.) Since $f \in \tilde{W}^{k,p}(R^m)$, we have $\partial^\alpha f \in L^p(R^m)$. Theorem 3.14.5 gives that $\partial^\alpha u \in D^{2,p}(R^m)$ and

$$\|\nabla^2 \partial^\alpha u\|_{L^p(R^m)} \leq C_2 \|\partial^\alpha f\|_{L^p(R^m)}$$

where C_2 depends only on m and p. Thus $u \in D^{k+2,p}(R^m)$ and

$$\|\nabla^{k+2} u\|_{L^p(R^m)} \leq C_3 \|f\|_{D^{k,p}(R^m)}, \tag{3.25}$$

where C_3 depends only on m and p. We now show that $u \in \tilde{W}^{k+2,p}(R^m)$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $B(0;1)$ and $\varphi = 0$ on $R^m \setminus B(0;2)$. Then there exists a constant C_4 such that $|\partial^\beta \varphi| \leq C_4$ for $|\beta| \leq k+2$. Denote $\varphi_n(x) = \varphi(x/n)$, $u_n = u\varphi_n$. Then $u_n \in \mathcal{C}_c^\infty(R^m)$. Clearly, $u_n = u$ on $B(0;1)$. If β is a multiindex with $|\beta| \leq k+2$, then $|\partial \varphi_n| \leq C_4 n^{-|\beta|}$ and $|\partial^\alpha u(x)| = O(|x|^{2-m-|\alpha|+1/2})$ as $|x| \to \infty$. (Compare Proposition 3.1.6.) So, there exists a constant C_5 such that

$$|\nabla^{k+2}[(1 - \varphi_n(x))u(x)]| \leq C_5 n^{-m-k+1/2}, \qquad n < |x| < 2n.$$

Since $k \geq 1$, we have

$$\|u - u_n\|^p_{D^{k+2,p}(R^m)} = \int\limits_{B(0;2n)\setminus B(0;n)} |\nabla^{k+2}(u - u_n)|^p \leq \frac{C_5 \mathcal{H}_m(B(0;n))}{n^{mp+p/2}} \to 0.$$

Therefore $u \in \tilde{W}^{k+2,p}(R^m)$.

Let now $f \in \tilde{W}^{k,p}(R^m)$ be general. Choose $f_n \in \mathcal{C}_c^\infty(R^m)$ such that $f_n \to f$ in $D^{k,p}(\Omega)$. We can suppose that $\|f_n\|_{D^{k,p}(R^m)} \leq 2\|f\|_{D^{k,p}(R^m)}$. Denote $v_n = -Vf_n$. We have proved that $v_n \in \tilde{W}^{k+2,p}(R^m)$. According to (3.25), we have

$$\|\nabla^{k+2} v_n\|_{L^p(R^m)} \leq 2C_3 \|f\|_{D^{k,p}(R^m)}. \tag{3.26}$$

Suppose first that $p(k+2) < m$. Denote $q = mp/[m - (k+2)p]$. According to Lemma 1.25.7,

$$\|v_n\|_{L^q(R^m)} \leq C_6 \|\nabla^{k+2} v_n\|_{L^p(R^m)} \leq 2C_3 C_6 \|f\|_{D^{k,p}(R^m)},$$

where C_6 depends only on m and p. Since $q > p$, Corollary 1.14.2 yields

$$\|v_n\|_{L^p(B(0;1))} \leq \tilde{C}_6 \|v_n\|_{L^q(B(0;1))} \leq 2C_3 C_6 \tilde{C}_6 \|f\|_{D^{k,p}(R^m)}$$

where $\tilde{C}_6$ depends only on p, q, and m. This and the inequality (3.26) give that v_n is a bounded sequence in the space $\tilde{W}^{k+2,p}(R^m)$. According to Theorem 1.5.10 and Lemma 1.25.3, there exist $v \in \tilde{W}^{k+2,p}(R^m)$ and a subsequence $v_{n(j)}$ such that $v_{n(j)}$ converges to v weakly. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\langle v, \Delta\varphi \rangle = \lim_{j\to\infty} \langle v_{n(j)}, \Delta\varphi \rangle = \lim_{j\to\infty} \langle \Delta v_{n(j)}, \varphi \rangle = \lim_{j\to\infty} \langle f_{n(j)}, \varphi \rangle = \langle f, \varphi \rangle.$$

Thus $\Delta v = f$.

Let now $p(k+2) \geq m$. Choose $l \in \{1, \dots, k+2\}$ such that $p(k+2-l) < m$, $p(k+3-l) \geq m$. According to Lemma 1.25.7, there exist constants $q \in (p, \infty)$ and C_7 such that

$$\|\nabla^l w\|_{L^q(R^m)} \leq C_7 \|\nabla^{k+2} w\|_{L^p(R^m)} \qquad \forall w \in \tilde{W}^{k+2,p}(R^m).$$

According to (3.26), we have $v_n \in D^{l,q}(R^m)$ and

$$\|\nabla^l v_n\|_{L^q(R^m)} \leq 2C_3 C_7 \|f\|_{D^{k,p}(R^m)}. \tag{3.27}$$

Denote by $\tilde{P}_t(R^m)$ the space of polynomials of degree at most t. According to Lemma 1.25.5, there exists a constant C_8 such that

$$\inf_{P \in \tilde{P}_{l-1}(R^m)} \|w + P\|_{D^{l,q}(R^m)} \leq C_8 \|\nabla^l w\|_{L^q(R^m)} \qquad \forall w \in D^{l,q}(R^m).$$

Hence there exist polynomials $P_n \in \tilde{P}_{l-1}(R^m)$ such that

$$\|v_n + P_n\|_{D^{l,q}(R^m)} \leq 2C_8 \|\nabla^l v_n\|_{L^q(R^m)}.$$

Corollary 1.14.2 yields

$$\|v_n + P_n\|_{L^p(\omega)} \leq C_9 \|v_n + P_n\|_{L^q(\omega)} \leq 2C_8 C_9 \|\nabla^l v_n\|_{L^q(R^m)},$$

where C_9 depends only on p, q, and ω. According to (3.27),

$$\|v_n + P_n\|_{L^p(\omega)} \leq 4C_3 C_7 C_8 C_9 \|f\|_{D^{k,p}(R^m)}.$$

Since $\nabla^{k+2}(v_n + P_n) = \nabla^{k+2} v_n$, (3.27) yields that $\{v_n + P_n\}$ is a bounded sequence of $D^{k+2,p}(R^m)$. Since $p \geq m/(k+3-l)$, Proposition 1.25.8 yields that $P_n \in \tilde{W}^{k+2,p}(R^m)$. Thus $v_n + P_n \in \tilde{W}^{k+2,p}(R^m)$. Since the mapping $\Delta : \tilde{W}^{k+2,p}(R^m) \to \tilde{W}^{k,p}(R^m)$ is bounded, $\Delta(v_n + P_n)$ is a bounded sequence in $\tilde{W}^{k,p}(R^m)$. According to Theorem 1.5.10 and Lemma 1.25.3, there exist $w \in \tilde{W}^{k+2,p}(R^m)$, $\tilde{w} \in \tilde{W}^{k,p}(R^m)$ and a subsequence $v_{n(j)} + P_{n(j)}$ such that $v_{n(j)} + P_{n(j)}$ converges to w weakly, and $\Delta(v_{n(j)} + P_{n(j)})$ converges to $\tilde{w}$ weakly. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\langle w, \Delta\varphi\rangle = \lim_{j\to\infty} \langle v_{n(j)} + P_{n(j)}, \Delta\varphi\rangle = \lim_{j\to\infty} \langle \Delta(v_{n(j)} + P_{n(j)}), \varphi\rangle = \langle \tilde{w}, \varphi\rangle.$$

Thus $\Delta w = \tilde{w}$. If $l \leq 2$ then $\Delta(v_{n(j)} + P_{n(j)}) = \Delta v_{n(j)} = f_j \to f$, because $P_{n(j)} \in \tilde{P}_{l-1}(R^m)$. Thus $\tilde{w} = f$ and $\Delta w = f$. Let now $l \geq 3$. Since $f_{n(j)} + \Delta P_{n(j)} = \Delta(v_{n(j)} + P_{n(j)})$ converges to $\tilde{w}$ weakly, and $f_{n(j)} \to f$, we deduce that $\Delta P_{n(j)}$ converges weakly to $\tilde{w} - f$. Since $\Delta P_{n(j)} \in \tilde{P}_{l-3}(R^m)$, Proposition 1.5.14 gives that $\tilde{w} - f \in \tilde{P}_{l-3}(R^m)$. According to Lemma 3.17.1, there exists $Q \in \tilde{P}_{l-1}(R^m)$ such that $\Delta Q = \tilde{w} - f$. Clearly, $\Delta(w - Q) = f$. Since $p \geq m/(k + 3 - l)$, Proposition 1.25.8 yields that $Q \in \tilde{W}^{k+2,p}(R^m)$. Thus $w - Q \in \tilde{W}^{k+2,p}(R^m)$.

3.18 Regularity of Solutions of the Poisson Equation

In this section, we study regularity of solutions of the equation $\Delta^k u = f$ dependent on the regularity of f.

Proposition 3.18.1 *Let Ω be an open subset of R^m, $m \geq 2$, $u \in \mathcal{C}^0(\Omega)$, $f \in \mathcal{C}^k(\Omega)$, $k \in N_0$, $-\Delta u = f$ in Ω in the sense of distributions, ω be a bounded open set such that $\overline{\omega} \subset \Omega$, $0 < \lambda < 1$. Then $u \in \mathcal{C}^{k+1,\lambda}(\omega)$. If, moreover, $f \in \mathcal{C}^{k,\lambda}(\Omega)$, then $u \in \mathcal{C}^{k+2,\lambda}(\omega)$.*

Proof We can suppose that Ω is bounded. Fix an open set U such that $\overline{\omega} \subset U \subset \overline{U} \subset \Omega$. Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi = 1$ on U. Define $\varphi f = 0$ on $R^m \setminus \Omega$. Then $\varphi f \in \mathcal{C}^0(R^m)$ has compact support and $\varphi f = f$ in U. Proposition 3.15.1 gives that $V(\varphi f) \in \mathcal{C}^{1,\lambda}(\Omega) \cap \mathcal{C}^1(R^m)$. Since $-\Delta V(\varphi f) = \varphi f$ in the sense of distributions (see Theorem 3.1.5), we have $\Delta u = \Delta V(\varphi f)$ in U. So, $u \in \mathcal{C}^{1,\lambda}(\omega)$ by Proposition 3.1.7.

Suppose now that $f \in \mathcal{C}^{k,\lambda}(\Omega)$, $0 < \lambda < 1$. Then $\varphi f \in \mathcal{C}^{k,\lambda}(\Omega)$. Theorem 3.14.2 gives that $V(\varphi f) \in \mathcal{C}^{2,\lambda}(\Omega)$. If α is a multiindex with $|\alpha| \leq k$, then $\partial^\alpha(\varphi f) \in \mathcal{C}^{0,\lambda}(R^m)$ has compact support in Ω. Therefore $\partial^\alpha V(\varphi f) = \partial^\alpha h_m * (\varphi f) = h_m * [\partial^\alpha(\varphi f)] \in \mathcal{C}^{2,\lambda}(\Omega)$ (see Theorem 3.14.2). Since $V(\varphi f) \in \mathcal{C}^{k+2,\lambda}(\Omega)$ we have $u \in \mathcal{C}^{k+2,\lambda}(\omega)$ by Proposition 3.1.7.

Let now $f \in \mathcal{C}^k(\Omega)$, $k \geq 1$. Then $f \in \mathcal{C}^{k-1,\lambda}(U)$ and $u \in \mathcal{C}^{k+1,\lambda}(\omega)$.

Proposition 3.18.2 *Let Ω be an open subset of R^m, $m \geq 2$, $u \in L^1_{loc}(\Omega)$, $f \in W^{k,p}(\Omega)$, $k \in N_0$, $j \in N$, $1 < p < \infty$, $-\Delta^j u = f$ in Ω in the sense of distributions, ω be a bounded open set such that $\overline{\omega} \subset \Omega$. Then $u \in W^{k+2j,p}(\omega)$.*

Proof Suppose first that $j = 1$. Fix a bounded open set U such that $\overline{\omega} \subset U \subset \overline{U} \subset \Omega$. Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi = 1$ on U. Define $\varphi f = 0$ on $R^m \setminus \Omega$. If α is a multiindex with $|\alpha| \leq k$, then $\partial^\alpha(\varphi f) \in L^p(R^m)$ has compact support in Ω. Therefore $\partial^\alpha V(\varphi f) = \partial^\alpha h_m * (\varphi f) = h_m * [\partial^\alpha(\varphi f)] \in W^{2,p}(\Omega)$ by Theorem 3.14.5

and Proposition 1.25.2. Since $-\Delta V(\varphi f) = \varphi f$ in the sense of distributions (see Theorem 3.1.5), we have $\Delta u = \Delta V(\varphi f)] = 0$ in U. Thus $u \in W^{2,p}(\Omega)$ by Proposition 3.1.7.

Let now the proposition holds true for $j-1 \in N$. We have proved that $\Delta^{j-1}u \in W^{k+2,p}_{loc}(\Omega)$. Hence $u \in W^{k+2j,p}_{loc}(\Omega)$.

Proposition 3.18.3 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, u be a distribution in Ω, $1 \leq p < m/(m-1)$. If Δu in the sense of distribution is a real measure, then $u \in W^{1,p}_{loc}(\Omega)$.*

Proof Denote $\mu = -\Delta u$. Fix $x \in \Omega$, $r > 0$ such that $B(x;3r) \subset \Omega$. Put $\nu = \mu | B(x;2r)$. Then $V\nu \in W^{1,p}(B(x;r))$ by Proposition 3.13.1. We have $\Delta(u - V\nu) = \mu - \nu = 0$ in $B(x;2r)$ by Theorem 3.1.5. So, $u - V\nu \in H(B(x;2r)) \subset \mathcal{C}^\infty(B(x;2r)) \subset W^{1,p}(B(x;r))$ (see Theorem 2.18.2 and Corollary 2.2.8). Thus $u \in W^{1,p}(B(x;r))$.

Proposition 3.18.4 *Let $m \geq 2$, $1 < p < \infty$. If $u \in D^{2,p}(R^m)$, then*

$$\|\nabla^2 u\|_{L^p(R^m)} \leq C_1 \|\Delta u\|_{L^p(R^m)}, \tag{3.28}$$

where a constant C_1 depends only on p and m. If, moreover, Ω is a bounded open set and $u \in \overset{\circ}{W}{}^{2,p}(\Omega)$, then

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \|\Delta u\|_{L^p(\Omega)}, \tag{3.29}$$

where a constant C_2 depends only on p and Ω.

Proof Put $f = -\Delta u$, $v = u - V^{m,p}f$. Then v is a polynomial of degree less than or equal to 1 by Corollary 3.16.3. According to Theorem 3.16.1, there exists a constant C_1 dependent only on p and m such that $\|\nabla^2 V^{m,p} f\|_{L^p(R^m)} \leq C_1 \|f\|_{L^p(R^m)}$. Since $\nabla^2 u = \nabla^2 V^{m,p} f$ and $-\Delta u = f$ we obtain (3.28).

Let Ω be a bounded open set. According to Lemma 1.21.10, there exists a constant C such that $\|u\|_{W^{2,p}(\Omega)} \leq C \|\nabla^2 u\|_{L^p(\Omega)}$ for all $u \in \overset{\circ}{W}{}^{2,p}(\Omega)$. This and (3.28) give (3.29).

Proposition 3.18.5 *Let $K \subset R^m$ be a compact set. Let f be a distribution in R^m supported in K and $\Omega \subset R^m$ be a bounded open set. Let $1 < p, q < \infty$, $s \in R^1$. If $f \in W^{s,p}(R^m)$, then $Vf \in W^{s+2,p}(\Omega)$ and*

$$\|Vf\|_{W^{s+2,p}(\Omega)} \leq C \|f\|_{W^{s,p}(R^m)}$$

where a constant C depends only on K, Ω, p and s. If $f \in B^{p,q}_s(R^m)$ then $Vf \in B^{p,q}_{s+2}(\Omega)$ and

$$\|Vf\|_{B^{p,q}_{s+2}(\Omega)} \leq C \|f\|_{B^{p,q}_s(R^m)}$$

where a constant C depends only on K, Ω, p, q, and s.

Proof Fix a bounded domain ω with smooth boundary such that $\overline{\Omega} \cup \operatorname{spt} f \subset \omega$. Let $k \in N_0$, $f \in W^{k,p}(R^m)$. Using regularization we infer that $f \in \overset{\circ}{W}{}^{k,p}(\omega)$. We have $Vf \in W^{k+2,p}(\Omega)$ by Proposition 3.18.2, Theorem 3.10.3, and Theorem 3.11.5. The operator V is a closed linear operator from $\overset{\circ}{W}{}^{k,p}(\omega)$ to $W^{k+2,p}(\Omega)$ by Proposition 3.13.2. Thus V is a bounded linear operator from $\overset{\circ}{W}{}^{k,p}(\omega)$ to $W^{k+2,p}(\Omega)$ by Theorem 1.4.15.

Fix $\varphi \in \mathcal{C}_c^\infty(\omega)$ such that $\varphi = 1$ on a neighborhood of K. Define $V_\varphi g = \varphi V(g\varphi)$. If $k \in N_0$, then V_φ is a bounded linear operator from $W^{k,p}(R^m)$ to $W^{k+2,p}(R^m)$ by Lemma 1.23.9, Theorem 1.23.5, and Theorem 1.22.1. Suppose that $s > 0$. Fix $k \in N$ such that $s < k$. Put $\theta = s/k$. Proposition 1.23.10 forces $(W^{0,p}(R^m), W^{k,p}(R^m))_{\theta,q} = B_s^{p,q}(R^m)$, and $(W^{2,p}(R^m), W^{k+2,p}(R^m))_{\theta,q} = B_{s+2}^{p,q}(R^m)$. So, V_φ is a bounded linear operator from $B_s^{p,q}(R^m)$ to $B_{s+2}^{p,q}(R^m)$ by Proposition 1.10.3. If $f \in B_s^{p,q}(R^m)$ then $Vf \in B_{s+2}^{p,q}(R^m)$. In particular, if $s \notin N$, $f \in W^{s,p}(R^m) = B_s^{p,p}(R^m)$ then $Vf \in W^{s+2,p}(R^m) = B_{s+2}^{p,p}(R^m)$ (see Theorem 1.23.3).

Let $t < 0$. Put $p' = p/(p-1)$, $q' = q/(q-1)$. Proposition 1.5.11 gives that the adjoint operator V'_φ is a bounded operator from $[B_{-t+2}^{p',q'}(R^m)]'$ to $[B_{-t}^{p',q'}(R^m)]'$. But $[B_{-t+2}^{p',q'}(R^m)]' = B_{t-2}^{p,q}(R^m)$, $[B_{-t}^{p',q'}(R^m)]' = B_t^{p,q}(R^m)$ by Theorem 1.23.1. Since $V'_\varphi = V_\varphi$ we infer that V_φ is a bounded linear operator from $B_{t-2}^{p,q}(R^m)$ to $B_t^{p,q}(R^m)$. Let now $s \le 0$. Then

$$(B_{-3+2s}^{p,q}(R^m), B_3^{p,q}(R^m))_{1/2,q} = B_s^{p,q}(R^m),$$
$$(B_{-1+2s}^{p,q}(R^m), B_5^{p,q}(R^m))_{1/2,q} = B_{s+2}^{p,q}(R^m)$$

by Theorem 1.23.6. Since V_φ is a bounded linear operator from $B_{-3+2s}^{p,q}(R^m)$ to $B_{-1+2s}^{p,q}(R^m)$, and from $B_3^{p,q}(R^m)$ to $B_5^{p,q}(R^m)$, the interpolation argument gives that V_φ is a bounded linear operator from $B_s^{p,q}(R^m)$ to $B_{s+2}^{p,q}(R^m)$ (see Proposition 1.10.3). If $f \in B_s^{p,q}(R^m)$ then $Vf \in B_{s+2}^{p,q}(R^m)$. In particular, if $s < 0, f \in W^{s,p}(R^m) = B_s^{p,p}(R^m)$ then Theorem 1.23.3 gives $Vf \in W^{s+2,p}(R^m) = B_{s+2}^{p,p}(R^m)$.

Corollary 3.18.6 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, $s \in R^1$, $1 < p, q < \infty$, u be a distribution on Ω, $\Delta u \in B_s^{p,q}(\Omega)$. If ω is a bounded open set such that $\overline{\omega} \subset \Omega$ then $u \in B_{s+2}^{p,q}(\omega)$.*

Proof Choose a bounded open set ω_1 such that $\overline{\omega} \subset \omega_1 \subset \overline{\omega}_1 \subset \Omega$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi = 1$ on ω_1. Put $f = \varphi \Delta u$. Then f is a distribution with a compact support. Moreover, $f \in B_s^{p,q}(R^m)$ by Lemma 1.23.9. Proposition 3.18.5 gives that $Vf \in B_{s+2}^{p,q}(\omega_1)$. Since $\Delta u = \Delta(-Vf)$ in ω_1 by Theorem 3.1.5, we have $u \in B_{s+2}^{p,q}(\omega)$ by Proposition 3.1.7.

3.19 Supersolutions

We define a supersolution of the equation $-\Delta u = f$ as it is common in the Sobolev space $W^{1,p}(\Omega)$ in the theory of partial differential equations. Then we prove that the following are equivalent: 1) u is a supersolution of the equation $-\Delta u = 0$; 2) $-\Delta u \geq 0$ in the sense of distributions; and 3) u is superharmonic.

Definition 3.19.1 Let $\Omega \subset R^m$ be an open set, $f, g \in \mathcal{D}'(\Omega)$. We say that $f \leq g$ if $\langle f, \varphi\rangle \leq \langle g, \varphi\rangle$ for each nonnegative $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Definition 3.19.2 Let $\Omega \subset R^m$ be an open set, $m \geq 2$, g be a distribution in Ω, $u \in W_{loc}^{1,1}(\Omega)$. We say that u is a supersolution of the equation $-\Delta u = g$ in Ω if

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m \geq \langle g, \varphi\rangle, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega), \varphi \geq 0.$$

We say that u is a subsolution of the equation $-\Delta u = g$ in Ω if

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m \leq \langle g, \varphi\rangle, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega), \varphi \geq 0.$$

Theorem 3.19.3 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, $g \in \mathcal{D}'(\Omega)$, $u \in W_{loc}^{1,1}\Omega)$. Then $-\Delta u = g$ in Ω in the sense of distributions if and only if*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle g, \varphi\rangle \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

The function u is a subsolution of the equation $-\Delta u = g$ if and only if $-\Delta u \leq g$ in the sense of distribution. The function u is a supersolution of the equation $-\Delta u = g$ if and only if $-\Delta u \geq g$ in the sense of distribution. If $u \in D^{1,p}(\Omega)$, $1 \leq p \leq \infty$, then $\Delta u \in W^{-1,p}(\Omega) \cap D^{-1,p}(\Omega)$ and $\|\Delta u\|_{W^{-1,p}(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$, $\|\Delta u\|_{D^{-1,p}(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$.

Proof If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then

$$\langle -\Delta u, \varphi\rangle = -\sum_{j=1}^m \langle \partial_j(\partial_j u), \varphi\rangle = \sum_{j=1}^m \langle \partial_j u, \partial_j \varphi\rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m.$$

According to the definitions, u is a subsolution of the equation $-\Delta u = g$ if and only if $-\Delta u \leq g$ in the sense of distribution; u is a supersolution of the equation $-\Delta u = g$ if and only if $-\Delta u \geq g$ in the sense of distribution.

Let now $u \in D^{1,p}(\Omega)$, $1/p + 1/p' = 1$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, Hölder's inequality (Proposition 1.14.1) yields

$$|\langle -\Delta u, \varphi\rangle| = \left| \int_\Omega \nabla u \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m \right| \le \|\nabla u\|_{L^p(\Omega)} \|\nabla \varphi\|_{L^{p'}(\Omega)}.$$

Since $\mathcal{C}_c^\infty(\Omega)$ is a dense subset of $\overset{\circ}{W}{}^{1,p'}(\Omega)$ and $\overset{\circ}{D}{}^{1,p'}(\Omega)$, we obtain $\Delta u \in W^{-1,p}(\Omega) \cap D^{-1,p}(\Omega)$ and $\|\Delta u\|_{W^{-1,p}(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}$, $\|\Delta u\|_{D^{-1,p}(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}$.

Theorem 3.19.4 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$, $u \in \mathcal{D}'(\Omega)$, $1 \le p < m/(m-1)$. Then the following assertions are equivalent:*

1. *$-\Delta u \ge 0$ in Ω in the sense of distributions.*
2. *u is representable by a superharmonic function in Ω.*
3. *u is representable by $v \in L^1_{loc}(\Omega)$ such that for almost all $x \in \Omega$*

$$v(x) \ge \fint_{B(x;r)} v \, \mathrm{d}\mathcal{H}_m \qquad \forall\, r \in (0, \operatorname{dist}(x, \partial\Omega)).$$

4. *$u \in W^{1,p}_{loc}(\Omega)$ is a supersolution of the equation $-\Delta u = 0$.*

Proof 1) $\Rightarrow$ 2, 4)) : $\mu = -\Delta u$ is a nonnegative Borel measure by Lemma 1.20.3. Fix $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)/2$. Put $\nu = \mu | B(x; 2r)$ the restriction of μ onto $B(x; 2r)$. Fix a nonnegative $\Psi \in \mathcal{C}_c^\infty(\Omega)$ such that $\Psi = 1$ on $B(x; 2r)$. Then $\nu(R^m) = \nu(B(x; 2r)) \le \langle \mu, \Psi \rangle < \infty$, and ν is a finite nonnegative Borel measure with compact support. Proposition 3.11.3 and Proposition 3.10.2 give that the volume potential $V\nu$ is a superharmonic function in R^m. Since $-\Delta V\nu = \nu$ in R^m by Theorem 3.1.5, we have $-\Delta(u - V\nu) = \mu - \nu = 0$ in $B(x; 2r)$. Thus $u - V\nu$ is representable by a harmonic function in $B(x; 2r)$ by Theorem 2.18.2. So, u is superharmonic in $B(x; 2r)$. Corollary 3.3.8 forces that u is superharmonic in Ω. Proposition 3.3.4, Proposition 3.18.3, and Theorem 3.19.3 give that $u \in W^{1,p}_{loc}(\Omega)$ is a supersolution of the equation $-\Delta u = 0$.

2) $\Rightarrow$ 3) : A superharmonic function is locally integrable by Proposition 3.3.4.

3) $\Rightarrow$ 1) : Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi \ge 0$. Choose a bounded open set G such that $\operatorname{spt}\varphi \subset G \subset \overline{G} \subset \Omega$. Put $w = v$ on G, $w = 0$ on $R^m \setminus G$. Choose $\Phi \in \mathcal{C}^\infty(R^1)$ such that $\Phi \ge 0$, $\operatorname{spt}\Phi \subset (0, 1)$ and

$$\int_{R^m} \Phi(|y|) \, d\mathcal{H}_m(y) = 1.$$

For $\epsilon > 0$ put $\Phi_\epsilon(y) = \Phi(|y|/\epsilon)\epsilon^{-m}$ and $w_\epsilon = w * \Phi_\epsilon$. Then $w_\epsilon \in \mathcal{C}^\infty(R^m)$. Lemma 3.8.1 gives that w_ϵ is a superharmonic function in $G_\epsilon = \{x \in G; \operatorname{dist}(x, \partial G) > \epsilon\}$. If there is $x \in G_\epsilon$ such that $-\Delta w_\epsilon(x) < 0$, then there is an open neighborhood V of x such that $-\Delta(-w_\epsilon) > 0$ on V. We have proved that $-w_\epsilon$ is superharmonic on V. Since w_ϵ is superharmonic and subharmonic on $V \cap G_\epsilon$, Mean value theorem (Theorem 2.6.1) forces $\Delta w_\epsilon = 0$ on $G_\epsilon \cap V$, what is a contradiction. Therefore $-\Delta w_\epsilon \geq 0$ on G. Since $\operatorname{spt}\varphi$ is a compact subset of G, we have $\operatorname{spt}\varphi \subset G_\epsilon$ for sufficiently small ϵ (see Remark 1.2.10). Thus

$$\langle -\Delta u, \varphi\rangle = \int\limits_{\Omega} v(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m = \int\limits_{G} w(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m$$

$$= \lim_{\epsilon\searrow 0}\int\limits_{G} w_\epsilon(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m = \lim_{\epsilon\searrow 0}\int\limits_{G}(-\Delta w_\epsilon)\varphi\,\mathrm{d}\mathcal{H}_m \geq 0.$$

4) $\Rightarrow$ 1) : This is a consequence of Theorem 3.19.3.

Corollary 3.19.5 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$, $u \in L^1_{loc}(\Omega)$. Then u is superharmonic in Ω if and only if $-\Delta u \geq 0$ in Ω in the sense of distributions and*

$$u(x) = \lim_{r\downarrow 0} \fint\limits_{B(x;r)} u\,\mathrm{d}\mathcal{H}_m \qquad \forall x \in \Omega. \tag{3.30}$$

In particular, if $u \in \mathcal{C}^2(\Omega)$ then u is superharmonic if and only if $-\Delta u \geq 0$ in Ω.

Proof Suppose first that u is superharmonic. Then $-\Delta u \geq 0$ by Theorem 3.19.4 and (3.30) holds by Proposition 3.7.1.

Let now $-\Delta u \geq 0$ and (3.30) holds. According to Theorem 3.19.4, there exists a superharmonic function v such that $u = v$ almost everywhere in Ω. By virtue of Proposition 3.7.1,

$$u(x) = \lim_{r\searrow 0} \fint\limits_{B(x;r)} u\,\mathrm{d}\mathcal{H}_m = \lim_{r\searrow 0} \fint\limits_{B(x;r)} v\,\mathrm{d}\mathcal{H}_m = v(x).$$

Proposition 3.19.6 *Let $\Omega \subset R^m$ be an open set and u, v be superharmonic functions on Ω. If $\Delta u = \Delta v$ in the sense of distributions, then there exists $h \in H(\Omega)$ such that $u = v + h$. In particular, if u is superharmonic in Ω, then u is harmonic if and only if $\Delta u = 0$ in the sense of distributions.*

Proof $w = u - v \in L^1_{loc}(\Omega)$ by Proposition 3.3.4. Since $\Delta w = 0$ in the sense of distribution, there exists $h \in H(\Omega)$ such that $w = h$ almost everywhere in Ω (see Theorem 2.18.2). By virtue of Proposition 3.7.1,

$$u(x) = \lim_{r\downarrow 0} \fint\limits_{B(x;r)} u\,\mathrm{d}\mathcal{H}_m = \lim_{r\downarrow 0} \fint\limits_{B(x;r)} v\,\mathrm{d}\mathcal{H}_m + \lim_{r\downarrow 0} \fint\limits_{B(x;r)} h\,\mathrm{d}\mathcal{H}_m = v(x) + h(x).$$

3.20 Eigenvalues of Green's Potential

In this section we study eigenvalues of the Green potentials. These results will be useful when we shall discuss eigenvalues for the Dirichlet problem.

Theorem 3.20.1 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, $1 < p \leq \infty$.*

- *Then $G_\Omega : f \mapsto G_\Omega f$ is a compact linear operator on $L^p(\Omega)$ and on $\mathcal{C}^{0,0}(\Omega)$.*
- *Moreover, there exists a sequence of real numbers λ_k such that $|\lambda_k| \geq |\lambda_{k+1}| > 0$, $\lambda_k \to 0$ as $k \to \infty$, $\{\lambda_k; k \in N\}$ is the set of all eigenvalues in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$, the spectrum $\sigma(G_\Omega) = \{0\} \cup \{\lambda_k; k \in N\}$ in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$.*
- *The kernel* $\mathrm{Ker}(\lambda_k I - G_\Omega)$ *is the same in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$. Moreover, there exist a nonnegative integer $n(k)$ and $\varphi_1^k, \dots, \varphi_{n(k)}^k \in \mathcal{C}^{0,0}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ which form a basis of* $\mathrm{Ker}(\lambda_k I - G_\Omega)$. *If $\varphi \in \mathrm{Ker}(\lambda_k I - G_\Omega)$, $\psi \in \mathrm{Ker}(\lambda_j I - G_\Omega)$, $j \neq k$, then*

$$\int_\Omega \varphi \psi \, \mathrm{d}\mathcal{H}_m = 0. \tag{3.31}$$

  We can choose φ_j^k in a such way that $\{\varphi_j^k; k \in N, 1 \leq j \leq n(k)\}$ is a complete orthonormal sequence in $L^2(\Omega)$.
- *If Ω is connected, then we can choose $\varphi_1^1 > 0$ in Ω and* $\dim \mathrm{Ker}(\lambda_1 I - G_\Omega) = 1$.

Proof $G_\Omega \in \mathcal{C}^0(\{[x,y] \in \Omega \times \Omega; x \neq y\})$ by Proposition 3.9.12. Moreover, there exists a constant c such that $0 \leq G_\Omega(x,y) \leq c|x-y|^{1-m}$ (see Propositions 3.9.14 and 3.9.15). Therefore, G_Ω is a compact linear operator on $L^p(\Omega)$ and on $\mathcal{C}^{0,0}(\Omega)$ by Propositions 1.26.6 and 1.26.3.

Suppose now that $\varphi \in L^p(\Omega)$, $G_\Omega \varphi \equiv 0$. Then $0 = -\Delta G_\Omega \varphi = \varphi$ in Ω by Theorem 3.11.5. Thus 0 is not an eigenvalue of G_Ω.

Proposition 3.9.12 gives that $G_\Omega(x,y) = G_\Omega(y,x)$. If $\varphi, \psi \in L^2(\Omega)$, we get by Fubini's theorem

$$\int_\Omega \varphi G_\Omega \psi \, \mathrm{d}\mathcal{H}_m = \int_{\Omega \times \Omega} G_\Omega \psi(x,y)\varphi(x)\psi(y) \, \mathrm{d}\mathcal{H}_{2m}(x,y) = \int_\Omega \psi G_\Omega \varphi \, \mathrm{d}\mathcal{H}_m.$$

So, if $\varphi, \psi \in L^2(\Omega, C)$, then

$$\int_\Omega (G_\Omega \varphi)\overline{\psi} \, \mathrm{d}\mathcal{H}_m = \int_\Omega \varphi \overline{G_\Omega \psi} \, \mathrm{d}\mathcal{H}_m$$

and G_Ω is a self-adjoint operator in $L^2(\Omega, C)$. If $G_\Omega \varphi = \lambda \varphi$, $G_\Omega \psi = \mu \psi$ with $\mu \neq \lambda$ then(3.31) holds true by Proposition 1.9.17. According to Proposition 1.9.16, all eigenvalues of G_Ω in $L^2(\Omega, C)$ are real. So, for an eigenvalue λ there exists a real eigenfunction φ. According to Corollary 1.9.19, there exists a complete orthonormal sequence φ_j of eigenfunctions of G in $L^2(\Omega)$ with eigenvalues λ_j such that $\{\lambda_j\}$ is a set of all eigenvalues, $\lambda_j \to 0$, and $\{\varphi_j; \lambda_j = \lambda_k\}$ is a basis of $\mathrm{Ker}(G - \lambda_k I)$.

If $\lambda \in C \setminus \{0\}$, then $\lambda \in \sigma(G_\Omega)$ (in $L^p(\Omega, C)$ or $\mathcal{C}^{0,0}(\Omega, C)$) if and only if λ is an eigenvalue in this space (see Theorem 1.7.7). Moreover, $\mathrm{Ker}(G_\Omega - \lambda I)$ is the same in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$ (see Lemma 1.8.4 and Theorem 1.8.3). Therefore $\sigma(G_\Omega) \setminus \{0\} = \{\lambda_k; k \in N\}$ in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$. Since $\lambda_k \to 0$ as $k \to \infty$, Lemma 1.6.5 gives that $0 \in \sigma(G_\Omega)$ in $L^p(\Omega, C)$ and in $\mathcal{C}^{0,0}(\Omega, C)$.

We have proved that $\varphi_j^k \in \mathcal{C}^0(\Omega)$. Suppose now that $\varphi \in \mathcal{C}^r(\Omega)$ with $r \in N_0$. Since

$$- \Delta\varphi_j^k = -\lambda_k^{-1} \Delta G_\Omega \varphi_j^k = \lambda_k^{-1} \varphi_j^k \tag{3.32}$$

(see Theorem 3.11.5), Proposition 3.18.1 gives that $\varphi \in \mathcal{C}^{r+1}(\Omega)$. Hence $\varphi \in \mathcal{C}^\infty(\Omega)$.

Let now Ω be connected. Then $G_\Omega > 0$ in Ω by Proposition 3.9.7. By virtue of Proposition 1.9.16,

$$|\lambda_1| = \left| \int_\Omega \varphi_j^1 G_\Omega \varphi_j^1 \, \mathrm{d}\mathcal{H}_m \right| \le \int_\Omega |\varphi_j^1| G_\Omega |\varphi_j^1| \, \mathrm{d}\mathcal{H}_m$$

$$\le \sup \left\{ \left| \int_\Omega \varphi G_\Omega \varphi \, \mathrm{d}\mathcal{H}_m \right| ; \|\varphi\|_{L^2(\Omega)} = 1 \right\} = |\lambda_1|.$$

Therefore φ_j^1 does not change the sign. So, we can choose an orthonormal basis $\varphi_1^1, \dots, \varphi_{n(1)}^1$ of $\mathrm{Ker}(\lambda_1^1 I - G_\Omega)$ such that $\varphi_j^1 \ge 0$. It follows from (3.32) that $-\Delta\varphi_j^1 = \lambda_1^{-1}\varphi_j^1 \ge 0$. The function φ_j^1 is superharmonic by Theorem 3.19.4. If there is $x \in \Omega$ such that $\varphi_j^1(x) = 0$, then $\varphi_j^1 \equiv 0$ by Proposition 3.4.1. Therefore $\varphi_j^1 > 0$ in Ω. If $\dim \mathrm{Ker}(\lambda_1 I - G_\Omega) > 1$, then

$$\int_\Omega \varphi_1^1 \varphi_2^1 \, \mathrm{d}\mathcal{H}_m > 0,$$

what is a contradiction with the orthogonality of φ_1^1 and φ_2^1.

3.21 Modulus of the Gradient

Proposition 3.21.1 *Let $\Omega \subset R^m$, $m \ge 2$, be an open set, $u \in \mathcal{C}^0(\Omega)$, $f \in \mathcal{C}_{loc}^{1,\lambda}(R^1)$ be nondecreasing,* $0 < \lambda < 1$. *If $\Delta u = f(u)$ in Ω in the sense of distributions (in particular, if $u \in H(\Omega)$), then $u \in \mathcal{C}^3(\Omega)$ and $|\nabla u|^2$ is a subharmonic function in Ω.*

If, moreover, Ω *is bounded and* $u \in \mathcal{C}^1(\overline{\Omega})$*, then*

$$\sup_{x\in\Omega} |\nabla u(x)| = \sup_{x\in\partial\Omega} |\nabla u(x)|. \tag{3.33}$$

Proof Since $\Delta u = f(u) \in \mathcal{C}^0(\Omega)$, Proposition 3.18.1 gives $u \in \mathcal{C}^{1,\lambda}_{loc}(\Omega)$. Thus $\Delta u = f(u) \in \mathcal{C}^{1,\lambda}_{loc}(\Omega)$, which forces $u \in \mathcal{C}^3(\Omega)$ by Proposition 3.18.1. Easy calculation yields

$$\partial_j|\nabla u|^2 = \sum_{k=1}^{m} 2(\partial_k u)(\partial_j\partial_k u),$$

$$-\Delta|\nabla u|^2 = -\sum_{j=1}^{m}\sum_{k=1}^{m} 2[(\partial_j\partial_k u)^2 + (\partial_k u)(\partial_j\partial_j\partial_k u)]$$

$$\le -2\sum_{k=1}^{m}(\partial_k u)(\partial_k \Delta u) = -2\sum_{k=1}^{m}(\partial_k u)^2 f'(u) \le 0.$$

$|\nabla u|^2$ is a subharmonic function by Theorem 3.19.4.

Let Ω be bounded and $u \in \mathcal{C}^1(\overline{\Omega})$. Denote $M = \sup\{|\nabla u(x)|^2; x \in \partial\Omega\}$. Since $|\nabla u|^2$ is a superharmonic function, $|\nabla u(x)|^2 \le M$ in Ω by Proposition 3.4.1.

3.22 Representation of Superharmonic Functions

In this section we show that a superharmonic function is representable by the sum of a Green potential (or a volume potential) corresponding to a nonnegative measure and a harmonic function.

Theorem 3.22.1 *Let* $\Omega \subset R^m$*,* $m \ge 2$*, be an open set with the Green function* $G_\Omega(x,y)$*,* U *be a bounded open subset of* Ω *such that* $\overline{U} \subset \Omega$*, and* u *be a superharmonic function on* Ω*. Then* $\mu = -\Delta u$ *is a nonnegative Radon measure on* Ω*. If we denote* $\mu_U = \mu|U$ *the restriction of* μ *onto* U*, then there exists* $h_U \in H(U)$ *such that* $u = G_\Omega\mu_U + h_U$ *in* U*. If* $G_\Omega\mu$ *is a superharmonic function in* Ω *then there exists* $h \in H(\Omega)$ *such that* $u = G_\Omega\mu + h$ *in* Ω*. This function* h *is the greatest harmonic minorant of* u*.*

Proof $\mu = -\Delta u \ge 0$ in the sense of distributions by Theorem 3.19.4. Lemma 1.20.3 gives that μ is a nonnegative Radon measure on Ω. Since $\overline{U}$ is compact, we have $\mu_U(\Omega) = \mu(U) < \infty$. So, $G_\Omega\mu_U$ is a superharmonic function in Ω (see Proposition 3.11.3) and $-\Delta G\mu_U = \mu_U$ in Ω (see Theorem 3.11.5). Since u and $G_U\mu_U$ are superharmonic functions, they are from $L^1_{loc}(\Omega)$ by Proposition 3.3.4. Since $-\Delta(u - G_\Omega\mu_U) = \mu - \mu_U$, Theorem 2.18.2 gives that there exists $h_U \in H(U)$

such that $u - G_\Omega \mu_U = h_U$ almost everywhere in U. Since u and $G_\Omega \mu_U + h_U$ are superharmonic functions in U and $u = G_\Omega \mu_U + h_U$ almost everywhere in U, Corollary 3.7.2 gives that $u = G_\Omega \mu_U + h_U$ in U.

Let now $G_\Omega \mu$ be a superharmonic function in Ω. Then $-\Delta G_\Omega \mu = \mu$ in Ω by Theorem 3.11.5. Since $-\Delta(u - G_\Omega \mu) = 0$, Theorem 2.18.2 gives that there exists $h \in H(\Omega)$ such that $u - G_\Omega \mu = h$ almost everywhere in Ω. Since u and $G_\Omega \mu + h$ are superharmonic functions in Ω and $u = G_\Omega \mu + h$ almost everywhere in Ω, Corollary 3.7.2 forces that $u = G_\Omega + h$ in Ω. Since 0 is the greatest harmonic minorant of $G_\Omega \mu = u - h$, we deduce that h is the greatest harmonic minorant of u.

Theorem 3.22.2 *Let $\Omega \subset R^m$, $m \geq 2$, be an open set with the Green function $G_\Omega(x, y)$, u be a superharmonic function on Ω, and $\mu = -\Delta u$. If there exists a harmonic minorant of u, then $u = G_\Omega \mu + h$ where h is the greatest harmonic minorant of u. If u is nonnegative, then $u = G_\Omega \mu$ if and only if the greatest harmonic minorant of u is the zero function. On the other hand, if for each component ω of Ω there exists $x \in \omega$ such that $G_\Omega \mu(x) < \infty$, then there exists a harmonic minorant of u.*

Proof Theorem 3.19.4 and Lemma 1.20.3 give that μ is a nonnegative measure.

Suppose that there exists a harmonic minorant of u. Then there exists the greatest harmonic minorant h of u by Theorem 3.6.5. Choose an increasing sequence of bounded open sets V_k such that $\overline{V_k} \subset \Omega$, $\Omega = \cup V_k$. Put $\mu_k = \mu | V_k$. Then $G_\Omega \mu_k$ is superharmonic in Ω by Proposition 3.11.3. According to Proposition 3.9.5, there exist Green's functions G_k of V_k and $G_k \leq G_{k+1} \leq G_\Omega$. Since $G_k \mu_k \leq G_\Omega \mu_k$, the function $G_k \mu_k$ is superharmonic by Proposition 3.11.2. Denote by h_k the greatest harmonic minorant of u on V_k. Clearly, $h_k \geq h_{k+1} \geq h$. Moreover, $u = G_k \mu_k + h_k$ on V_k by Theorem 3.22.1. Proposition 2.11.3 gives that $h_k \searrow v \in H(\Omega)$. Since v is a harmonic minorant of u and $h \leq \lim h_k = v$, we infer that $h = v$. Thus $u = G_k \mu_k + h_k \leq G_\Omega \mu + h_k \to G_\Omega \mu + h$. Fix $x \in \Omega$, $\alpha < G_\Omega \mu(x)$. Then there exists $n \in N$ such that $x \in V_n$ and $G_\Omega \mu_n(x) > \alpha$. Since $G_k \nearrow G_\Omega$ uniformly on $\overline{V_n} \times \overline{V_n}$ (see Theorem 3.9.13), we deduce that $G_k \mu_n(x) \to G_\Omega \mu_n(x)$ as $k \to \infty$. Hence there exists $k > n$ such that $G_k \mu_n(x) > \alpha$. Therefore $u(x) = G_k \mu_k(x) + h_k(x) \geq G_k \mu_n(x) + h(x) > \alpha + h(x)$. Since α was arbitrary, we infer $u(x) \geq G_\Omega \mu(x) + h(x)$.

Suppose now that u is nonnegative. If the greatest harmonic minorant of u is the zero function, then $u = G_\Omega \mu$. If $u = G_\Omega \mu$, then the greatest harmonic minorant of u is the zero function by Proposition 3.11.6.

Suppose now that for each component ω of Ω, there exists $x \in \omega$ such that $G_\Omega \mu(x) < \infty$. Then $G_\Omega \mu$ is a superharmonic function in Ω by Proposition 3.11.2 and Proposition 3.9.7. We have $-\Delta G_\Omega \mu = \mu$ by Theorem 3.11.5 and Proposition 3.3.4. According to Proposition 3.19.6, there exists $h \in H(\Omega)$ such that $u = G_\Omega \mu + h$. Since $u - h = G_\Omega \mu \geq 0$, the function h is a harmonic minorant of u.

Theorem 3.22.3 *Let $\Omega \subset R^2$ be a domain, u be a superharmonic function in Ω, and ω be a bounded open set such that $\overline{\omega} \subset \Omega$. Denote $\mu = -\Delta u$, $\mu_\omega = \mu | \omega$.*

Then there exists $h_\omega \in H(\omega)$ *such that* $u = V\mu_\omega + h_\omega$ *in* ω. *Put* $\mu = 0$ *on* $R^2 \setminus \Omega$. *If*

$$\int\limits_{R^2 \setminus B(0;1)} \ln |y| \, \mathrm{d}\mu < \infty \tag{3.34}$$

then there exists $h \in H(\Omega)$ *such that* $u = V\mu + h$ *in* Ω.

Proof Let Ω be bounded. According to Proposition 3.9.8, there exists the Green function $G_\Omega(x, y) = h_2(x - y) - g_\Omega(x, y)$ for Ω. According Theorem 3.22.1, μ_ω is a nonnegative Radon measure and there exists $h_1 \in H(\omega)$ such $u = G_\Omega \mu_\omega + h_1 = V\mu_\omega - g_\Omega \mu_\omega + h_1$ in ω. Lemma 3.11.4 gives that $g_\Omega \mu_\omega \in H(\omega)$.

Let now Ω be general and (3.34) holds true. Then $-\Delta V\mu = \mu$ by Theorem 3.10.3. Since $-\Delta(u - V\mu) = 0$, Theorem 2.18.2 gives that there exists $h \in H(\Omega)$ such that $u - V\mu = h$ almost everywhere in Ω. Since u and $V\mu + h$ are superharmonic functions (see Proposition 3.10.2 and Lemma 3.3.3) and $u = V\mu + h$ almost everywhere in Ω, Corollary 3.7.2 forces that $u = V\mu + h$.

3.23 Continuity of Superharmonic Functions

Proposition 3.23.1 *Let* $\Omega \subset R^m$ *be an open set and* u *be a superharmonic function on* Ω. *Define* $\mu = -\Delta u$ *in* Ω, $\mu = 0$ *on* $R^m \setminus \Omega$. *Denote* $E = \Omega \cap \operatorname{spt} \mu$. *If* $z \in E$ *and*

$$\lim_{E \ni x \to z} u(x) = u(z)$$

then

$$\lim_{x \to z} u(x) = u(z).$$

Proof Since u is lower semicontinuous, it is enough to prove that

$$\limsup_{x \to z} u(x) \le u(z).$$

We can suppose that $u(z) < \infty$. Choose $\rho > 0$ such that $B(z; \rho) \subset \Omega$. For $r \in (0, \rho)$ denote $\mu_r = \mu | B(z; r)$. According to Theorem 3.22.3 and Theorem 3.22.1, there exists $h_r \in H(B(z; r))$ such that $u = V\mu_r + h_r$ in $B(z; r)$. Since $u(z) < \infty$, we have $\mu(\{z\}) = \mu_r(\{z\}) = 0$.

Let now $x_k \to z$. Choose $z_k \in E_r = E \cap \overline{B(z; r)}$ such that $|x_k - z_k| = \operatorname{dist}(x_k, E_r)$. If $y \in E_r$ then $|z_k - y| \le |z_k - x_k| + |y - x_k| \le 2|y - x_k|$. So, there exist positive constant a_1, a_2 dependent only on m such that $h_m(z_k - y) \geq a_1 h_m(x_k - y) - a_2$. Hence

$$\limsup_{k\to\infty} u(x_k) = \limsup_{k\to\infty}[h_r(x_k) + V\mu_r(x_k)] \leq h_r(z) + a_1^{-1}a_2\mu(B(z;r))$$
$$+a_1^{-1}\limsup_{k\to\infty} V\mu_r(z_k) \leq u(z) + \frac{a_2}{a_1}\mu(B(z;r)) + a_1^{-1}\limsup_{k\to\infty}[u(z_k) - h_r(z_k)]$$
$$= u(z) + \frac{a_2}{a_1}\mu(B(z;r)) + a^{-1}[u(z) - h_r(z)] = u(z) + \frac{a_2}{a_1}\mu(B(z;r)) + a_1^{-1}V\mu_r(z).$$

Since $\mu(\{z\}) = 0$ and $V\mu_r(z) < \infty$, we obtain for $r \downarrow 0$

$$\limsup_{k\to\infty} u(x_k) \leq u(z).$$

Corollary 3.23.2 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let μ be a nonnegative Radon measure with support K, where K is a compact subset of Ω. If $G_\Omega\mu$ is finite-valued, then, for each positive number ϵ, there exists a compact subset L of K such that $\mu(K \setminus L) < \epsilon$ and $G_\Omega(\mu|L)$ is continuous.*

Proof $G_\Omega\mu$ is superharmonic by Proposition 3.11.2 and therefore measurable. According to the Luzin theorem (Theorem 1.13.1), there exists a compact subset L of K such that $\mu(K \setminus L) < \epsilon$ and $(G_\Omega\mu)|L$ is continuous. Denote $\mu_1 = \mu|L$, $\mu_2 = \mu|(K - L)$. Then $G_\Omega\mu_j$ is superharmonic by Proposition 3.11.2. If $z \in L$, then Proposition 3.3.6 forces

$$G_\Omega\mu_1(z) \leq \liminf_{x\to z} G_\Omega\mu_1(x)) \leq \limsup_{L\ni x\to z} G_\Omega\mu_1(x)$$
$$= \limsup_{L\ni x\to z}[G_\Omega\mu(x) - G_\Omega\mu_2(x)] = G_\Omega\mu(z) + \limsup_{L\ni x\to z}[-G_\Omega\mu_2(x)]$$
$$\leq G_\Omega\mu(z) - \liminf_{x\to z} G_\Omega\mu_2(x) \leq G_\Omega\mu(z) - G_\Omega\mu_2(z) = G_\Omega\mu_1(z).$$

Since $G_\Omega\mu_1|L$ is continuous, Proposition 3.23.1 gives that $G_\Omega\mu_1$ is continuous.

3.24 Domination Principle

Theorem 3.24.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω and μ be a nonnegative Radon measure supported on Ω. Suppose that $G_\Omega\mu$ is finite-valued. If u is a nonnegative superharmonic function on Ω such that $u \geq G_\Omega\mu$ on $\Omega \cap \operatorname{spt}\mu$, then $u \geq G_\Omega\mu$ on Ω. If ν is a nonnegative Radon measure on Ω such that $G_\Omega\nu \geq G_\Omega\mu$ on $\Omega \cap \operatorname{spt}\mu$, then $G_\Omega\nu \geq G_\Omega\mu$ on Ω.*

Proof According to Proposition 3.9.7, we can suppose that Ω is connected. Suppose first that $K = \operatorname{spt}\mu$ is a compact subset of Ω. Fix $z \in \Omega \setminus K$. According to Proposition 3.9.12 and Proposition 3.9.16, there exists a constant C such that $0 \le G_\Omega(z, y) \le C$ for $y \in K$. Fix $\epsilon > 0$. According to Corollary 3.23.2, there exists a compact subset L of K such that $\mu(K \setminus L) < \epsilon$ and $G_\Omega \tilde{\mu}$ is continuous on Ω where $\tilde{\mu} = \mu | L$. Define

$$w(x) = \begin{cases} 0, & x \in L, \\ \max(0, G_\Omega \tilde{\mu}(x) - u(x)), & x \in \Omega \setminus L. \end{cases}$$

The function $G_\Omega \tilde{\mu} - u$ is subharmonic on $\Omega \setminus L$ by Proposition 3.11.2 and Lemma 3.3.3. Corollary 3.3.10 gives that w is subharmonic in Ω. Since $G_\Omega \tilde{\mu}$ is superharmonic by Proposition 3.11.2 and $w \le G_\Omega \tilde{\mu}$, there exists $h \in H(\Omega)$ such that $0 \le w \le h \le G_\Omega \tilde{\mu}$ (see Proposition 3.6.4). Hence $h \equiv 0$ by Theorem 3.22.2. Thus $G_\Omega \tilde{\mu} \le u$ in Ω. Therefore

$$G_\Omega \mu(z) \le G_\Omega \tilde{\mu}(z) + G_\Omega(\mu - \tilde{\mu})(z) \le u(z) + C\epsilon.$$

Since ϵ is arbitrary, we infer $G_\Omega \mu(z) \le u(z)$.

Let now μ be arbitrary. Choose an increasing sequence of compact subsets K_n of Ω such that $\Omega = \cup K_n$. Denote $\mu_n = \mu | K_n$. Then

$$G_\Omega \mu(z) = \lim_{n \to \infty} G_\Omega \mu_n(z) \le u(z).$$

Proposition 3.11.2 gives that $G_\Omega \nu \equiv \infty$ or $G_\Omega \nu$ is superharmonic in Ω. Thus $G_\Omega \nu \ge G_\Omega \mu$ on Ω.

3.25 Sequences of Superharmonic Functions

In this section we discuss under which condition the limit of a sequence of superharmonic functions is superharmonic.

Proposition 3.25.1 *Let $\Omega \subset R^m$ be a domain and u_j be a sequence of superharmonic functions in Ω. Suppose that u_j is locally uniformly bounded from below. (That is true if $u_j \le u_{j+1}$.) Denote $u = \sup u_j$. Suppose that $u(x) < \infty$ for some $x \in \Omega$. If $u_j \to u$ as $j \to \infty$ then u is superharmonic in Ω.*

Proof Fix $z \in \Omega$. Since

$$u_j(z) \le \liminf_{y \to z} u_j(y) \le \liminf_{y \to z} u(y)$$

we have

$$u(z) = \sup_j u_j(z) \le \liminf_{y \to z} u(y).$$

Let $r > 0$, $\overline{B(z;r)} \subset \Omega$. Since there exists a constant c such that $u_j + c \geq 0$ on $B(z;r)$, Fatou's lemma (Lemma 1.13.5) gives

$$\oint_{B(z;r)} u \, \mathrm{d}\mathcal{H}_m \leq \liminf_{j\to\infty} \oint_{B(z;r)} u_j \, \mathrm{d}\mathcal{H}_m \leq \liminf_{j\to\infty} u_j(z) = u(z).$$

Remark that if $u_j \leq u_{j+1}$, then Lemma 3.2.2 gives that u_j are locally uniformly bounded from below.

3.26 Polar Sets of Potential Theory

Polar sets play the role of negligible sets for the Dirichlet problem. The solution of the Dirichlet problem does not change even if we change a boundary condition on a polar set. In this section we define polar sets and study their properties.

Definition 3.26.1 A set E in R^m, $m \geq 2$, is called polar if there exists a superharmonic function u on some open set Ω such that $E \subset \{x \in \Omega; u(x) = \infty\}$. If a proposition $P(x)$, concerning a point x in a set B, is true for all x in B apart from a polar set, then $P(x)$ is said to hold quasi-everywhere (q.e.) on B, or for quasi-every point x of B.

Lemma 3.26.2

- *If $y \in R^m$, $m \geq 2$, then $\{y\}$ is a polar set.*
- *If $A \subset B \subset R^m$, $m \geq 2$, and B is polar, then A is polar.*
- *If $A \subset R^m$ is polar, $m \geq 2$, then $\mathcal{H}_m(A) = 0$.*

Proof Fix $y \in R^m$. Put $u(y) = \infty$, $u(x) = h_m(x - y)$ for $x \in R^m \setminus \{y\}$. Then u is a superharmonic function in R^m by Example 3.3.9. Thus $\{y\}$ is polar.

Let now A be polar. Choose an open set Ω and a superharmonic function u on Ω such that $A \subset \Omega$ and $u = \infty$ on A. Since $u \in L^1_{loc}(\Omega)$ by Proposition 3.3.4, we deduce that $\mathcal{H}_m(A) = 0$.

Theorem 3.26.3 *Let $A \subset R^m$ be polar, $m \geq 2$, $z \in R^m \setminus A$. Then there exists a nonnegative Radon measure μ in R^m such that $\mu(R^m) < \infty$, $V\mu$ is a superharmonic function in R^m, $V\mu = \infty$ on A, and $V\mu(z) < \infty$.*

Proof We can suppose that $z = 0$. Fix a superharmonic function u on an open set Ω such that $A \subset \{x \in \Omega; u(x) = \infty\}$. Put $\omega = \{x \in \Omega; u(x) > 0\} \setminus \{0\}$. Put $\nu = -\Delta u$. Then ν is a nonnegative Radon measure on ω (see Theorem 3.22.1). Let B_k be a sequence of open balls such that $\overline{B_k} \subset \omega$ for each k and $\cup B_k = \omega$. Put $\nu_k = \nu|B_k$ for each k. Since ν is a Radon measure, $\nu_k(R^m) = \nu(B_k) < \infty$. Denote by $G_k(x, y) = h_m(x - y) - g_k(x, y)$ the Green function of B_k. (The Green function exists by Corollary 3.9.6 and Proposition 3.9.8.) Since $u \geq 0$ in B_k, there exists a nonnegative harmonic function h_k in B_k such that $u = G_k\nu_k + h_k$ (see

Theorem 3.22.2). This gives $G_k\nu_k = \infty$ on $B_k \cap A$. Since

$$x \mapsto \int_{B_k} g_k(x, y)\, \mathrm{d}\mu_k(y)$$

is a harmonic function in B_k by Lemma 3.11.4, we infer that $V\nu_k = \infty$ on $B_k \cap A$. Clearly, $V\nu_k(0)$ is finite. Put

$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k(1 + \nu_k(R^m) + |V\nu_k(0)|)}\nu_k.$$

Then $\mu(R^m) < \infty$ and $V\mu(0)$ is finite. So, $V\mu$ is a superharmonic function in R^m by Propositions 3.9.2, 3.11.2, and 3.10.2. Moreover, $V\mu = \infty$ on A.

Proposition 3.26.4 *Let $\Omega \subset R^m$ be an open set, G_Ω be the Green function of Ω, $A \subset \Omega$ polar, $x \in \Omega \setminus A$. Then there exists a nonnegative Radon measure μ on Ω such that $\mu(\Omega) < \infty$, $G_\Omega\mu = \infty$ on A, and $G_\Omega\mu(z) < \infty$.*

Proof We can suppose that $x = 0$. According to Theorem 3.26.3, there exists a nonnegative Radon measure ν in R^m such that $\nu(R^m) < \infty$, $V\nu$ is a superharmonic function in R^m, $V\nu = \infty$ on A, and $V\nu(z) < \infty$. For $n \in N$ denote $\omega_n = \{y \in \Omega; |y| < n, \mathrm{dist}(y, \partial\Omega) > 1/n\}$, $\nu_n = \nu|\omega_n$. According to Theorem 3.22.1, there exists $h_n \in H(\omega_n)$ such that $V\nu = G_\Omega\nu_n + h_n$ in ω_n. Thus $G_\Omega\nu_n(x) < \infty$, $G_\Omega\nu_n = \infty$ on $A \cap \omega_n$. Set $a_n = 2^{-n}/(1 + G_\Omega\nu_n(x))$,

$$\mu = \sum_{n=1}^{\infty} a_n\nu_n.$$

Then $\mu(\Omega) < \infty$, $G_\Omega\mu(x) < \infty$ and $G\mu = \infty$ on A.

Corollary 3.26.5 *A countable union of polar sets is polar.*

Proof Let A_k be a countable collection of polar sets. According to Theorem 3.26.3, there exist nonnegative Radon measures μ_k such that $\mu_k(R^m) \le 1$, $V\mu_k$ is a superharmonic function in R^m, and $V\mu_k = \infty$ on A_k. If $m = 2$, then we can suppose moreover that

$$\int_{R^2\setminus B(0;1)} \ln|y|\, \mathrm{d}\mu_k(y) \le 1$$

(see Proposition 3.10.2). Put

$$\mu = \sum_k 2^{-k}\mu_k.$$

Then $\mu(R^m) < \infty$. If $m = 2$, then

$$\int\limits_{R^2 \setminus B(0;1)} \ln|y| \, \mathrm{d}\mu(y) < \infty.$$

Propositions 3.10.2, 3.11.3, and 3.9.2 give that the potential $V\mu$ is a superharmonic function in R^m. Clearly $V\mu = \infty$ on $\cup A_k$.

3.27 Lower Semicontinuous Regularization

In this section, we study the lower semicontinuous regularization of a function f, i.e., the greatest lower semicontinuous minorant of f.

Definition 3.27.1 Let $\Omega \subset R^m$ be an open set, $f : \Omega \to \langle -\infty, \infty \rangle$. Then the lower semicontinuous regularization $\tilde{f}$ of f is defined by

$$\tilde{f}(x) = \min(f(x), \liminf_{y \to x} f(y)) = \sup_{r>0} \inf_{y \in \Omega \cap B(x;r)} f(y).$$

Lemma 3.27.2 *Let $\Omega \subset R^m$ be open and $\mathcal{A}$ be a set of functions from Ω to $\langle -\infty, \infty \rangle$. Denote $f = \inf\{u; u \in \mathcal{A}\}$. Then there exists at most countable subset $\mathcal{B}$ of $\mathcal{A}$ such that $\tilde{g} = \tilde{f}$, where $g = \inf\{u; u \in \mathcal{B}\}$.*

Proof $t \mapsto t/(1 + |t|)$ is an increasing mapping of $\langle -\infty, \infty \rangle$ onto $\langle -1, 1 \rangle$. By considering the function $u/(1 + |u|)$ in place of $u \in \mathcal{A}$, we may suppose that $\mathcal{A}$ is uniformly bounded. Choose a sequence of open balls $B(n)$ such that $B(n) \subset \Omega$, and for each $x \in \Omega$ and $r > 0$ there exists $n \in N$ with $x \in B(n) \subset B(x; r)$. For $n \in N$ choose $x_n \in B(n)$ such that

$$f(x_n) \leq \inf_{y \in B(n)} f(y) + 1/n.$$

Choose a function $u_n \in \mathcal{A}$ such that

$$u_n(x_n) \leq f(x_n) + 1/n.$$

Put $\mathcal{B} = \{u_n; n \in N\}$. Clearly, $f \leq g$ and therefore $\tilde{f} \leq \tilde{g}$. Fix $x \in \Omega$ and $k \in N$. Fix $r > 0$ such that $B(x; r) \subset \Omega$, $r < \operatorname{diam} B(n)/2$ for $n = 1, \ldots, k$. Then there exists $n \in N$ such that $x \in B(n) \subset B(x; r)$. Clearly, $n > k$. Since $x \in B(n)$ we have

$$\inf_{y \in B(x;r)} g(y) \leq g(x_n) \leq u(x_n) \leq f(x_n) + \frac{1}{k} \leq \inf_{y \in B(n)} f(y) + \frac{2}{k} \leq \tilde{f}(x) + \frac{2}{k}.$$

Thus $\tilde{g}(x) \leq \tilde{f}(x) + 2/k$.

Theorem 3.27.3 *Let $\Omega \subset R^m$ be an open set. Suppose that $\mathcal{A}$ is a set of superharmonic functions on Ω which is locally equibounded from below. Denote $f = \inf\{h; h \in \mathcal{A}\}$ and by $\tilde{f}$ the lower semicontinuous regularization of f. Then $\tilde{f}$ is superharmonic in Ω,*

$$\tilde{f}(x) = \liminf_{y \to x} f(y), \qquad x \in \Omega, \tag{3.35}$$

and $\tilde{f}(x) = f(x)$ for almost all $x \in \Omega$.

Proof If $h \in \mathcal{A}$, then

$$h(x) = \liminf_{y \to x} h(y) \geq \liminf_{y \to x} f(y)$$

by Proposition 3.3.6. Thus

$$f(x) = \inf_{h \in \mathcal{A}} h(x) \geq \liminf_{y \to x} f(y)$$

which gives (3.35). Clearly $\tilde{f} \leq f$. If $\tilde{f}(x) > a$, then there exists $r > 0$ such that $f > a$ on $B(x; r)$. So, $\tilde{f} > a$ on $B(x; r)$ by(3.35). Thus $\tilde{f}$ is lower semicontinuous.

According to Lemma 3.27.2 there exists a sequence $\{f_k\} \subset \mathcal{A}$ such that for $g = \inf f_k$ we have $\tilde{f} = \tilde{g}$. Denote $g_k = \min(f_1, \dots, f_k)$. Then g_k are superharmonic by Lemma 3.3.3, and $g_k \downarrow g$. Fix $\varphi \in \mathcal{C}_c^\infty(\Omega)$, $\varphi \geq 0$. Theorem 3.19.4, Proposition 3.3.4, and the Lebesgue lemma (Lemma 1.13.2) give

$$\int_\Omega g(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m = \lim_{k \to \infty} \int_\Omega g_k(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m \geq 0.$$

Therefore $-\Delta g \geq 0$ in the sense of distributions. According to Theorem 3.19.4, there exists a superharmonic function h such that $h(x) = g(x)$ for $x \in \Omega \setminus A$ with $\mathcal{H}_m(A) = 0$. Proposition 3.7.1 forces

$$h(x) = \lim_{r \downarrow 0} \fint_{B(x;r)} h\, \mathrm{d}\mathcal{H}_m = \lim_{r \downarrow 0} \fint_{B(x;r)} g\, \mathrm{d}\mathcal{H}_m \leq \lim_{r \downarrow 0} \fint_{B(x;r)} g_k\, \mathrm{d}\mathcal{H}_m \leq g_k(x).$$

Thus $h \leq g$. By virtue of Proposition 3.3.6,

$$h(x) = \liminf_{y \to x} h(y) \leq \liminf_{y \to x} g(y) = \tilde{g}(x),$$

$$h(x) = \liminf_{\Omega \setminus A \ni y \to x} h(y) = \liminf_{\Omega \setminus A \ni y \to x} g(y) \geq \liminf_{y \to x} g(y) = \tilde{g}(x).$$

Since $h = \tilde{g}$ we infer that $g = \tilde{g}$ almost everywhere and $\tilde{g}$ is superharmonic. Since $\tilde{g} = \tilde{f} \leq f \leq g$ we deduce that $\tilde{f} = f$ almost everywhere.

3.28 Reduced Functions

For a nonnegative superharmonic function u on a Green open set Ω and a subset E of Ω, we define R_u^E, the reduced function of u relative to E, and $\tilde{R}_u^E$, the regularized reduced function of u relative to E, and study their properties.

Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let u be a nonnegative superharmonic function on Ω and $E \subset \Omega$. Define the *reduced function* of u relative to E in Ω by

$$R_u^E = \inf\{v \text{ superharmonic in } \Omega; v \geq 0 \text{ in } \Omega, v \geq u \text{ on } E\}.$$

The lower semicontinuous regularization $\tilde{R}_u^E$ of R_u^E is called the *regularized reduced function* of u relative to E in Ω. Theorem 3.27.3 gives that $\tilde{R}_u^E$ is superharmonic in Ω and $\tilde{R}_u^E = R_u^E$ almost everywhere in Ω. By virtue of Proposition 3.3.6,

$$\tilde{R}_u^E(x) = \liminf_{y \to x} R_u^E(y).$$

Theorem 3.28.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \geq 2$, $E, F \subset \Omega$, and u, v be nonnegative superharmonic functions on Ω.*

1. *If $E \subset F$, then $R_u^E \leq R_u^F$ and $\tilde{R}_u^E \leq \tilde{R}_u^F$.*
2. *If $u \leq v$ on E, then $R_u^E \leq R_v^E$ and $\tilde{R}_u^E \leq \tilde{R}_v^E$.*
3. *If $\lambda > 0$ then $R_{\lambda u}^E = \lambda R_u^E$ and $\tilde{R}_{\lambda u}^E = \lambda \tilde{R}_u^E$.*
4. *$\tilde{R}_u^E$ is positive or identically zero on each component of Ω.*
5. *$R_{u+v}^E \leq R_u^E + R_v^E$, $\tilde{R}_{u+v}^E \leq \tilde{R}_u^E + \tilde{R}_v^E$.*
6. *The functions R_u^E and $\tilde{R}_u^E$ are equal and harmonic on $\Omega \setminus \overline{E}$.*
7. *If $E \subset F$ and E is open then $R_u^F = u$ on F and $\tilde{R}_u^F = u$ on E.*
8. *If E is open then $\tilde{R}_u^E = R_u^E$.*

Proof (1), (2), and (3) follow immediately from the definition.

4) If ω is a component of Ω, then $\tilde{R}_u^E$ is a nonnegative superharmonic function on ω. If there exists $x \in \omega$ such that $u(x) = 0$, then $u = 0$ on ω by the minimum principle (Proposition 3.4.1).

5) Let f and g be nonnegative superharmonic functions on Ω such that $f \geq u$, $g \geq v$ on E. Then $f + g$ is a nonnegative superharmonic function on Ω such that $u + v \leq f + g$ (see Lemma 3.3.3). Therefore $R_{u+v}^E \leq f + g$. Taking infimum over f and g, we obtain $R_{u+v}^E \leq R_u^E + R_v^E$. So, $\tilde{R}_{u+v}^E \leq R_{u+v}^E \leq R_u^E + R_v^E$. Fix $x \in \Omega$, $r > 0$ such that $\overline{B(x;r)} \subset \Omega$. Since $\tilde{R}_u^E = R_u^E$, $\tilde{R}_v^E = R_v^E$ almost everywhere in Ω by Theorem 3.27.3

$$\oint_{B(x;r)} \tilde{R}_{u+v}^E \, d\mathcal{H}_m \leq \oint_{B(x;r)} R_u^E + \oint_{B(x;r)} R_v^E = \oint_{B(x;r)} (\tilde{R}_u^E + \tilde{R}_v^E) \, d\mathcal{H}_m.$$

Since $\tilde{R}^E_{u+v}$ and $\tilde{R}^E_u + \tilde{R}^E_v$ are superharmonic functions, Proposition 3.7.1 yields

$$\tilde{R}^E_{u+v}(x) = \lim_{r\downarrow 0} \fint_{B(x;r)} \tilde{R}^E_{u+v} \le \lim_{r\downarrow 0} \fint_{B(x;r)} (\tilde{R}^E_u + \tilde{R}^E_v) = \tilde{R}^E_u(x) + \tilde{R}^E_v(x).$$

6) Denote $\mathcal{F} = \{v \ge 0$ superharmonic on $\Omega; v \ge u$ on $E\}$. If $w, v \in \mathcal{F}$ then $\min(w, v) \in \mathcal{F}$ by Lemma 3.3.3. If $v \in \mathcal{F}$ and $B = B(z;\rho)$ is a ball with $\overline{B} \subset \Omega \setminus \overline{E}$, denote

$$v_B(x) = \begin{cases} v(x), & x \in \Omega \setminus B, \\ \int\limits_{\partial B(z;\rho)} P(x,y)v(y)\, \mathrm{d}\mathcal{H}_{m-1}(y), & x \in B, \end{cases} \tag{3.36}$$

where $P(x, y)$ is the Poisson kernel corresponding to the ball B. Then $v_B \in \mathcal{F}$ by Proposition 3.6.2. Lemma 3.6.3 forces that $R^E_u \in H(\Omega \setminus \overline{E})$. Hence $\tilde{R}^E_u = R^E_u$ on $\Omega \setminus \overline{E}$.

7) Clearly, $R^F_u = u$ on F. Proposition 3.3.6 gives that $\tilde{R}^F_u = \tilde{u} = u$ on E.

8) We have $\tilde{R}^E_u = u$ on E. Since $\tilde{R}^E_u$ is a nonnegative superharmonic function on Ω such that $\tilde{R}^E_u \ge u$ on E, we deduce that $\tilde{R}^E_u \ge R^E_u$. On the other hand, $\tilde{R}^E_u \le R^E_u$ by the definition.

Lemma 3.28.2 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \ge 2$, and $E \subset \Omega$. The following are equivalent:*

1. *E is polar.*
2. *$\tilde{R}^E_u \equiv 0$ for each nonnegative superharmonic function u in Ω.*
3. *There exists a positive superharmonic function u on Ω with $\tilde{R}^E_u \equiv 0$.*

Proof 1) $\Rightarrow$ 2): Fix $z \in \Omega \setminus E$. According to Proposition 3.26.4, there exists a nonnegative Radon measure μ on Ω such that $\mu(\Omega) < \infty$, $G_\Omega\mu = \infty$ in E, $G_\Omega\mu(z) < \infty$. For $n \in N$ define $v_n = min(u, G_\Omega\mu/n)$. Then $v_n \ge 0$, $v_n = u$ on E and v_n is superharmonic by Proposition 3.11.3 and Lemma 3.3.3. Thus

$$0 \le R^E_u(z) \le \lim_{n\to\infty} v_n(z) = 0.$$

Hence $\tilde{R}^E_u = 0$ on $\Omega \setminus E$. Since $\tilde{R}^E_u$ is superharmonic and $\mathcal{H}_m(E) = 0$ by Lemma 3.26.2, Proposition 3.3.6 forces $\tilde{R}^E_u \equiv 0$.

2) $\Rightarrow$ 3): We can set $u \equiv 1$.

3) $\Rightarrow$ 1): Fix a component ω of Ω. Since $R^E_u = u > 0$ on E and $0 = \tilde{R}^E_u = R^E_u$ almost everywhere on ω there exists $z \in \omega \setminus E$ such that $R^E_u(z) = 0$. For every $n \in N$ there exists a nonnegative superharmonic function v_n on Ω such that $v_n(z) < 2^{-n}$ and $v_n \ge u > 0$ on E. Denote

$$v = \sum_{n=1}^{\infty} v_n$$

in ω. Then $v(z) < \infty$, and $v = \infty$ on $E \cap \omega$. The function v is superharmonic in ω by Proposition 3.25.1 and Lemma 3.3.3. Hence E is polar.

Proposition 3.28.3 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \geq 2$, $E \subset \Omega$, and u be a nonnegative superharmonic function on Ω. If u is a continuous real function on an open set containing E, then*

$$R_u^E = \inf\{R_u^\omega; E \subset \omega \subset \Omega, \omega \text{ open}\}.$$

Proof Let $E \subset \tilde{\Omega} \subset \Omega$, $\tilde{\Omega}$ open, and u be finite and continuous on $\tilde{\Omega}$. Denote $f = \inf\{R_u^\omega; E \subset \omega \subset \Omega, \omega \text{ open }\}$. Then $R_u^E \leq f$ by Theorem 3.28.1. Let now v be a nonnegative superharmonic function on Ω such that $v \geq u$ on E. Fix $n \in N$. Since $v - u$ is lower semicontinuous on $\tilde{\Omega}$, the set $\omega = \{x \in \tilde{\Omega}; v(x) + 1/n - u(x) > 0\}$ is open. Since $E \subset \omega$ we have $v + 1/n \geq R_u^\omega \geq f$. Since v and n are arbitrary, we deduce $R_u^E \geq f$.

Proposition 3.28.4 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \geq 2$, $A \subset \overline{A} \subset \Omega$, A be bounded. If u is a nonnegative superharmonic function on Ω, then $\mu = -\Delta \tilde{R}_u^A$ is a nonnegative Radon measure supported in $\overline{A}$ and $\tilde{R}_u^A = G_\Omega \mu$.*

Proof According to Proposition 3.9.7, we can suppose that Ω is connected. Choose bounded open sets ω and U such that $\overline{A} \subset \omega \subset \overline{\omega} \subset U \subset \overline{U} \subset \Omega$. Denote $v = \tilde{R}_u^{\overline{\omega}}$. Then $v = u$ on ω, $v \in H(\Omega \setminus \overline{\omega})$ by Theorem 3.28.1. Fix $y \in \Omega$. Since $G_\Omega(\cdot, y)$ is superharmonic and positive by Proposition 3.9.7, Lemma 3.2.2 gives that there exists a positive constant c such that $G_\Omega(\cdot, y) \geq c$ on ∂U. Hence there exists a positive constant a such that $v(x) \leq a G_\Omega(x, y)$ for $x \in \partial U$. Define

$$w(x) = \begin{cases} v(x), & x \in U \\ \min(v(x), a G_\Omega(x, y)), & x \in \Omega \setminus U. \end{cases}$$

Since v is superharmonic by Theorem 3.27.3, Corollary 3.3.10 gives that w is superharmonic. Since $w = u$ on A, we have $w \geq R_u^A \geq \tilde{R}_u^A$. Let now h be a harmonic minorant of $\tilde{R}_u^A$. Then $h \leq w$. In particular $h(x)/a \leq G_\Omega(x, y)$ on $\Omega \setminus U$. Proposition 3.4.1 forces $h(x)/a \leq G_\Omega(x, y)$ on Ω. According to the definition of the Green function $h \leq 0$. Since $\tilde{R}_u^A$ is superharmonic by Theorem 3.27.3, Theorem 3.22.2 gives that $\mu = -\Delta \tilde{R}_u^A$ is a nonnegative Radon measure on Ω and $\tilde{R}_u^A = G_\Omega \mu$. Since $\tilde{R}_u^A$ is harmonic in $\Omega \setminus \overline{A}$ by Theorem 3.28.1, one has $\operatorname{spt} \mu \subset \overline{A}$.

3.29 Potentials and Polar Sets

Proposition 3.29.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \geq 2$. Let E be a relatively closed non-polar subset of Ω. Then there exists a nontrivial nonnegative Radon measure μ with compact support* $\operatorname{spt} \mu \subset E$ *such that $G_\Omega \mu$ is bounded and continuous in Ω.*

Proof Denote $K_n = \{x \in E; |x| \le n, \operatorname{dist}(x, \partial\Omega) \ge 1/n\}$. Then K_n is compact. If K_n is polar for all $n \in N$, then E is polar by Corollary 3.26.5. Thus we can suppose that E is compact. Put $\nu = -\Delta \tilde{R}_1^E$. Then ν is a nonnegative Radon measure supported in E and $\tilde{R}_1^E = G_\Omega \mu$ (see Proposition 3.28.4). Lemma 3.28.2 gives that $G_\Omega \nu = \tilde{R}_1^E \not\equiv 0$. Therefore, ν is nontrivial. According to Corollary 3.23.2, there exists a compact $K \subset E$ such that $\mu = \nu | K$ is nontrivial and $G_\Omega \mu$ is continuous in Ω. Clearly, $0 \le G_\Omega \mu \le G_\Omega \nu = \tilde{R}_1^E \le 1$.

Proposition 3.29.2 *Let $E \subset R^2$ be a closed nonpolar set. Then there exists a nontrivial nonnegative Radon measure μ with compact support* $\operatorname{spt} \mu \subset E$ *such that $V\mu$ is upper bounded in R^2.*

Proof Denote $E_n = \{x \in E; |x| \le n\}$. According to Corollary 3.26.5, there exists $n \in N$ such that E_n is nonpolar. Set $\Omega = B(0; 3n)$. Then there exists the Green function $G_\Omega(x, y) = h_m(x - y) - g_\Omega(x, y)$ (see Proposition 3.9.8). According to Proposition 3.29.1, there exists a nontrivial nonnegative Radon measure μ with compact support $\operatorname{spt} \mu \subset E_n$ such that $G_\Omega \mu$ is bounded in Ω. We have $V\mu = G_\Omega \mu + g_\Omega \mu$ in Ω. The function $g_\Omega \mu$ is harmonic in Ω by Lemma 3.11.4. Hence $V\mu$ is bounded in $B(0; 2n)$. Clearly, $V\mu \le 0$ in $R^2 \setminus B(0; 2n)$.

3.30 Superharmonic Functions and Removable Singularity

In this section we study under which condition we can extend a superharmonic function on $\Omega \setminus F$ onto a superharmonic function on Ω.

Theorem 3.30.1 *Let A be a relatively closed polar subset of an open set $\Omega \subset R^m$, $m \ge 2$. Suppose that u is a superharmonic function in $\Omega \setminus A$ such that u is bounded from below on $(\Omega \setminus A) \cap F$ for each compact subset F of Ω. Then u has a unique superharmonic extension to Ω.*

Proof $\mathcal{H}_m(A) = 0$ by Lemma 3.26.2. So, the uniqueness of a superharmonic extension of u is a consequence of Proposition 3.7.1.

Put

$$u(x) = \liminf_{\Omega \setminus A \ni y \to x} u(y), \qquad x \in A.$$

Clearly, u is lower semicontinuous in Ω. If $x \in A$ and $u(x) = \infty$ then

$$u(x) \ge \oint_{B(x;r)} u \, \mathrm{d}\mathcal{H}_m$$

for $0 < r < \operatorname{dist}(x; \partial\Omega)$. In particular, if $u = \infty$ on A, then u is superharmonic in Ω by Proposition 3.3.7.

Let now $x \in A$ and $u(x) < \infty$. Fix $\{x_n\} \subset \Omega \setminus A$ such that $x_n \to x$ and $u(x_n) \to u(x)$ as $n \to \infty$. According to Theorem 3.26.3, there exists a superharmonic function v in R^m such that $v = \infty$ on A. Choose positive constants r_n such that $B(x_n, 2r_n) \cap A = \emptyset$ and $B(x_n; 2r_n) \cap B(x_k; 2r_k) = \emptyset$ for $k \neq n$. Let w be on $B(x_k; r_k)$ the Poisson integral corresponding to v and $B(x_k, r_k)$. Put $w = v$ on $\Omega \setminus \cup B(x_k; r_k)$. Then w is a superharmonic function in R^m, $w = \infty$ on A and $w(x_k) < \infty$ (see Proposition 3.6.2). We have proved that $u + \epsilon w$ is a superharmonic function in Ω for arbitrary $\epsilon > 0$. Since w is locally integrable by Proposition 3.3.4, one has for $0 < r < \operatorname{dist}(x_k, \partial\Omega)$

$$u(x_k) = \lim_{\epsilon \searrow 0}[u(x_k) + \epsilon w(x_k)] \geq \lim_{\epsilon \searrow 0} \oint_{B(x_k;r)} [u + \epsilon w] \, \mathrm{d}\mathcal{H}_m = \oint_{B(x_k;r)} u \, \mathrm{d}\mathcal{H}_m .$$

If $0 < r < \operatorname{dist}(x, \partial\Omega)$, then

$$u(x) = \lim_{k \to \infty} u(x_k) \geq \liminf_{k \to \infty} \oint_{B(0;r)} u(y + x_k) \, \mathrm{d}\mathcal{H}_m(y)$$

$$\geq \oint_{B(0;r)} \liminf_{k \to \infty} u(y + x_k) \, \mathrm{d}\mathcal{H}_m \geq \oint_{B(0;r)} u(y + x) \, \mathrm{d}\mathcal{H}_m .$$

u is superharmonic in Ω by Proposition 3.3.7.

Corollary 3.30.2 *Let A be a relatively closed polar subset of an open set $\Omega \subset R^m$. Suppose that $u \in H(\Omega \setminus A)$, u is bounded on $F \setminus A$ for each compact subset F of Ω. Then u has a unique harmonic extension to Ω.*

Proof According to Theorem 3.30.1, there exist a superharmonic function v and a subharmonic function w in Ω such that $w = u = v$ on $\Omega \setminus A$. Since $\mathcal{H}_m(A) = 0$ by Lemma 3.26.2, Proposition 3.7.1 gives that $v = w$. Thus $v \in H(\Omega)$ by Mean value theorem (Theorem 2.6.1.)

3.31 Green's Function and Polar Sets

If $\Omega \subset R^m$ with $m > 2$, then there exists Green's function corresponding to Ω. In this section we prove that there exists Green's function for $\Omega \subset R^2$ if and only if $R^2 \setminus \Omega$ is not polar.

Proposition 3.31.1 *Let A be a relatively closed polar subset of an open set $\Omega \subset R^m$, $m \geq 2$. Then there exists the Green function G_Ω of Ω if and only if there exists the Green function $G_{\Omega \setminus A}$ of $\Omega \setminus A$. Moreover, $G_\Omega = G_{\Omega \setminus A}$ on $(\Omega \setminus A) \times (\Omega \setminus A)$.*

Proof If there exists G_Ω, then there exists $G_{\Omega\setminus A}$ by Proposition 3.9.5. Suppose now that there exists $G_{\Omega\setminus A}$.Fix $y \in \Omega \setminus A$. Choose $r > 0$ such that $\overline{B(y;r)} \subset \Omega \setminus A$. Clearly, $g_{\Omega\setminus A}(\cdot, y)$ is bounded on $\overline{B(y;r)}$. The function $G_{\Omega\setminus A}(\cdot, y)$ is bounded on $\Omega \setminus A \setminus \overline{B(y;r)}$ by Proposition 3.9.16. Thus $g_{\Omega\setminus A}(\cdot, y)$ is bounded on $F \setminus A$ for each compact subset F of Ω. According to Corollary 3.30.2, there exists a harmonic extension $\tilde{g}$ of $g_{\Omega\setminus A}(\cdot, y)$ to Ω. From the continuity we infer that $\tilde{g}(x) \le h_m(x-y)$ for $x \in \Omega$. If $u \in H(\Omega)$, $u(x) \le h_m(x-y)$ on Ω, then $u(x) \le g_{\Omega\setminus A}(x, y)$ for $x \in \Omega \setminus A$. From the continuity we deduce that $u(x) \le \tilde{g}(x)$ for $x \in \Omega$. Thus $\tilde{g} = g_\Omega(\cdot, y)$.

If $m > 2$ then there exists G_Ω by Corollary 3.9.6. Let now $m = 2$, $y \in A$. Fix $r > 0$ such that $B(y; 3r) \subset \Omega$. Since $\mathcal{H}_m(A) = 0$ (see Lemma 3.26.2) there exists $z \in B(y;r) \setminus A$. Clearly there exists a positive constant C such that $h_m(x-y) \ge g_\Omega(x, z) - C$ on $\overline{B(y; 2r)}$. If $x \in \Omega \setminus \overline{B(y; 2r)}$ then $g_\Omega(x, z) \le h_m(x-z) = h_m(x-y) + (2\pi)^{-1} \ln[|x-y|/|x-z|] \le h_m(x-y) + (2\pi)^{-1} \ln 2$. Therefore $g_\Omega(x, z) - C - (2\pi)^{-1} \ln 2$ is a harmonic minorant of $h_m(x-y)$. So, $g_\Omega(\cdot, y)$ exists by Proposition 3.6.4.

Theorem 3.31.2 *Let $\Omega \subset R^2$ be a nonempty open set. The following are equivalent:*

1. *There exists the Green function G_Ω of Ω.*
2. *$R^2 \setminus \Omega$ is nonpolar.*
3. $\ln^+ |x| = \max(0, \ln |x|)$ *has a harmonic majorant on* Ω.

Proof 1) ⇒ 2) follows from Proposition 3.31.1 and Proposition 3.9.3.

2) ⇒ 3): According to Proposition 3.29.2, there exist a positive constant c_1 and a nontrivial nonnegative Radon measure μ with compact support $K \subset R^2 \setminus \Omega$ such that $V\mu \le c_1$. Since

$$\lim_{|x|\to\infty} [V\mu(x) + (\ln |x|)\mu(K)/(2\pi)] \to 0,$$

there exist positive constants c_2, c_3 such that $\ln |x| \le c_2 - c_3 V\mu(x)$. Thus $c_2 - c_3 V\mu + c_3 c_1$ is a harmonic majorant of $\ln^+ |x|$ on Ω (see Remark 2.1.4).

3) ⇒ 1): Fix $z \in \Omega$. Since $\ln |x-z| - \ln |x| \to 0$ as $|x| \to \infty$, there exists a harmonic majorant of $\ln^+ |x-z|$ on Ω. Denote $\mathcal{A} = \{u \in H(\Omega); u \ge (\ln |x-z|)/(2\pi)\}$. Then $\mathcal{A} \ne \emptyset$. Denote $g = \inf\{u; u \in \mathcal{A}\}$. Then $g \in H(\Omega)$ by Proposition 3.6.4. Then $g_\Omega(x, z) = -g(x)$ where $G_\Omega(x, z) = h_m(x-z) - g_\Omega(x, z)$.

3.32 The Green Capacity of a Compact Set

In several next sections, we study the Green capacity of sets. Remark that a capacity is a more general notion than a measure. We begin with the Green capacity of compact sets.

We shall suppose that $\Omega \subset R^m$ is an open set with the Green function G_Ω. Let $K \subset \Omega$ be compact. The function $\tilde{R}_1^K$ is a superharmonic function on Ω which is harmonic in $\Omega \setminus K$ (see Theorems 3.27.3 and 3.28.1). Denote $\mu_\Omega^K = -\Delta \tilde{R}_1^K$. Then μ_Ω^K is a nonnegative Radon measure supported on K and $\tilde{R}_1^K = G_\Omega \mu_\Omega^K$ (see Proposition 3.28.4). Since $\tilde{R}_1^K = 1$ on $K \setminus \partial K$ by Theorem 3.28.1 we deduce $\operatorname{spt} \mu_\Omega^K \subset \partial K$. Denote $C_\Omega(K) = \mu_\Omega^K(\Omega)$ the Green capacity of K (with respect to Ω). Clearly, $\tilde{R}_1^\emptyset \equiv 0$, and thus $\mathcal{C}_\Omega(\emptyset) = 0$.

Lemma 3.32.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let μ, ν be nonnegative Radon measures on Ω,* $\operatorname{spt} \mu \subset \Omega$ *be compact. If $G_\Omega \mu \le G_\Omega \nu$ in Ω, then $\mu(\Omega) \le \nu(\Omega)$.*

Proof Choose a bounded open set ω such that $\operatorname{spt} \mu \subset \omega \subset \overline{\omega} \subset \Omega$. Clearly $G_\Omega \mu_\Omega^{\overline{\omega}} = \tilde{R}_1^{\overline{\omega}} = 1$ in ω. Since $G_\Omega(x, y) = G_\Omega(y, x)$ by Proposition 3.9.12, Fubini's theorem gives

$$\mu(\Omega) = \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \, \mathrm{d}\mu = \int_\Omega G_\Omega \mu \mathrm{d}\mu_\Omega^{\overline{\omega}} \le \int_\Omega G_\Omega \nu \mathrm{d}\mu_\Omega^{\overline{\omega}} = \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \, \mathrm{d}\nu \le \int_\Omega 1 \, \mathrm{d}\nu.$$

Proposition 3.32.2 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $K, L \subset \Omega$ are compact sets and $K \subset L$, then $C_\Omega(K) \le C_\Omega(L)$.*

Proof Since $K \subset L$ we have $G_\Omega \mu_\Omega^K = \tilde{R}_1^K \le \tilde{R}_1^L = G_\Omega \mu_\Omega^L$. Therefore $C_\Omega(K) = \mu_\Omega^K(\Omega) \le \mu_\Omega^L(\Omega) = C_\Omega(L)$ by Lemma 3.32.1.

Lemma 3.32.3 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $K \subset \Omega$ be compact. Let μ , $\{\mu_k\}$ be bon-negative Radon measures on Ω such that* $\operatorname{spt} \mu \subset K$, $\operatorname{spt} \mu_k \subset K$. *Suppose that $G_\Omega \mu_k$ is a monotone sequence such that $G_\Omega \mu_k \to G_\Omega \mu$ on $\Omega \setminus K$. Then $\mu_k(\Omega) \to \mu(\Omega)$.*

Proof Choose a bounded open set ω such that $K \subset \omega \subset \overline{\omega} \subset \Omega$. Then $G_\Omega \mu_\Omega^{\overline{\omega}} = \tilde{R}_1^{\overline{\omega}} = 1$ on $\omega \supset K$. Since $\operatorname{spt} \mu_\Omega^{\overline{\omega}} \subset \partial\omega \subset \Omega \setminus K$ and $G_\Omega(x, y) = G_\Omega(y, x)$ (see Proposition 3.9.7), Fubini's theorem and Theorem on monotone convergence (Theorem 1.13.3) give

$$\lim_{k\to\infty} \mu_k(\Omega) = \lim_{k\to\infty} \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \, \mathrm{d}\mu_k = \lim_{k\to\infty} \int_\Omega G_\Omega \mu_k \, \mathrm{d}\mu_\Omega^{\overline{\omega}}$$

$$= \int_\Omega G_\Omega \mu \, \mathrm{d}\mu_\Omega^{\overline{\omega}} = \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \, \mathrm{d}\mu = \mu(\Omega).$$

Proposition 3.32.4 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $K(n)$ is a decreasing sequence of compact subsets of Ω and $K = \cap K(n)$ then $C_\Omega(K(n)) \downarrow C_\Omega(K)$.*

Proof Clearly the sequence $G_\Omega \mu_\Omega^{K(n)} = \tilde{R}_1^{K(n)}$ is decreasing. Moreover $G_\Omega \mu_\Omega^{K(n)} = R_1^{K(n)} \geq R_1^K = G_\Omega \mu_\Omega^K$. Fix $x \in \Omega$. If $a > R_1^K(x)$, then there exists an open neighborhood ω of K such that $R_1^\omega(x) < a$ (see Proposition 3.28.3). There exists $k \in N$ such that $K(n) \subset \omega$ for $n \geq k$. Hence $R_1^{K(n)}(x) \to R_1^K(x)$. Since $\tilde{R}_1^{K(n)} = R_1^{K(n)}$ on $\Omega \setminus K(n)$ and $\tilde{R}_1^K = R_1^K$ on $\Omega \setminus K$ by Theorem 3.28.1, we deduce that $G_\Omega \mu_\Omega^{K(n)} = \tilde{R}^{K(n)} \downarrow \tilde{R}_1^K = G_\Omega \mu_\Omega^K$ on $\Omega \setminus K$. Since $\operatorname{spt} \mu_\Omega^K \subset K(1)$, $\operatorname{spt} \mu_\Omega^{K(n)} \subset K(1)$, Lemma 3.32.3 gives

$$\mathcal{C}_\Omega(K(n)) = \mu_\Omega^{K(n)}(\Omega) \to \mu_\Omega^K(\Omega) = \mathcal{C}_\Omega(K).$$

The monotony follows from Proposition 3.32.2.

Proposition 3.32.5 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let $K, L \subset \Omega$ be compact sets. Then*

$$\tilde{R}_1^{K \cup L} + \tilde{R}_1^{K \cap L} \leq \tilde{R}_1^K + \tilde{R}_1^L. \tag{3.37}$$

Proof Let v_K, v_L be nonnegative superharmonic functions on Ω such that $v_K \geq 1$ on K, $v_L \geq 1$ on L. If $x \in K$ then $v_K(x) \geq 1 \geq \tilde{R}_1^{K \cup L}$, $v_L(x) \geq \tilde{R}_1^{K \cap L}(x)$. So, $v_L(x) + v_K(x) \geq \tilde{R}_1^{K \cup L}(x) + \tilde{R}_1^{K \cap L}(x)$. Similarly, $v_L(x) + v_K(x) \geq \tilde{R}_1^{K \cup L}(x) + \tilde{R}_1^{K \cap L}(x)$ for $x \in L$. Theorem 3.24.1 gives $G_\Omega(\mu_\Omega^{K \cup L} + \mu_\Omega^{L \cap K}) = \tilde{R}_1^{K \cup L} + \tilde{R}_1^{K \cap L} \leq v_L + v_K$ in Ω. Since v_K and v_L were arbitrary we obtain (3.37).

Proposition 3.32.6 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let $K, L \subset \Omega$ be compact sets. Then*

$$\mathcal{C}_\Omega(K \cup L) + \mathcal{C}_\Omega(K \cap L) \leq \mathcal{C}_\Omega(K) + \mathcal{C}_\Omega(L).$$

Proof Choose a bounded open set ω such that $K \cup L \subset \omega \subset \overline{\omega} \subset \Omega$. Clearly, $\tilde{R}_1^{\overline{\omega}} = 1$ in ω. Since μ_Ω^K, μ_Ω^L, $\mu_\Omega^{K \cup L}$, and $\mu_\Omega^{K \cap L}$ are supported in ω, Fubini's theorem and Proposition 3.32.5 give

$$\mathcal{C}_\Omega(K \cup L) + \mathcal{C}_\Omega(K \cap L) = \int_\omega \tilde{R}_1^{\overline{\omega}} \,\mathrm{d}(\mu_\Omega^{K \cup L} + \mu_\Omega^{K \cap L}) = \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \,\mathrm{d}(\mu_\Omega^{K \cup L} + \mu_\Omega^{K \cap L})$$

$$= \int_\Omega (G_\Omega \mu_\Omega^{K \cup L} + G_\Omega \mu_\Omega^{K \cap L}) \,\mathrm{d}\mu_\Omega^{\overline{\omega}} = \int_\Omega (\tilde{R}_1^{K \cup L} + \tilde{R}_1^{K \cap L}) \,\mathrm{d}\mu_\Omega^{\overline{\omega}} \leq \int_\Omega (\tilde{R}_1^K$$

$$+\tilde{R}_1^L) \,\mathrm{d}\mu_\Omega^{\overline{\omega}} = \int_\Omega G_\Omega \mu_\Omega^{\overline{\omega}} \,\mathrm{d}(\mu_\Omega^K + \mu_\Omega^L) = \int_\omega \tilde{R}_1^{\overline{\omega}} \,\mathrm{d}(\mu_\Omega^K + \mu_\Omega^L) = \mathcal{C}_\Omega(K) + \mathcal{C}_\Omega(L).$$

3.33 Inner and Outer Green Capacity

Now we define inner Green capacity and outer Green capacity and study their properties. We define capacitable sets as sets for which the outer Green capacity and the inner Green capacity coincide. We shall see that compact sets are capacitable.

Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $E \subset \Omega$ we define the inner Green capacity of E by

$$\underline{\mathcal{C}}_\Omega(E) = \sup\{\mathcal{C}_\Omega(K); K \subset E, K \text{ compact}\}$$

and the outer Green capacity of E by

$$\overline{\mathcal{C}}_\Omega(E) = \inf\{\underline{\mathcal{C}}_\Omega(\omega); E \subset \omega, \omega \text{ open}\}.$$

These set functions take values in $\langle 0, \infty\rangle$. If $E \subset F \subset \Omega$, then clearly $\underline{\mathcal{C}}_\Omega(E) \leq \underline{\mathcal{C}}_\Omega(F)$ and $\overline{\mathcal{C}}_\Omega(E) \leq \overline{\mathcal{C}}_\Omega(F)$. Since $\underline{\mathcal{C}}_\Omega(E) \leq \underline{\mathcal{C}}_\Omega(\omega)$ for each open subset of Ω such that $E \subset \omega$, we deduce that $\underline{\mathcal{C}}_\Omega(E) \leq \overline{\mathcal{C}}_\Omega(E)$.

A set $E \subset \Omega$ is called *capacitable* if $\underline{\mathcal{C}}_\Omega(E) = \overline{\mathcal{C}}_\Omega(E)$. Clearly any open subset of Ω is capacitable. If E is capacitable, we shall write $\mathcal{C}_\Omega(E) = \underline{\mathcal{C}}_\Omega(E) = \overline{\mathcal{C}}_\Omega(E)$ and call it the Green capacity of E. We shall see that this definition of the Green capacity is consistent with the earlier definition of the Green capacity for compact sets.

Lemma 3.33.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $K \subset \Omega$ is compact, then K is capacitable and $\mathcal{C}_\Omega(K) = \underline{\mathcal{C}}_\Omega(K) = \overline{\mathcal{C}}_\Omega(K)$.*

Proof We have $\mathcal{C}_\Omega(K) = \underline{\mathcal{C}}_\Omega(K)$ by Proposition 3.32.2. Denote $\omega_k = \{x; \operatorname{dist}(x, K) < 1/k\}$. According to Remark 1.2.10, there exists $k_0 \in N$ such that $\overline{\omega_k} \subset \Omega$ for $k \geq k_0$. Since $K = \cap \overline{\omega_k}$, Proposition 3.32.4 forces

$$\mathcal{C}_\Omega(K) \leq \overline{\mathcal{C}}_\Omega(K) \leq \mathcal{C}_\Omega(\overline{\omega_k}) \to \mathcal{C}_\Omega(K).$$

Lemma 3.33.2 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω and ω be a bounded open set such that $\overline{\omega} \subset \Omega$. Denote $\mu_\omega = -\Delta \tilde{R}_1^\omega$. Then $\underline{\mathcal{C}}_\Omega(\omega) = \mu_\omega(\Omega)$.*

Proof Denote $\omega_n = B(0; n) \cap \{x \in \omega; \operatorname{dist}(x, \partial\omega) > 1/n\}$, $K(n) = \overline{\omega}_n$. Then $K(n)$ is an increasing sequence of compact sets such that $\omega = \cup K(n)$. If $K \subset \omega$ is a compact, then there exists $n \in N$ such that $K \subset \omega_1 \cup \cdots \cup \omega_n$. Hence $K \subset K(n)$ and by virtue of Proposition 3.32.2

$$\underline{\mathcal{C}}_\Omega(\omega) = \lim_{k\to\infty} \mathcal{C}_\Omega(K(n)). \tag{3.38}$$

$\tilde{R}_1^{K(n)}$ is an increasing sequence of superharmonic functions on Ω. Denote $v = \lim \tilde{R}_1^{K(n)}$. Since $\tilde{R}_1^{K(n)} \leq \tilde{R}_1^\omega$, Proposition 3.25.1 gives that v is superharmonic in Ω. Theorem 3.27.3 gives that $\tilde{R}_1^{K(n)} = R_1^{K(n)} = 1$ almost everywhere on $K(n)$. Hence $v \geq 1$ almost everywhere on ω. So, $v \geq 1$ on ω by Proposition 3.3.6.

Since v is a nonnegative superharmonic function such that $v \geq 1$ on ω, we have $v \geq R_1^{\omega} \geq \tilde{R}_1^{\omega} \geq v$.

Choose a bounded open set U such that $\overline{\omega} \subset U \subset \overline{U} \subset \Omega$. Since $G_\Omega \mu_{\overline{U}} = \tilde{R}_1^{\overline{U}} = 1$ on U, we have by Proposition 3.9.12, Fubini's theorem, Theorem 1.13.3, and Proposition 3.28.4

$$\begin{aligned}\underline{\mathcal{C}}_\Omega(\omega) &\overset{(3.38)}{=} \lim_{k\to\infty} \int\limits_U G_\Omega \mu_{\overline{U}} \,\mathrm{d}\mu_{K(n)} = \lim_{k\to\infty} \int\limits_U G_\Omega \mu_{K(n)} \,\mathrm{d}\mu_{\overline{U}} \\ &= \lim_{k\to\infty} \int\limits_U \tilde{R}_1^{K(n)} \,\mathrm{d}\mu_{\overline{U}} = \int\limits_U \tilde{R}_1^{\omega} \,\mathrm{d}\mu_{\overline{U}} = \int\limits_U G_\Omega \mu_\omega \,\mathrm{d}\mu_{\overline{U}} \\ &= \int\limits_U G_\Omega \mu_{\overline{U}} \,\mathrm{d}\mu_\omega = \mu_\omega(\Omega).\end{aligned}$$

Lemma 3.33.3 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If ω_k is an increasing sequence of open subsets of Ω and $\omega = \cup \omega_n$, then $\mathcal{C}_\Omega(\omega_k) \uparrow \mathcal{C}_\Omega(\omega)$.*

Proof $\mathcal{C}_\Omega(\omega_k) \leq \mathcal{C}_\Omega(\omega_{k+1}) \leq \mathcal{C}_\Omega(\omega)$. If $K \subset \omega$ is a compact set, then there exists $n \in N$ such that $K \subset \omega_n$. Thus $\mathcal{C}_\Omega(K) \leq \mathcal{C}_\Omega(\omega_n) \leq \lim \mathcal{C}_\Omega(\omega_k)$. Since K was arbitrary we have $\mathcal{C}_\Omega(\omega) \leq \lim \mathcal{C}_\Omega(\omega_k)$.

Lemma 3.33.4 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If ω_1, ω_2 are open subsets of Ω, then*

$$\mathcal{C}_\Omega(\omega_1 \cup \omega_2) + \mathcal{C}_\Omega(\omega_1 \cap \omega_2) \leq \mathcal{C}_\Omega(\omega_1) + \mathcal{C}_\Omega(\omega_2). \tag{3.39}$$

Proof Let $K \subset \omega_1 \cap \omega_2$, $L \subset \omega_1 \cup \omega_2$ be compact. Then there exist disjoint open subsets W_1, W_2 such that

$$L \setminus \omega_2 \subset W_1 \subset \omega_1, \qquad L \setminus \omega_1 \subset W_2 \subset \omega_2.$$

Define $L_1 = L \setminus W_2$, $L_2 = L \setminus W_1$. Then $L_j \subset \omega_j$ and $L_1 \cup L_2 = L \setminus (W_1 \cap W_2) = L$. It follows from Proposition 3.32.6 that

$$\begin{aligned}\mathcal{C}_\Omega(L) + \mathcal{C}_\Omega(K) &\leq \mathcal{C}_\Omega(K \cup L) + \mathcal{C}_\Omega((K \cup L_1) \cap (K \cup L_2)) \\ &\leq \mathcal{C}_\Omega(K \cup L_1) + \mathcal{C}_\Omega(K \cup L_2) \leq \mathcal{C}_\Omega(\omega_1) + \mathcal{C}_\Omega(\omega_2).\end{aligned}$$

If we take the supremum over all possible choices of K and L, we obtain (3.39).

Proposition 3.33.5 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let E_n be an increasing sequence of subsets of Ω, $E = \cup E_n$. Then $\overline{\mathcal{C}}_\Omega(E_n) \uparrow \overline{\mathcal{C}}_\Omega(E)$.*

Proof Clearly,

$$\lim_{n\to\infty} \overline{\mathcal{C}}_{\Omega}(E_n) \le \overline{\mathcal{C}}_{\Omega}(E)$$

We can suppose that $\overline{\mathcal{C}}_{\Omega}(E_n) < \infty$ for all n.

Fix $\epsilon > 0$. For each n we chose an open set ω_n such that $E_n \subset \omega_n \subset \Omega$ and $\mathcal{C}_{\Omega}(\omega_n) < \overline{\mathcal{C}}(E_n) + 2^{-n}\epsilon$. Denote

$$\Omega_k = \bigcup_{n=1}^{k} \omega_n.$$

We prove

$$\mathcal{C}_{\Omega}(\Omega_k) < \overline{\mathcal{C}}_{\Omega}(E_k) + (1 - 2^{-k})\epsilon. \tag{3.40}$$

This inequality is obvious when $k = 1$. Suppose that this inequality is true for k. According to Lemma 3.33.4,

$$\begin{aligned} \mathcal{C}_{\Omega}(\Omega_{k+1}) + \mathcal{C}_{\Omega}(\Omega_k \cap \omega_{k+1}) &\le \mathcal{C}_{\Omega}(\Omega_k) + \mathcal{C}_{\Omega}(\omega_{k+1}) \\ &< \overline{\mathcal{C}}_{\Omega}(E_k) + (1 - 2^{-k})\epsilon + \overline{\mathcal{C}}_{\Omega}(E_{k+1}) + 2^{-(k+1)}\epsilon. \end{aligned}$$

Since $E_k \subset \Omega_k \cap \omega_{k+1}$, we have

$$\begin{aligned} \mathcal{C}_{\Omega}(\Omega_{k+1}) + \overline{\mathcal{C}}_{\Omega}(E_k) &\le \mathcal{C}_{\Omega}(\Omega_{k+1}) + \mathcal{C}_{\Omega}(\Omega_k \cap \omega_{k+1}) \\ &< \overline{\mathcal{C}}_{\Omega}(E_k) + \overline{\mathcal{C}}_{\Omega}(E_{k+1}) + (1 - 2^{-(k+1)})\epsilon. \end{aligned}$$

This gives

$$\mathcal{C}_{\Omega}(\Omega_{k+1}) < \overline{\mathcal{C}}_{\Omega}(E_{k+1}) + (1 - 2^{-(k+1)})\epsilon.$$

Lemma 3.33.3 and (3.40) force

$$\begin{aligned} \overline{\mathcal{C}}_{\Omega}(E) \le \mathcal{C}_{\Omega}(\bigcup_n \Omega_n) &= \lim_{n\to\infty} \mathcal{C}_{\Omega}(\Omega_n) \\ \le \lim_{n\to\infty} [\overline{\mathcal{C}}_{\Omega}(E_n) + (1 - 2^{-n})\epsilon] &= \lim_{n\to\infty} \overline{\mathcal{C}}_{\Omega}(E_n) + \epsilon. \end{aligned}$$

Since ϵ can be arbitrary small, we obtain $\overline{\mathcal{C}}_{\Omega}(E_n) \uparrow \overline{\mathcal{C}}_{\Omega}(E)$.

Proposition 3.33.6 *Let $\Omega \subset R^m$ be an open set with the Green function G_{Ω}. If $E_1, E_2 \subset \Omega$, then*

$$\overline{\mathcal{C}}_{\Omega}(E_1 \cup E_2) + \overline{\mathcal{C}}_{\Omega}(E_1 \cap E_2) \le \overline{\mathcal{C}}_{\Omega}(E_1) + \overline{\mathcal{C}}_{\Omega}(E_2). \tag{3.41}$$

Proof Let ω_j be an open set such that $E_j \subset \omega_j \subset \Omega, j = 0, 1$. Then

$$\overline{\mathcal{C}}_\Omega(E_1 \cup E_2) + \overline{\mathcal{C}}_\Omega(E_1 \cap E_2) \le \mathcal{C}_\Omega(\omega_1 \cup \omega_2) + \mathcal{C}_\Omega(\omega_1 \cap \omega_2) \le \mathcal{C}_\Omega(\omega_1) + \mathcal{C}_\Omega(\omega_2)$$

by Lemma 3.33.4. If we take infima over all possible choices of ω_1 and ω_2, we obtain (3.41).

Proposition 3.33.7 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $\{E_n; n \in K\}$ is at most countable collection of subsets of Ω, then*

$$\overline{\mathcal{C}}_\Omega(\bigcup_{n\in K} E_n) \le \sum_{n\in K} \overline{\mathcal{C}}_\Omega(E_n).$$

Proof It follows from Proposition 3.33.6

$$\overline{\mathcal{C}}_\Omega(\bigcup_{n=1}^{k} E_n) \le \overline{\mathcal{C}}_\Omega(\bigcup_{n=1}^{k-1} E_n) + \overline{\mathcal{C}}_\Omega(E_k) \le \dots \le \sum_{n=1}^{k} \overline{\mathcal{C}}_\Omega(E_n).$$

If K is infinite, then Proposition 3.33.5 gives

$$\overline{\mathcal{C}}_\Omega(\bigcup_{n=1}^{\infty} E_n) = \lim_{k\to\infty} \overline{\mathcal{C}}_\Omega(\bigcup_{n=1}^{k} E_n) \le \sum_{n=1}^{\infty} \overline{\mathcal{C}}_\Omega(E_n).$$

Proposition 3.33.8 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let E be a bounded set such that $\overline{E} \subset \Omega$. Denote $\mu_E = -\Delta \tilde{R}_1^E$. Then $\overline{\mathcal{C}}_\Omega(E) = \mu_E(\Omega)$.*

Proof We have $\tilde{R}_1^E = G_\Omega \mu_E$ by Proposition 3.28.4. Denote by $\mathcal{A}$ the system of all bounded open sets ω such that $E \subset \omega \subset \overline{\omega} \subset \Omega$. If $\omega \in \mathcal{A}$ then $G_\Omega \mu_E = \tilde{R}_1^E \le \tilde{R}_1^\omega = G_\Omega \mu_\omega$. Lemma 3.32.1 gives $\mu_E(\Omega) \le \mu_\omega(\Omega)$. Since $\mathcal{C}_\Omega(\omega) = \mu_\omega(\Omega)$ by Lemma 3.33.2, we deduce

$$\mu_E(\Omega) \le \inf\{\mu_\omega(\Omega); \omega \in \mathcal{A}\} = \inf\{\mathcal{C}_\Omega(\omega); \omega \in \mathcal{A}\} = \overline{\mathcal{C}}_\Omega(E).$$

Proposition 3.28.3 gives

$$R_1^E = \inf\{R_1^\omega; \omega \in \mathcal{A}\}.$$

According to Lemma 3.27.2, there exists a sequence $\omega(n) \in \mathcal{A}$ such that $\tilde{R}_1^E = \tilde{f}$ where $f = \inf\{R_1^{\omega(n)}; n \in N\}$. Denote

$$\Omega(n) = \bigcap_{k=1}^{n} \omega(k), \qquad g = \inf\{R_1^{\Omega(n)}; n \in N\}.$$

Since $f \geq g \geq R_1^E$ we deduce that $\tilde{g} = \tilde{R}_1^E$. Clearly, $G_\Omega \mu_{\Omega(n)} = \tilde{R}_1^{\Omega(n)}$ is a decreasing sequence and $G_\Omega \mu_{\Omega(n)} = \tilde{R}_1^{\Omega(n)} \geq \tilde{R}_1^E = G_\Omega \mu_E$. Theorem 3.28.1 gives $\tilde{R}_1^{\Omega(n)} = R_1^{\Omega(n)} \in H(\Omega \setminus \overline{\Omega(1)})$. Since $R_1^{\Omega(n)} \downarrow g \geq 0$, Proposition 2.11.3 forces $g \in H(\Omega \setminus \overline{\Omega(1)})$. Thus $g = \tilde{g} = \tilde{R}_1^E$ in $\Omega \setminus \overline{\Omega(1)}$. Since $G_\Omega \mu_{\Omega(n)} \downarrow G_\Omega \mu_E$ in $\Omega \setminus \overline{\Omega(1)}$ and $\operatorname{spt} \mu_{\Omega(n)} \subset \overline{\Omega(1)}$, $\operatorname{spt} \mu_E \subset \overline{\Omega(1)}$, Lemma 3.32.3 gives $\mu_{\Omega(n)}(\Omega) \to \mu_E(\Omega)$. Hence

$$\mu_E(\Omega) = \inf\{\mu_{\Omega(n)}(\Omega); n \in N\} \geq \inf\{\mu_\omega(\Omega); \omega \in \mathcal{A}\} = \overline{\mathcal{C}}_\Omega(E).$$

3.34 Capacitable Sets

In this section we show that every Borel set is capacitable.

Definition 3.34.1 Denote by N^N the set of all sequences of positive integers and by $\mathcal{K}(\Omega)$ the collection of all compact subsets of Ω. A subset A of R^m is called analytic if there exists a map $T : \cup_k N^k \to \mathcal{K}(R^m)$ such that

$$A = \bigcup_{\{k_1,k_2,\dots\} \in N^N} [T(k_1) \cap T(k_1,k_2) \cap T(k_1,k_2,k_3) \cap \dots]. \tag{3.42}$$

Lemma 3.34.2 *Let $\Omega \subset R^m$ be open. If $A \subset \Omega$ is analytic, then there exists a map $T : \cup_k N^k \to \mathcal{K}(\Omega)$ such that (3.42) holds.*

Proof Suppose that $T : \cup_k N^k \to \mathcal{K}(R^m)$ is such that (3.42) holds. Choose an increasing sequence K_j of compact sets such that $\Omega = \cup_j K_j$. Denote $\tilde{T}(k) = K_k$. For $n > 1$ and $\{k_1,\dots,k_n\} \in N^n$ define $\tilde{T}(k_1,\dots,k_n) = K_{k_1} \cap T(k_2,\dots,k_n)$. Then $\tilde{T} : \cup_k N^k \to \mathcal{K}(\Omega)$. Clearly,

$$[\tilde{T}(k_1) \cap \tilde{T}(k_1,k_2) \cap \dots] \subset [K_{k_1} \cap T(k_2) \cap T(k_2,k_3) \cap \dots] \subset A.$$

If $x \in A$ then there exist $k_1 \in N$ and a sequence $\{k_2,k_2,\dots\} \in N^N$ such that $x \in K_{k_1}$ and $x \in T(k_2) \cap T(k_2,k_3) \cap \dots$. We have

$$x \in K_{k_1} \cap [T(k_2) \cap T(k_2,k_3) \cap \dots] = [\tilde{T}(k_1) \cap \tilde{T}(k_1,k_2) \cap \dots].$$

Lemma 3.34.3 *If A_k is a sequence of analytic sets, then $\cup_k A_k$ is analytic.*

Proof Let $T_n : \cup_k N^k \to \mathcal{K}(R^m)$ be such that

$$A_n = \bigcup_{\{k_1,k_2,\dots\} \in N^N} [T_n(k_1) \cap T_n(k_1,k_2) \cap T_n(k_1,k_2,k_3) \cap \dots].$$

Choose a mapping $S(k) = [S_1(k), S_2(k)]$ from N onto N^2. Define a mapping $T : \cup_k N^k \to \mathcal{K}(R^m)$ by

$$T(k_1, \dots, k_n) = T_{S_1(k_1)}(S_2(k_1), k_2, \dots, k_n).$$

Since

$$A_n = \bigcup_{\substack{\{k_1, k_2, \dots\} \in N^N \\ S_1(k_1) = n}} [T(k_1) \cap T(k_1, k_2) \cap T(k_1, k_2, k_3) \cap \dots],$$

we deduce

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{\{k_1, k_2, \dots\} \in N^N} [T(k_1) \cap T(k_1, k_2) \cap T(k_1, k_2, k_3) \cap \dots].$$

Lemma 3.34.4 *If A_k is a sequence of analytic sets, then $\cap_k A_k$ is analytic.*

Proof Let $T_n : \cup_k N^k \to \mathcal{K}(R^m)$ be such that

$$A_n = \bigcup_{\{k_1, k_2, \dots\} \in N^N} [T_n(k_1) \cap T_n(k_1, k_2) \cap T_n(k_1, k_2, k_3) \cap \dots].$$

Let S be a mapping from the space of infinite matrix onto the space of sequences given by the diagonal procedure, i.e.,

$$S(\{a_{ij}\}_{i,j=1}^{\infty}) = (a_{11}, a_{21}, a_{12}, \dots, a_{k,1}, a_{k-1,2}, \dots, a_{2,k-1}, a_{1,k}, \dots).$$

We write $S(\{a_{ij}\}) = (S_1\{a_{ij}\}, S_2\{a_{ij}\}, \dots)$. Denote by $f : N^2 \to N$ the corresponding mapping of indices and $g = f^{-1}$ the inverse mapping. (If $S\{a_{ij}\} = (b_1, b_2, \dots)$, then $b_k = a_{ij}$ if $k = f(i,j)$, i.e., $g(k) = (g_1(k), g_2(k)) = (i,j)$.) Define now the mapping $T : \cup_k N^k \to \mathcal{K}(R^m)$ by

$$T(m_1, \dots, m_k) = T_{g_1(k)}(\{m_j; 1 \le j \le k; g_1(j) = g_1(k)\}).$$

If $x \in R^m$, then $x \in \cap_j A_j$ if for each $i \in N$, there exists a sequence of positive integers $m_{i1}, m_{i2}, m_{i3}, \dots$ such that

$$x \in T_i(a_{i1}) \cap T_i(a_{i1}, a_{i2}) \cap T_i(a_{i1}, a_{i2}, a_{i3}) \cap \dots,$$

i.e., if

$$x \in T(S_1\{a_{ij}\}) \cap T(S_1\{a_{ij}\}, S_2\{a_{ij}\}) \cap T(S_1\{a_{ij}\}, S_2\{a_{ij}\}, S_3\{a_{ij}\}) \cap \dots.$$

Thus

$$\bigcap_n A_n = \bigcup_{\{k_1,k_2,\dots\}\in N^N} [T(k_1)\cap T(k_1,k_2)\cap T(k_1,k_2,k_3)\cap\dots].$$

Theorem 3.34.5 *Every Borel set in R^m is analytic.*

Proof Denote by $\mathcal{B}$ the collection of analytic sets A for which $R^m\setminus A$ is also analytic. If A is compact, then we can put $T(k_1,\dots,k_n)=A$. Hence A is analytic. If A is open, we define $A_k=\{x\in A\cap\overline{B(0;k)};\operatorname{dist}(x,\partial A)\geq 1/k\}$. Then A_k are compact. Since $A=\cup_k A_k$, the set A is analytic by Lemma 3.34.3. The set $F=R^m\setminus A$ is closed. The sets $F_k=F\cap\overline{B(0;k)}$ are compact. Since $F=\cup_k F_k$, the set F is analytic by Lemma 3.34.3. Thus $A\subset\mathcal{B}$. Let now $A_k\in\mathcal{B}$, $A=\cup_k A_k$. Then A is analytic by Lemma 3.34.3. Since

$$R^m\setminus A = R^m\setminus\bigcup_{k=1}^{\infty}A_k=\bigcap_{k=1}^{\infty}(R^m\setminus A_k),$$

the set $R^m\setminus A$ is analytic by Lemma 3.34.4. Thus $A\in\mathcal{B}$. So, every Borel set is in $\mathcal{B}$.

Lemma 3.34.6 *Let $A\subset R^m$, $T:\cup_k N^k\to\mathcal{K}(R^m)$ be such that (3.42) holds. Fix a sequence $\{k_n\}$. For $j\in N$, define*

$$E_j=\bigcup_{\{m_n\in N^N;m_n\leq k_n when\ n\leq j\}} T(m_1)\cap T(m_1,m_2)\cap\dots,$$

$$F_j=\bigcup_{\{m_n\in N^N;m_n\leq k_n when\ n\leq j\}} T(m_1)\cap T(m_1,m_2)\cap\dots\cap T(m_1,m_2,\dots,m_j),$$

and $F=\cap F_j$. Then E_j is a decreasing sequence of sets, F_j is a decreasing sequence of compact sets, $E_j\subset A\cap F_j$, $F\subset A$.

Proof It is clear that E_j and F_j are decreasing sequences of sets, and $E_j\subset A\cap F_j$. Each set F_j is a finite union of compact sets, so it is compact.

We now show that $F\subset A$. Let $x\in F$. Then, for any $j\in N$, there exists a sequence m_{ij} such that $m_{ij}\in\{1,2,\dots,k_i\}$ for $i\leq j$, and such that

$$x\in T(m_{1j})\cap T(m_{1j},m_{2j})\cap\dots\cap T(m_{1j},\dots,m_{jj}).$$

Since $m_{1j}\in\{1,\dots,k_1\}$ there exists $j(1)\in N$ such that $m_{1j}=m_{1j(1)}$ for infinitely many j. Since $m_{2j}\in\{1,\dots,k_2\}$ there exists $j(2)\in N$ such that $m_{1j}=m_{1j(1)}=m_{1j(2)}$, $m_{2j}=m_{2j(2)}$ for infinitely many j. By this way for any $l\in N$ can be found $j(l)\in N$ such that $m_{1j}=m_{1j(1)}=m_{1j(l)}$, $m_{2j}=m_{2j(2)}=m_{2j(l)}$, $\dots$, $m_{l-1,j}=m_{l-1,j(l-1)}=m_{l-1,j(l)}$, $m_{lj}=m_{lj(l)}$ for infinitely many j. Setting $m_n=m_{nj(n)}$, we get

$$x\in T(m_1)\cap T(m_1,m_2)\cap T(m_1,m_2,m_3)\cap\dots.$$

Thus $x\in A$.

Theorem 3.34.7 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Then every analytic subset of Ω (in particular, every Borel subset of Ω) is capacitable.*

Proof Every Borel set is analytic by Theorem 3.34.5. Let now A be analytic. According to Lemma 3.34.2, there exists $T : \cup_k N^k \to \mathcal{K}(\Omega)$ such that (3.42) holds. For $k_1, \dots, k_n \in N$, denote

$$H(k_1, \dots, k_n) = \bigcup_{\{m_j \in N^N; m_j \le k_j \text{when } j \le n\}} T(m_1) \cap T(m_1, m_2) \cap \dots .$$

Fix $a < \overline{\mathcal{C}}_\Omega(A)$. Since $H(k)$ is an increasing sequence of sets such that $A = \cup_k H(k)$, Proposition 3.33.5 gives that $\overline{\mathcal{C}}_\Omega(H(k)) \uparrow \overline{\mathcal{C}}_\Omega(A)$. So, we can choose $k_1 \in N$ such that $\overline{\mathcal{C}}_\Omega(H(k_1)) > a$. Suppose that we have $k_1, \dots, k_n \in N$ such that $\overline{\mathcal{C}}_\Omega(H(k_1, \dots, k_n)) > a$. Since $H(k_1, \dots, k_n, k)$ is an increasing sequence of sets such that $H(k_1, \dots, k_n) = \cup_k H(k_1, \dots, k_n, k)$, Proposition 3.33.5 gives that $\overline{\mathcal{C}}_\Omega(H(k_1, \dots, k_n, k)) \uparrow \overline{\mathcal{C}}_\Omega(H(k_1, \dots, k_n))$. So, we can choose $k_{n+1} \in N$ such that $\overline{\mathcal{C}}_\Omega(H(k_1, k_2, \dots, k_{n+1})) > a$. Let now E_j, F_j and F are sets from Lemma 3.34.6. Since $H(k_1, \dots, k_n) = E_n \subset F_n$ we have $\overline{\mathcal{C}}_\Omega(F_n) > a$. Since F_n is compact, it is capacitable and $\mathcal{C}_\Omega(F_n) = \overline{\mathcal{C}}_\Omega(F_n) > a$ (see Lemma 3.33.1). Since F_n is a decreasing sequence of compact sets and $F = \cap_n F_n$, Proposition 3.32.4 yields $\mathcal{C}_\Omega(F) = \lim \mathcal{C}_\Omega(F_n) \ge a$. Since $F \subset A$ we infer that $\underline{\mathcal{C}}_\Omega(A) \ge a$. If we let $a \to \overline{\mathcal{C}}_\Omega(A)$ we obtain $\underline{\mathcal{C}}_\Omega(A) \ge \overline{\mathcal{C}}_\Omega(A)$. The reverse inequality is always true.

3.35 Capacity and Polar Sets

Theorem 3.35.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $E \subset \Omega$. Then E is polar if and only if $\overline{\mathcal{C}}_\Omega(E) = 0$.*

Proof Choose a sequence of bounded open sets $\Omega(n)$ such that $\overline{\Omega(n)} \subset \Omega$ and $\Omega = \cup\{\Omega(n); n \in N\}$. The set $E \cap \Omega(n)$ is polar if and only if $\tilde{R}_1^{E\cap\Omega(n)} \equiv 0$ (see Lemma 3.28.2).

Suppose first that E is polar. Then $E \cap \Omega(n)$ is polar, too. So, $\mu_{E\cap\Omega(n)} = -\Delta R_1^{E\cap\Omega(n)} = 0$. Proposition 3.33.8 gives that $\overline{\mathcal{C}}_\Omega(E \cap \Omega(n)) = \mu_{E\cap\Omega(n)}(\Omega) = 0$. By virtue of Proposition 3.33.7

$$0 \le \overline{\mathcal{C}}_\Omega(E) \le \sum_{n=1}^{\infty} \overline{\mathcal{C}}_\Omega(E \cap \Omega(n)) = 0.$$

Suppose now that $\overline{\mathcal{C}}_\Omega(E) = 0$. Then $\mu_{E\cap\Omega(n)}(\Omega) = \overline{\mathcal{C}}_\Omega(E \cap \Omega(n)) = 0$ by Proposition 3.33.8. Since $\tilde{R}_1^{E\cap\Omega(n)} = G_\Omega \mu_{E\cap\Omega(n)} = 0$ by Proposition 3.28.4, we infer that the set $E \cap \Omega(n)$ is polar. Corollary 3.26.5 forces that $E = \cup\{E \cap \Omega(n); n \in N\}$ is polar.

3.36 $\tilde{R}_u^E = R_u^E$ Quasi-everywhere

In this section, we show that $\tilde{R}_u^E = R_u^E$ quasi-everywhere for a nonnegative superharmonic function u.

Proposition 3.36.1 *Let $\Omega \subset R^m$ be an open set. Suppose that $\mathcal{A}$ is a set of superharmonic functions on Ω which is locally equibounded from below. Denote $f = \inf\{h; h \in \mathcal{A}\}$, and by $\tilde{f}$ the lower semicontinuous regularization of f. Then $\tilde{f} = f$ quasi-everywhere on Ω.*

Proof If $\mathcal{A}$ is finite, then f is superharmonic by Lemma 3.3.3. Proposition 3.3.6 gives that $\tilde{f} = f$.

Let now $\mathcal{A}$ be infinite. We can suppose that $\mathcal{A}$ is a sequence $\{h_j\}$ (see Lemma 3.27.2). Denote

$$f_k = \min_{1 \le j \le k} h_j.$$

Then f_j is a superharmonic function (see Lemma 3.3.3) and $f_j \downarrow f$.

Fix $z \in \Omega$. Choose $r > 0$ such that $\overline{B(z;3r)} \subset \Omega$. We can find a constant c such that $0 \le f + c \le f_j + c$ on $\overline{B(z;3r)}$. Denote

$$v_j = \tilde{R}_{f_j+c}^{B(z;r)}$$

the regularized reduced function of $f_j + c$ relative to $B(z;r)$ in $\omega = B(z;3r)$. Then $v_j = f_j + c$ on $B(z;r)$ by Theorem 3.28.1. Denote $\mu_j = -\Delta v_j$. Then μ_j is a nonnegative Radon measure supported on $\overline{B(z;r)}$ and $v_j = G_\omega \mu_j$ on ω (see Proposition 3.28.4). (The Green function G_ω exists by Corollary 3.9.6 and Proposition 3.9.8.) Denote $\lambda = -\Delta \tilde{R}_1^{B(z;2r)}$ where $\tilde{R}_1^{B(z;2r)}$ is the regularized reduced function of 1 relative to $B(z;2r)$ in ω. Then λ is a nonnegative Radon measure supported in $\overline{B(z;2r)}$ and $\tilde{R}_1^{B(z;2r)} = G_\omega \lambda$ by Proposition 3.28.4. Moreover, $G_\omega \lambda = 1$ on $B(z;2r)$ by Theorem 3.28.1. Since $G_\omega(x,y) = G_\omega(y,x)$ by Proposition 3.9.12, we obtain by Fubini's theorem

$$\mu_j(\overline{B(z;r)}) = \int_\omega G_\omega \lambda \,\mathrm{d}\mu_j = \int_\omega G_\omega \mu_j \,\mathrm{d}\lambda = \int_\omega v_j \,\mathrm{d}\lambda \le \int_\omega v_1 \,\mathrm{d}\lambda$$

$$= \int_\omega G_\omega \mu_1 \,\mathrm{d}\lambda = \int_\omega G_\omega \lambda \,\mathrm{d}\mu_1 = \mu_1(\overline{B(z;r)}) < \infty.$$

Thus μ_j is a bounded sequence in $\mathcal{C}'(\overline{B(z;r)})$. Theorem 1.16.3 gives that $\mathcal{C}'(\overline{B(z;r)})$ is the dual space of $\mathcal{C}^0(\overline{B(z;r)})$. According to Theorem 1.5.9, there exists a subsequence (again denoted by μ_j) and $\mu \in \mathcal{C}'(\overline{B(z;r)})$ such that

$$\int g \, \mathrm{d}\mu_j \to \int g \, \mathrm{d}\mu \qquad \forall g \in \mathcal{C}^0(\overline{B(z;r)}).$$

If $g \geq 0$ then $\int g \, \mathrm{d}\mu = \lim_{j\to\infty} \int g \, \mathrm{d}\mu_j \geq 0$. Thus μ is a nonnegative measure by Theorem 1.16.3.

Fix $x \in B(z;r)$. If $k \in N$ then $\min(G_\omega(x,y),k) \in \mathcal{C}^0(\Omega)$ by Proposition 3.9.12. Thus

$$f(x) + c = \lim_{j\to\infty} v_j(x) = \lim_{j\to\infty} G_\omega \mu_j(x)$$

$$\geq \limsup_{j\to\infty} \int\limits_{\overline{B(z;r)}} \min(G_\omega(x,y),k) \, \mathrm{d}\mu_j(y) = \int\limits_{\overline{B(z;r)}} \min(G_\omega(x,y),k) \, \mathrm{d}\mu(y).$$

For $k \to \infty$ we obtain $f + c \geq G_\omega\mu$ (see Theorem 1.13.3). The function $G_\omega\mu$ is superharmonic by Proposition 3.11.3. Proposition 3.3.6 forces $f(x)+c \geq \tilde{f}(x)+c \geq \tilde{G}_\omega\mu(x) = G_\omega\mu(x)$.

Denote

$$A := \{x \in B(z;r); f(x) + c > G_\omega\mu(x)\}.$$

Then $\{x \in B(z;r); f(x) > \tilde{f}(x)\} \subset A$. Let $K \subset A$ be compact. Suppose that K is nonpolar. According to Proposition 3.29.1, there exists a nontrivial nonnegative Radon measure ν supported in K such that $G_\omega\nu$ is bounded and continuous in ω. By virtue of Fubini's theorem

$$\int\limits_\omega G_\omega\mu_1 \, \mathrm{d}\nu = \int\limits_\omega G_\omega\nu \, \mathrm{d}\mu_1 < \infty.$$

Since $G_\omega\mu_j = v_j \leq v_1 = G_\omega\mu_1$, Lebesgue lemma (Lemma 1.13.2) and Fubini's theorem yield

$$\int\limits_\omega (f + c) \, \mathrm{d}\nu = \lim_{j\to\infty} \int\limits_\omega G_\omega\mu_j \, \mathrm{d}\nu = \lim_{j\to\infty} \int\limits_\omega G_\omega\nu \, \mathrm{d}\mu_j$$

$$= \int\limits_\omega G_\omega\nu \, \mathrm{d}\mu = \int\limits_\omega G_\omega\mu \, \mathrm{d}\nu.$$

Therefore $\int[(f + c) - G_\omega\mu] \, \mathrm{d}\nu = 0$. Since $(f + c) - G_\omega\mu > 0$ on $K \supset \operatorname{spt}\nu$, we infer that $\nu \equiv 0$ what is a contradiction. Thus K is polar. Theorem 3.35.1 gives $\mathcal{C}_\omega(K) = 0$.

For t rational define $C_t = \{x \in B(z;r); f(x) + c > t\}$, $D_t = \{x \in B(z;r); G_\omega\mu(x) < t\}$. Since $G_\omega\mu$ is superharmonic (see Proposition 3.11.2), it is a Borel measurable function. Hence D_t is Borel. Since f_j is superharmonic, the set $C_{t,j} = \{x \in B(z;r); f_j(x) + c > t\}$ is Borel. Thus $C_t = \cap_j C_{t,j}$ is Borel. Hence

$$A = \bigcap_{t<\tau; t,\tau \text{ rational}} C_\tau \cap D_t$$

is Borel. The set A is capacitable by Theorem 3.34.7. Thus $\mathcal{C}_\omega(A) = \underline{\mathcal{C}}_\omega(A) = 0$. Theorem 3.35.1 gives that A is polar.

Denote $D = \{x \in \Omega; f(x) > \tilde{f}(x)\}$. We have proved that for each $z \in \Omega$, there exists $r(z) > 0$ such that $D \cap B(z; r(z))$ is polar. Choose a sequence of compact sets K_j such that $\Omega = \cup K_j$. For a fixed j, there exist $z_1, \dots, z_k \in \Omega$ such that $K_j \subset B(z_1; r(z_1)) \cup \cdots \cup B(z_k, r(z_k))$. The set $K_j \cap D \subset [B(z_1; r(z_1)) \cap D] \cup \cdots \cup [B(z_k; r(z_k) \cap D]$ is polar by Corollary 3.26.5. Hence $D = \cup D \cap K_j$ is polar by Corollary 3.26.5.

Corollary 3.36.2 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. Let u be a nonnegative superharmonic function on Ω, and $E \subset \Omega$. Then $\tilde{R}_u^E = R_u^E$ quasi-everywhere in Ω.*

Proof We use the definition of R_u^E and Proposition 3.36.1.

3.37 The Trace of Green's Function

Theorem 3.37.1 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω. If $y \in \Omega$ then the function $G_\Omega(\cdot, y)$ has limit 0 at quasi-every point of $\partial\Omega$.*

Proof Fix $z \in \partial\Omega$. We show that there exists $r(z) > 0$ such that the open set $\omega(z) = \Omega \cup B(z; r(z))$ has Green's function. If $m > 2$ then we can choose $r(z)$ arbitrary by Corollary 3.9.6. Let now $m = 2$. Suppose that the set $R^2 \setminus (\Omega \cup B(z; 1/k))$ is polar for all $k \in N$. Then

$$R^2 \setminus \Omega = \bigcup_{k \in N} [R^2 \setminus (\Omega \cup B(z; 1/k))]$$

is polar by Corollary 3.26.5. Since there exists G_Ω, the set $R^2 \setminus \Omega$ is nonpolar by Theorem 3.31.2. That is a contradiction. So, there exists $k \in N$ such that the set $[R^2 \setminus (\Omega \cup B(z; 1/k))$ is polar. Theorem 3.31.2 gives that there exists the Green function of $R^2 \setminus (\Omega \cup B(z; 1/k))$.

Define

$$f := \tilde{R}_{G_{\omega(z)}(\cdot, y)}^{\omega(z) \setminus \Omega}$$

with respect to $\omega(z)$. Then $f = G_{\omega(z)}(\cdot, y)$ quasi-everywhere on $\omega(z) \cap \partial\Omega$ by Proposition 3.36.1. Since $f \leq G_{\omega(z)}(\cdot, y) = h_m(\cdot - y) - g_{\omega(z)}(\cdot, y)$, we deduce $f + g_{\omega(z)}(\cdot, y) \leq h_m(\cdot - y)$. The function $f + g_{\omega(z)}(\cdot, y)$ is harmonic in Ω by Theorem 3.28.1. Thus $f + g_{\omega(z)}(\cdot, y) \leq g_\Omega(\cdot, y)$ in Ω. The function f is superharmonic in $\omega(z)$ and thus $\liminf_{x \to w} f(x) = f(w)$ by Proposition 3.3.6. If $w \in \partial\Omega \cap B(z; r(z))$, then

$$\limsup_{\Omega \ni x \to w} G_\Omega(x, y) = \limsup_{\Omega \ni x \to w} [h_m(x - y) - g_\Omega(x, y)]$$

$$\leq \limsup_{x \to w} [h_m(x - y) - g_{\omega(z)}(x, y) - f(x)] = G_{\omega(z)}(w, y) - f(w).$$

Since $f = G_{\omega(z)}(\cdot, y)$ quasi-everywhere on $\omega(z) \cap \partial\Omega$ and $G_\Omega \geq 0$, we infer that

$$\lim_{x \to w} G_\Omega(x, y) = 0$$

quasi-everywhere on $\partial\Omega \cap B(z; r(z))$.

We can choose a sequence $z_j \in \partial\Omega$ such that $\partial\Omega \subset \cup_{j \in N} B(z_j, r(z_j))$. Since a countable union of polar sets is polar by Corollary 3.26.5, we infer that $G_\Omega(\cdot, y)$ has limit 0 at quasi-every point of $\partial\Omega$.

3.38 $\tilde{W}^{1,p}(R^m)$ and $D^{-1,p}(R^m)$

In the theory of boundary value problems, we need spaces $D^{1,p}(\Omega)$, $\tilde{W}^{1,p}(\Omega)$ and behavior of the Laplace operator on these spaces. In this section, we prove all we shall need to know about these spaces in the future. We characterize the range of the Laplace operator on $\tilde{W}^{1,p}(R^m)$ and on $\tilde{W}^{1,p}(\Omega)$ for a domain with compact Lipschitz boundary. We show that $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega) + \{c; c \in R\}$, in particular $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m) + \{c; c \in R\}$. If $p < m$, then $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m) \oplus \{c; c \in R\}$, and we characterize functions from $\tilde{W}^{1,p}(R^m)$ as functions from $D^{1,p}(R^m) \cap L^{pm/(m-p)}(R^m)$. If $p \geq m$ then $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m)$. We prove for an unbounded domain $\Omega \subset R^m$ with compact Lipschitz boundary that $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega) \oplus \{c; c \in R\}$ for $p < m$, and $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$ for $p \geq m$. Moreover, if $u \in D^{1,p}(\Omega)$ and $p < m$, then there exist finite limits

$$\lim_{r \to \infty} \oint_{B(0;r)} u \, \mathrm{d}\mathcal{H}_m, \quad \lim_{r \to \infty} \oint_{\partial B(0;r)} u \, \mathrm{d}\mathcal{H}_{m-1}$$

and these limits coincide. If $u \in \tilde{W}^{1,p}(\Omega)$ then these limits are equal to zero.

Let $1 < p < \infty$, $p' = p/(p-1)$, $m \geq 2$, $\Omega \subset R^m$ be an open set. Recall that $D^{1,p}(\Omega)$ is the space of distributions in Ω with derivatives of order 1 in $L^p(\Omega)$. If Ω

is a domain, fix a nonempty bounded open set ω such that $\overline{\omega} \subset \Omega$ and define a norm

$$\|u\|_{D^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Then different choices of ω give equivalent norms. $\overset{\circ}{D}{}^{1,p}(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $D^{1,p}(R^m)$ and $D^{-1,p'}(\Omega)$ is the dual space of $\overset{\circ}{D}{}^{1,p}(\Omega)$. We denote by $D^{1,p}(\Omega)/R$ the factor space $D^{1,p}(\Omega)/\{c; c \in R\}$. If Ω is connected, then $D^{1,p}(\Omega)/R$ is a Banach space equipped with the norm $\|\nabla u\|_{L^p(\Omega)}$ (see Lemma 1.25.5). Denote $\tilde{W}^{k,p}(\Omega) = \{\varphi|\Omega; \varphi \in \overset{\circ}{D}{}^{k,p}(R^m)\}$ equipped with the norm

$$\|f\|_{\tilde{W}^{k,p}(\Omega)} = \inf\{\|\varphi\|_{D^{k,p}(R^m)}; \varphi \in \overset{\circ}{D}{}^{k,p}(R^m), f = \varphi|\Omega\}.$$

Denote by $\mathcal{E}_p(R^m)$ the closure of $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ in $D^{-1,p}(R^m)$. If $p' < m$ (i.e., if $p > m/(m-1)$), then $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$. If $p' \geq m$ (i.e., if $p \leq m/(m-1)$), then $\mathcal{E}_p(R^m) = \{\nu \in D^{-1,p}(R^m); \langle \nu, 1\rangle = 0\}$. (See Corollary 2.18.6.)

Lemma 3.38.1 *Let* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $m \in N$, $m \geq 2$. *Then there exists a positive constant* C *such that*

$$\|f\|_{D^{-1,p}(R^m)} \leq \sup_{\substack{\varphi \in \mathcal{C}_c^\infty(R^m) \\ \varphi \not\equiv 0}} \frac{\langle f, \varphi\rangle}{\|\nabla\varphi\|_{L^{p'}(R^m)}} \leq C\|f\|_{D^{-1,p}(R^m)} \tag{3.43}$$

for all $f \in \mathcal{E}_p(R^m)$.

Proof Since $\|\nabla\varphi\|_{L^{p'}(R^m)} \leq \|\varphi\|_{D^{1,p'}(R^m)}$, we obtain the left inequality. We now prove the second inequality.

Suppose first that $p' < m$. Then $\|\nabla\varphi\|_{L^{p'}(R^m)}$ is an equivalent norm on $\overset{\circ}{D}{}^{1,p'}(R^m)$ by Proposition 1.25.8. This forces (3.43).

Let now $p' \geq m$. Then $1 \in \overset{\circ}{D}{}^{1,p'}(R^m)$ by Proposition 1.25.8. According to Lemma 1.25.5, there exists a positive constant M such that

$$\inf_{c\in R} \|u + c\|_{D^{1,p'}(R^m)} \leq M\|\nabla u\|_{L^{p'}(R^m)} \qquad \forall u \in D^{1,p'}(R^m).$$

Fix $f \in \mathcal{E}_p(R^m)$. Denote $X = \{\varphi \in \mathcal{C}_c^\infty(R^m); \varphi \not\equiv 0\}$. Since $\langle f, 1\rangle = 0$

$$\|f\|_{D^{-1,p}(R^m)} = \sup_{\varphi\in X} \frac{\langle f, \varphi\rangle}{\|\varphi\|_{D^{1,p'}(R^m)}} = \sup_{\varphi\in X}\sup_{c\in R} \frac{\langle f, \varphi + c\rangle}{\|\varphi + c\|_{D^{1,p'}}}$$

$$= \sup_{\varphi\in X} \frac{\langle f, \varphi\rangle}{\inf\{\|\varphi + c\|_{D^{1,p'}(R^m)}; c \in R\}} \geq \sup_{\varphi\in X} \frac{\langle f, \varphi\rangle}{M\|\nabla\varphi\|_{L^{p'}(R^m)}}.$$

Proposition 3.38.2 *Let* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $m \in N$, $m \geq 2$. *Then there exists a positive constant* M *such that for each* $u \in W^{1,1}_{loc}(R^m)$

$$\|\nabla u\|_{L^p(R^m)} \leq M \sup_{\substack{\varphi \,\in\, \mathcal{C}_c^\infty(R^m) \\ \varphi \,\not\equiv\, 0}} \int\limits_{R^m} \frac{\nabla u \cdot \nabla \varphi}{\|\nabla \varphi\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m. \tag{3.44}$$

Proof Denote $X = \{\varphi \in \mathcal{C}_c^\infty(R^m); \varphi \not\equiv 0\}$. Fix $j \in \{1, \ldots, m\}$. Since $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ is a dense subset of $L^{p'}(R^m)$ (see Corollary 2.18.5), we obtain by virtue of Theorem 1.14.3

$$\|\partial_j u\|_{L^p(R^m)} = \sup_{\substack{v \,\in\, L^{p'}(R^m) \\ v \,\not\equiv\, 0}} \int\limits_{R^m} \frac{v\,\partial_j u}{\|v\|_{L^{p'}(R^m)}} = \sup_{\varphi \in X} \int\limits_{R^m} \frac{(\Delta\varphi)(\partial_j u)}{\|\Delta\varphi\|_{L^{p'}(R^m)}}\,.$$

According to Proposition 3.18.4, there exists a positive constant c dependent only on p and m such that

$$\|\nabla^2 \varphi\|_{L^{p'}(R^m)} \leq c\|\Delta\varphi\|_{L^{p'}(R^m)} \qquad \forall \varphi \in \mathcal{C}_c^\infty(R^m).$$

Using the definition of the derivative in the sense of distributions and Theorem 3.19.3,

$$\|\partial_j u\|_{L^p(R^m)} \leq \sup_{\varphi \in X} \frac{c\langle \partial_j u, \Delta\varphi\rangle}{\|\nabla \partial_j \varphi\|_{L^{p'}(R^m)}} = \sup_{\varphi \in X} \frac{c\langle -\Delta u, \partial_j \varphi\rangle}{\|\nabla \partial_j \varphi\|_{L^{p'}(R^m)}}$$
$$= \sup_{\varphi \in X} \int\limits_{R^m} \frac{c\nabla \partial_j \varphi \cdot \nabla u}{\|\nabla \partial_j \varphi\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m \leq c \sup_{v \in X} \int\limits_{R^m} \frac{\nabla u \cdot \nabla v}{\|\nabla v\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m\,.$$

Proposition 3.38.3 *Let* $1 < p < \infty$, $m \in N$, $m \geq 2$. *Then* $\Delta : u \mapsto \Delta u$ *is an isomorphism from* $D^{1,p}(R^m)/R$ *onto* $\mathcal{E}_p(R^m)$.

Proof $\Delta : D^{1,p}(R^m)/R \to D^{-1,p}(R^m)$ is bounded by Theorem 3.19.3. Put $p' = p/(p-1)$. If $p' < m$ then $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$. Let now $p' \geq m$. Then $1 \in \mathring{D}^{1,p'}(R^m)$ by Proposition 1.25.8. Let $u \in D^{1,p}(R^m)$. Since

$$\langle -\Delta u, \varphi\rangle = \int\limits_{R^m} \nabla u \cdot \nabla\varphi \,\mathrm{d}\mathcal{H}_m \qquad \forall \varphi \in \mathring{D}^{1,p'}(R^m)$$

by Theorem 3.19.3, we deduce that $\langle -\Delta u, 1\rangle = 0$. Thus $\Delta u \in \mathcal{E}_p(R^m)$.

According to Lemma 3.38.1 and Proposition 3.38.2, there exist positive constants C and M such that(3.44) and (3.43) hold. By virtue of Theorem 3.19.3,

$$\|\Delta u\|_{D^{-1,p}(R^m)} \geq C^{-1} \sup_{\substack{\varphi \in \mathcal{C}_c^\infty(R^m) \\ \varphi \not\equiv 0}} \frac{\langle \Delta u, \varphi \rangle}{\|\nabla \varphi\|_{L^{p'}(R^m)}}$$

$$= C^{-1} \sup_{\substack{\varphi \in \mathcal{C}_c^\infty(R^m) \\ \varphi \not\equiv 0}} \int_{R^m} \frac{\nabla u \cdot \nabla \varphi}{\|\nabla \varphi\|_{L^{p'}(R^m)}} \, d\mathcal{H}_m \geq C^{-1} M^{-1} \|\nabla u\|_{L^p(R^m)}.$$

Therefore there exists a positive constant c such that $\|\Delta u\|_{D^{-1,p}(R^m)} c \geq \|u\|_{D^{1,p}(R^m)/R}$. Since $\{\Delta u; u \in \mathcal{C}_c^\infty(R^m)\}$ is a dense subset of $\mathcal{E}_p(R^m)$, we deduce that Δ maps $D^{1,p}(R^m)/R$ onto $\mathcal{E}_p(R^m)$ and Δ^{-1} is a bounded linear operator from $\mathcal{E}_p(R^m)$ onto $D^{1,p}(R^m)/R$.

Proposition 3.38.4 *Let $m \in N$, $m \geq 2$, $1 < p < \infty$. If $p < m$ then $D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m) \oplus \{c; c \in R\}$. If $m \leq p$ then $D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m)$. Let $\Omega \subset R^m$ be an open set. Then $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega) + \{c; c \in R\}$. If Ω is bounded or $m \leq p$ then $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega)$.*

Proof Suppose first that $1 < p < m$. Then $1 \notin \overset{\circ}{D}{}^{1,p}(R^m)$ and $\|\nabla u\|_{L^p(R^m)}$ is an equivalent norm on $\overset{\circ}{D}{}^{1,p}(R^m)$ by Proposition 1.25.8. So, we consider $\overset{\circ}{D}{}^{1,p}(R^m)$ to be a closed subspace of $D^{1,p}(R^m)/R$. Since $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ is a dense subset of $\mathcal{E}_p(R^m)$ by Corollary 2.18.6 and Δ is an isomorphism from $D^{1,p}(R^m)/R$ onto $\mathcal{E}_p(R^m)$ (see Proposition 3.38.3), we deduce that $\{\Delta\varphi; \varphi \in \overset{\circ}{D}{}^{1,p}(R^m)\} = \mathcal{E}_p(R^m)$ and $\overset{\circ}{D}{}^{1,p}(R^m) = \{\Delta^{-1}f; f \in \mathcal{E}_p(R^m)\} = D^{1,p}(R^m)/R$. Thus $D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m) \oplus \{c; c \in R\}$.

Let now $m \leq p < \infty$. Denote by X the factor space $\overset{\circ}{D}{}^{1,p}(R^m)/\{c; c \in R\}$. Then X is a closed subspace of $D^{1,p}(R^m)/R$ by Proposition 1.3.13. Since $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ is a dense subset of $\mathcal{E}_p(R^m)$ by Corollary 2.18.6 and Δ is an isomorphism from $D^{1,p}(R^m)/R$ onto $\mathcal{E}_p(R^m)$ (see Proposition 3.38.3), we deduce that $\{\Delta\varphi; \varphi \in X\} = \mathcal{E}_p(R^m)$ and $X = \{\Delta^{-1}f; f \in \mathcal{E}_p(R^m)\} = D^{1,p}(R^m)/R$. Thus $D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m)$.

The rest is trivial.

Proposition 3.38.5 *Let $u \in D^{1,p}(R^m)$, $m \geq 2$, $1 < p < m$. Then the following assertions are equivalent*

1. $u \in \overset{\circ}{D}{}^{1,p}(R^m)$.
2. $u \in L^{pm/(m-p)}(R^m)$.
3. *There exists* $1 < q < \infty$ *such that* $u \in L^q(R^m)$.
4.

$$\lim_{r \to \infty} \oint_{B(0;r)} |u| \, d\mathcal{H}_m = 0.$$

5.

$$\lim_{r\to\infty} \oint_{\partial B(0;r)} |u| \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Proof 1) ⇒ 2): See Lemma 1.25.7.

2) ⇒ 3): That is trivial.

3) ⇒ 4): By virtue of Hölder's inequality (Proposition 1.14.1)

$$\oint_{B(0;r)} |u| \, \mathrm{d}\mathcal{H}_m \le \|u\|_{L^q(B(0;r))} \left[\int_{B(0;r)} \mathcal{H}_m(B(0;r))^{-q/(q-1)} \, \mathrm{d}\mathcal{H}_m \right]^{(q-1)/q}$$
$$\le \|u\|_{L^q(R^m)} \mathcal{H}_m(B(0;r))^{-1/q} \to 0.$$

4) ⇒ 5): Define $\|v\| = \|v\|_{L^1(B(0;1))} + \|\nabla v\|_{L^p(B(0;1))}$. Then $\| \ \|$ is an equivalent norm on $W^{1,p}(B(0;1))$ by Proposition 1.21.9 and Hölder's inequality (Proposition 1.14.1). According to Theorem 1.24.1, there exists a constant C such that $\|v\|_{L^1(\partial B(0;1))} \le C\|v\|$ for each $v \in W^{1,p}(B(0;1))$. For $r > 0$, define $v_r(x) = u(rx)$. Then

$$\oint_{\partial B(0;r)} |u| \, \mathrm{d}\mathcal{H}_{m-1} = \oint_{\partial B(0;1)} |v_r| \, \mathrm{d}\mathcal{H}_{m-1} \le C\|v_r\|$$
$$= C \left[\mathcal{H}_m(B(0;1)) \oint_{B(0;r)} |u| \, \mathrm{d}\mathcal{H}_m + \int_{B(0;r)} \frac{|r\nabla u|^p}{r^m} \, \mathrm{d}\mathcal{H}_m \right] \to 0.$$

5) → 1): According to Proposition 3.38.4, there exists a constant c such that $u - c \in \overset{\circ}{D}{}^{1,p}(R^m)$. We have proved that

$$\lim_{r\to\infty} \oint_{\partial B(0;r)} |u - c| \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Hence

$$c = \lim_{r\to\infty} \left[\oint_{\partial B(0;r)} (c - u) \, \mathrm{d}\mathcal{H}_{m-1} + \oint_{\partial B(0;r)} u \, \mathrm{d}\mathcal{H}_{m-1} \right] = 0.$$

Corollary 3.38.6 *Let* $u \in D^{1,p}(R^m)$, $1 < p < m$, $m \geq 2$. *Then there exists a constant* c *such that*

$$c = \lim_{r \to \infty} \oint_{B(0;r)} u \,\mathrm{d}\mathcal{H}_m = \lim_{r \to \infty} \oint_{\partial B(0;r)} u \,\mathrm{d}\mathcal{H}_{m-1}. \tag{3.45}$$

Proof According to Proposition 3.38.4, there exists a constant c such that $u - c \in \overset{\circ}{D}{}^{1,p}(R^m)$. The rest is a consequence of Proposition 3.38.5.

Example 3.38.7 If $m < p$ then there exists $u \in D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m)$ such that (3.45) does not hold for any $c \in R^1$. We can choose, for example, $u(x) = \ln |x|$ for $|x| > 1$, $u(x) = 0$ elsewhere.

Lemma 3.38.8 *Let* $u \in \overset{\circ}{D}{}^{1,p}(R^m)$, $v \in \overset{\circ}{D}{}^{1,q}(R^m)$, $1 < p, q < m$, $m \geq 2$. *If* $\Delta u = \Delta v$ *then* $u = v$.

Proof Since $\Delta(u - v) = 0$ in the sense of distribution, the function $u - v$ is harmonic in R^m by Theorem 2.18.2. Fix $x \in R^m$. By virtue of Mean value theorem (Theorem 2.6.1) and Proposition 3.38.5

$$u(x) - v(x) = \lim_{r \to \infty} \oint_{B(x;r)} (u - v) \,\mathrm{d}\mathcal{H}_m = 0.$$

Proposition 3.38.9 *Let* $\mu \in \mathcal{E}_p(R^m)$ *be a real measure,* $m \geq 2$, $1 < p < m$. *If* $V\mu(z)$ *is well defined and finite for some* $z \in R^m$, *then* $V\mu \in \overset{\circ}{D}{}^{1,p}(R^m)$.

Proof $V\mu \in W^{1,1}_{loc}(R^m)$ by Proposition 3.13.1. If $m = 2$ then $p < m/(m-1)$ and $\mu(R^2) = 0$ (see Corollary 2.18.6). If $x \in R^m$, then Propositions 3.12.1 and 3.12.3 give

$$\lim_{r \to \infty} \oint_{\partial B(x;r)} V\mu \,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

Proposition 3.38.3 gives that there exists $u \in D^{1,p}(R^m)$ such that $-\Delta u = \mu$. Since $D^{1,p}(R^m) = \overset{\circ}{D}{}^{1,p}(R^m) \oplus \{c; c \in R\}$ by Proposition 3.38.4, we can suppose that $u \in \overset{\circ}{D}{}^{1,p}(R^m)$. If $x \in R^m$, then Proposition 3.38.5 forces that

$$\lim_{r \to \infty} \oint_{\partial B(x;r)} u \,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

Since $-\Delta V\mu = \mu$ (see Theorems 3.10.3, 3.11.5 and Proposition 3.9.2), we have $\Delta(V\mu - u) = 0$ in the sense of distributions in R^m. Hence $(V\mu - u) \in H(R^m)$ by

Theorem 2.18.2. Fix $x \in R^m$. By virtue of Mean value theorem (Theorem 2.6.1)

$$V\mu(x) - u(x) = \lim_{r\to\infty} \oint_{\partial B(x;r)} (V\mu - u)\, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Theorem 3.38.10 *Let $1 < p < m$, $m \in N$, $m \geq 2$. Then $-\Delta$ is an isomorphism from $\overset{\circ}{D}{}^{1,p}(R^m)$ onto $\mathcal{E}_p(R^m)$. The operator $(-\Delta)^{-1}$ does not depend on the choice of p. (If $1 < p, q < \infty$ and $f \in \mathcal{E}_p(R^m) \cap \mathcal{E}_q(R^m)$ then $(-\Delta)^{-1}f$ is the same for $(-\Delta)^{-1}$ as an operator from $\mathcal{E}_p(R^m)$ to $\overset{\circ}{D}{}^{1,p}(R^m)$ or an operator from $\mathcal{E}_q(R^m)$ to $\overset{\circ}{D}{}^{1,q}(R^m)$.) If $\mu \in \mathcal{E}_p(R^m)$ is a real measure such that $V\mu(x)$ is finite for some $x \in R^m$, then $V\mu \in \overset{\circ}{D}{}^{1,p}(R^m)$ and $V\mu = (-\Delta)^{-1}\mu$. So, $(-\Delta)^{-1}$ is a continuous extension of the volume potential V, and we shall write $V\mu$ instead of $(-\Delta)^{-1}\mu$ for all $\mu \in \mathcal{E}_p(R^m)$.*

Proof $\overset{\circ}{D}{}^{1,p}(R^m)$ is isomorphic to $D^{1,p}(R^m)/R$ by Proposition 3.38.4. So, $-\Delta$ is an isomorphism $\overset{\circ}{D}{}^{1,p}(R^m)$ onto $\mathcal{E}_p(R^m)$ by Proposition 3.38.3. The operator $(-\Delta)^{-1}$ does not depend on the choice of p by Lemma 3.38.8. If $\mu \in \mathcal{E}_p(R^m)$ is a real measure such that $V\mu(x)$ is finite for some $x \in R^m$, then $V\mu \in \overset{\circ}{D}{}^{1,p}(R^m)$ by Proposition 3.38.9. Theorems 3.10.3, 3.11.5 and Proposition 3.9.2 force that $-\Delta V\mu = \mu$.

Lemma 3.38.11 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m \geq 2$. If $1 < p < m$ then $D^{1,p}(\Omega) = D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega) \oplus \{c; c \in R\}$. If $m \leq p < \infty$ then $D^{1,p}(\Omega) = D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega)$.*

Proof Fix $u \in D^{1,p}(\Omega)$. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$. Since $u \in D^{1,p}(\Omega \cap B(0;r))$, Proposition 1.25.2 gives that $u \in \tilde{W}^{1,p}(\Omega \cap B(0;r))$. Therefore there exists $\tilde{u} \in \overset{\circ}{D}{}^{1,p}(R^m)$ such that $\tilde{u} = u$ in $\Omega \cap B(0;r)$. If we put $u = \tilde{u}$ in $R^m \setminus \Omega$, then $u \in D^{1,p}(R^m)$. Thus $u \in D^{1,p}(\overline{\Omega})$. If $m \leq p$ then $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega)$ by Proposition 3.38.4. Let now $1 < p < m$. Then $D^{1,p}(\overline{\Omega}) = \tilde{W}^{1,p}(\Omega) + \{c; c \in R\}$ by Proposition 3.38.4. Since $1 \notin \overset{\circ}{D}{}^{1,p}(R^m)$ by Proposition 1.25.8, we have $1 \in D^{1,p}(\Omega) \setminus \tilde{W}^{1,p}(\Omega)$. Therefore $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega) \oplus \{c; c \in R\}$.

Lemma 3.38.12 *Let $\Omega \subset R^m$ be a domain with compact nonempty Lipschitz boundary, $1 < p < \infty$. If $f \in D^{-1,p}(\Omega)$, then there exists $u \in \tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$ such that $-\Delta u = f$ in Ω in the sense of distributions.*

Proof If $p > m/(m-1)$ then $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$. According to Hahn-Banach theorem (Theorem 1.5.2), we can extend f as an element of $\mathcal{E}_p(R^m)$.

Let now $p \leq m/(m-1)$. Then $\mathcal{E}_p(R^m) = \{\nu \in D^{-1,p}(R^m); \langle \nu, 1\rangle = 0\}$. Put $p' = p/(p-1)$. Since $p' \geq m$, Proposition 3.38.4 gives that $\overset{\circ}{D}{}^{1,p'}(R^m) = D^{1,p'}(R^m)$. We have $\overset{\circ}{D}{}^{1,p'}(\Omega) = \{u \in \tilde{W}^{1,p'}(\Omega); u = 0 \text{ on } \partial\Omega\}$ (see Proposition 1.25.2 and Proposition 1.24.3). Thus $1 \in \overset{\circ}{D}{}^{1,p'}(R^m) \setminus \overset{\circ}{D}{}^{1,p'}(\Omega)$. Denote $X = \overset{\circ}{D}{}^{1,p'}(\Omega) \oplus \{c; c \in R^1\}$. Then X is a closed subspace of $\overset{\circ}{D}{}^{1,p'}(R^m)$ by Proposition 1.3.7. If $g \in \overset{\circ}{D}{}^{1,p'}(\Omega)$

and $c \in R^1$, define

$$\|g + c\| = \|g\|_{\tilde{W}^{1,p'}(R^m)} + |c|,$$

$f(g+c) = f(g)$. Then $\|\cdot\|$ is an equivalent norm on X by Proposition 1.3.6. Since f is a continuous linear operator on X with respect to $\|\cdot\|$, it is continuous with respect to the original norm. According to Hahn-Banach theorem (Theorem 1.5.2), we can extend f as an element of $D^{-1,p}(R^m)$. Since $\langle f, 1\rangle = 0$, we deduce that $f \in \mathcal{E}_p(R^m)$.

According to Proposition 3.38.3, there exists $v \in D^{1,p}(R^m)$ such that $-\Delta v = f$. Proposition 3.38.4 gives that there exist $u \in \tilde{W}^{1,p}(\Omega)$ and $c \in R^1$ such that $v = u + c$ in Ω. Clearly, $-\Delta u = f$ in Ω.

3.39 Real Measures with Finite Energy

Very important role in boundary value problems play distributions with finite energy as a special subspace of $D^{-1,2}(R^m)$. The next three sections are devoted to this topic. First we define real measures with finite energy in R^m for $m > 3$ and study their properties.

Let μ be a real Radon measure in R^m, $m > 2$. We say that μ has finite energy if

$$(|\mu|, |\mu|)_{\mathcal{E}} := \int_{R^m} V|\mu| \, \mathrm{d}|\mu| < \infty. \tag{3.46}$$

(If $f \in L_{loc}(R^m)$ we shall say that f has finite energy if $\mu_f = f\mathcal{H}_m$ has finite energy.)

Remark 3.39.1 Let ν, μ be nonnegative Radon measures in R^m, $m > 2$, such that $V\nu \le V\mu$ in R^m. Then $(\nu, \nu)_{\mathcal{E}} \le (\mu, \mu)_{\mathcal{E}}$.

Proof By virtue of Fubini's theorem

$$\int V\nu \, \mathrm{d}\nu \le \int V\mu \, \mathrm{d}\nu = \int V\nu \, \mathrm{d}\mu \le \int V\mu \, \mathrm{d}\mu \, .$$

Lemma 3.39.2 *Let $m > 2$. Define $f(x) = |x|^{1-m}$ in R^m. Then there exists a positive constant c such that $h_m = cf * f$.*

Proof $f * f$ has a sense because $f \ge 0$. Clearly, $(f * f)(x)$ depends only on $|x|$. Let $x \ne 0$. Using substitution

$$(f * f)(x) = \int_{R^m} |x - y|^{1-m} |y|^{1-m} \, \mathrm{d}\mathcal{H}_m(y)$$

$$= |x|^{2-2m} \int_{R^m} \left| \frac{x}{|x|} - \frac{y}{|x|} \right|^{1-m} \left| \frac{y}{|x|} \right|^{1-m} \mathrm{d}\mathcal{H}_m(y) = |x|^{2-m} (f * f)(x/|x|).$$

For $z = x/|x|$

$$0 < c := (f * f)(z) = \int_{R^m} |z - y|^{1-m}|y|^{1-m} \, \mathrm{d}\mathcal{H}_m(y)$$

$$\leq \int_{B(0;1/2)} 2^{m-1}|y|^{1-m} \, \mathrm{d}\mathcal{H}_m(y) + \int_{B(z;1/2)} 2^{m-1}|z - y|^{1-m} \, \mathrm{d}\mathcal{H}_m(y)$$

$$+ \int_{R^m \setminus B(z;1/2) \setminus B(0;1/2)} [|y|^{2-2m} + |z - y|^{2-2m}] \, \mathrm{d}\mathcal{H}_m(y) < \infty$$

Theorem 3.39.3 *Let $m > 2$. Then the set of all real Radon measures with finite energy in R^m is a vector space equipped with the inner product*

$$(\mu, \nu)_{\mathcal{E}} = \int_{R^m} V\mu \, \mathrm{d}\nu. \tag{3.47}$$

Proof Denote $f(x) = |x|^{1-m}$. Let c be a positive constant from Lemma 3.39.2. Suppose that μ is a real measure with finite energy. Using Lemma 3.39.2, substitution, and Fubini's theorem:

$$(\mu, \mu)_{\mathcal{E}} = \int_{R^m} \int_{R^m} \int_{R^m} cf(x - y - z)f(z) \, \mathrm{d}\mathcal{H}_m(z) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x)$$

$$= \int_{R^m} \int_{R^m} \int_{R^m} cf(x - w)f(w - y) \, \mathrm{d}\mathcal{H}_m(w) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x)$$

$$= c \int_{R^m} \left[\int_{R^m} f(x - w) \, \mathrm{d}\mu(x) \right] \left[\int_{R^m} f(w - y) \, \mathrm{d}\mu(y) \right] \mathrm{d}\mathcal{H}_m(w)$$

$$= c \int_{R^m} \left[\int_{R^m} f(x - w) \, \mathrm{d}\mu(x) \right]^2 \mathrm{d}\mathcal{H}_m(w) \geq 0.$$

Let now μ, ν be real measures with finite energy. Suppose first that μ, ν are nonnegative. Then Lemma 3.39.2, the preceding reasoning, Fubini's theorem, and

Hölder's inequality (Proposition 1.14.1) give

$$\begin{aligned}
|(\mu,\nu)_{\mathcal{E}}| &= \left|\int\limits_{R^m}\int\limits_{R^m}\int\limits_{R^m} cf(x-y-z)f(z)\,\mathrm{d}\mathcal{H}_m(z)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\right| \\
&= \left|\int\limits_{R^m}\int\limits_{R^m}\int\limits_{R^m} cf(x-w)f(w-y)\,\mathrm{d}\mathcal{H}_m(w)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)\right| \\
&= \left|c\int\limits_{R^m}\left[\int\limits_{R^m} f(x-w)\,\mathrm{d}\nu(x)\right]\left[\int\limits_{R^m} f(w-y)\,\mathrm{d}\mu(y)\right]\,\mathrm{d}\mathcal{H}_m(w)\right| \\
&\le \sqrt{c\int\limits_{R^m}\left[\int\limits_{R^m} f(x-w)\,\mathrm{d}\nu(x)\right]^2\,\mathrm{d}\mathcal{H}_m}\sqrt{c\int\limits_{R^m}\left[\int\limits_{R^m} f(y-w)\,\mathrm{d}\mu(y)\right]^2\,\mathrm{d}\mathcal{H}_m} \\
&= \sqrt{(\nu,\nu)_{\mathcal{E}}}\,\sqrt{(\mu,\mu)_{\mathcal{E}}} < \infty.
\end{aligned}$$

Let now μ, ν be arbitrary. By the same reasoning, we obtain $|(\mu,\nu)_{\mathcal{E}}| \le \sqrt{(\nu,\nu)_{\mathcal{E}}}\,\sqrt{(\mu,\mu)_{\mathcal{E}}} < \infty$.

Fubini's theorem gives that $(\mu,\nu)_{\mathcal{E}} = (\nu,\mu)_{\mathcal{E}}$. If $\alpha \in R$, then clearly $\alpha\mu$ has finite energy and $(\alpha\mu,\nu)_{\mathcal{E}} = \alpha(\mu,\nu)_{\mathcal{E}}$. Using preceding results $(|\mu+\nu|,|\mu+\nu|)_{\mathcal{E}} \le (|\mu|+|\nu|,|\mu|+|\nu|)_{\mathcal{E}} \le (|\mu|,|\mu|)_{\mathcal{E}}+(|\nu|,|\mu|)_{\mathcal{E}}+(|\mu|,|\nu|)_{\mathcal{E}}+(|\nu|,|\nu|)_{\mathcal{E}} < \infty$ and $\mu+\nu$ has finite energy.

Let $(\mu,\mu)_{\mathcal{E}} = 0$. If $\varphi \in \mathcal{C}_c^\infty(R^m)$ then $\varphi = V(-\Delta\varphi)$ by Corollary 2.2.7. Thus $(-\Delta\varphi)\mathcal{H}_m$ is a real measure with finite energy. We have

$$\begin{aligned}
\left|\int\limits_{R^m}\varphi\,\mathrm{d}\mu\right| &= \left|\int V(-\Delta\varphi)\,\mathrm{d}\mu\right| = |(-\Delta\varphi\mathcal{H}_m,\mu)_{\mathcal{E}}| \\
&\le \sqrt{(-\Delta\varphi\mathcal{H}_m,-\Delta\varphi\mathcal{H}_m)_{\mathcal{E}}}\,\sqrt{(\mu,\mu)_{\mathcal{E}}} = 0.
\end{aligned}$$

Since $\int\varphi\,\mathrm{d}\mu = 0$ for each $\varphi \in \mathcal{C}_c^\infty(R^m)$, we infer that $\mu = 0$.

Lemma 3.39.4 *Let μ be a nonnegative Radon measure with finite energy in R^m, $m > 2$. Suppose that $\{\mu_k\}$ is a sequence of nonnegative Radon measures in R^m such that $V\mu_k \uparrow V\mu$ as $k \to \infty$. Then μ_k have finite energy and $\|\mu-\mu_k\|_{\mathcal{E}} \to 0$.*

Proof By virtue of Fubini's theorem and Theorem on monotone convergence (Theorem 1.13.3)

$$(\mu_k,\mu_k)_{\mathcal{E}} \le \int\limits_{R^m} V\mu\,\mathrm{d}\mu_k = \int\limits_{R^m} V\mu_k\,\mathrm{d}\mu \uparrow (\mu,\mu)_{\mathcal{E}} < \infty.$$

This and Theorem 3.39.3 force

$$0 \le (\mu - \mu_k, \mu - \mu_k)_{\mathcal{E}} = (\mu, \mu)_{\mathcal{E}} + (\mu_k, \mu_k)_{\mathcal{E}} - (\mu_k, \mu)_{\mathcal{E}} - (\mu, \mu_k)_{\mathcal{E}}$$
$$\le 2(\mu, \mu)_{\mathcal{E}} - 2 \int\limits_{R^m} V\mu_k \, \mathrm{d}\mu \to 0.$$

Lemma 3.39.5 *Let μ, μ_k be real Radon measures with finite energy in R^m, $m > 2$, $\|\mu_k - \mu\|_{\mathcal{E}} \to 0$, $\|V\mu - V\mu_k\|_{L^1(B(0;r))} \to 0$ for each $r > 0$. Suppose $V\mu_k \in D^{1,2}(R^m)$, $\|\mu_k\|_{\mathcal{E}} = \|\nabla V\mu_k\|_{L^2(R^m)}$, $\|\mu_k - \mu_j\|_{\mathcal{E}} = \|\nabla V\mu_k - \nabla V\mu_j\|_{L^2(R^m)}$. Then $V\mu_k \to V\mu$ in $D^{1,2}(R^m)$ and $\|\mu\|_{\mathcal{E}} = \|\nabla V\mu\|_{L^2(R^m)}$.*

Proof $V\mu, V\mu_k \in W^{1,1}_{loc}(R^m)$ by Propositions 3.9.2, 3.11.2, 3.11.3, and 3.13.1. Fix $r > 0$. Proposition 1.21.9 gives that $V\mu_k$ is a Cauchy sequence in $W^{1,2}(B(0;r))$ and so in $D^{1,2}(R^m)$. Hence there exists $w \in D^{1,2}(R^m)$ such that $V\mu_k \to w$ in $D^{1,2}(R^m)$. If $r > 0$ then $\|w - V\mu_k\|_{L^1(B(0;r))} \to 0$ by Hölder's inequality (Proposition 1.14.1). Since $\|V\mu - V\mu_k\|_{L^1(B(0;r))} \to 0$ we have $V\mu = w \in D^{1,2}(R^m)$, $V\mu_k \to V\mu$ in $D^{1,2}(R^m)$ and $\|\mu\|_{\mathcal{E}} = \lim \|\mu_k\|_{\mathcal{E}} = \lim \|\nabla V\mu_k\|_{L^2(R^m)} = \|\nabla V\mu\|_{L^2(R^m)}$.

Theorem 3.39.6 *Let μ be a real Radon measure with finite energy in R^m, $m > 2$. Then $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$ and*

$$\|\mu\|_{\mathcal{E}} \equiv \sqrt{(\mu, \mu)_{\mathcal{E}}} = \|\nabla V\mu\|_{L^2(R^m)}.$$

Moreover, there exists a sequence $f_k \in \mathcal{C}_c^\infty(R^m)$ such that $Vf_k \in \mathcal{C}_c^\infty(R^m)$ and $\|\mu - f_k\|_{\mathcal{E}} \to 0$ as $k \to \infty$.

Proof $V\mu \in W^{1,1}_{loc}(R^m)$ by Propositions 3.9.2, 3.11.2, 3.11.3, and 3.13.1.

Suppose first that $\mu \in \mathcal{C}_c^\infty(R^m)$. Then $-\Delta V\mu = \mu$ by Theorem 3.11.5, $V\mu \in \mathcal{C}^\infty(R^m)$ by Proposition 3.18.1. Clearly

$$V\mu(x) = O(|x|^{2-m}), \quad |\nabla V\mu(x)| = O(|x|^{1-m}) \quad \text{as } |x| \to \infty.$$

By virtue of Gauss-Green theorem (Theorem 1.19.1)

$$0 = \lim_{r\to\infty} \int\limits_{\partial B(0;r)} (V\mu) \frac{\partial V\mu}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{r\to\infty} \int\limits_{B(0;r)} [|\nabla V\mu|^2 + (V\mu)\Delta V\mu] \, \mathrm{d}\mathcal{H}_m$$
$$= \lim_{r\to\infty} \int\limits_{B(0;r)} [|\nabla V\mu|^2 - \mu(V\mu) \, \mathrm{d}\mathcal{H}_m = \|\nabla V\mu\|^2_{L^2(R^m)} - \|\mu\|^2_{\mathcal{E}}.$$

Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $B(0;1)$, $\varphi = 0$ on $R^m \setminus B(0;2)$. Choose a constant $C \ge 1$ such that $|\varphi| \le C$, $|\nabla\varphi| \le C$, $|V\mu(x)| \le C|x|^{2-m}$, $|\nabla V\mu(x)| \le C|x|^{1-m}$. Put $\varphi_k(x) = \varphi(x/k)$, $v_k(x) = \varphi_k V\mu(x)$, $f_k = -\Delta v_k$. Then $f_k \in \mathcal{C}_c^\infty(R^m)$,

$Vf_k = v_k \in \mathcal{C}_c^\infty(R^m)$ (see Corollary 2.2.7). Clearly $\nabla(V\mu - \varphi_k V\mu) = 0$ on $B(0;k)$ and on $R^m \setminus B(0;2k)$. If $k \le |x| \le 2k$ then $|\nabla(V\mu - \varphi_k V\mu)| \le |(1-\varphi_k(x))\nabla V\mu(x)| + |\nabla\varphi_k(x)||V\mu(x)| \le (C+1)Ck^{1-m} + (C/k)Ck^{2-m}$. Thus

$$\|\mu - f_k\|_{\mathcal{E}}^2 = \|\nabla(V\mu - \varphi_k V\mu)\|_{L^2(R^m)}^2 \le 9C^4\mathcal{H}_m(B(0;1))k^{2-m} \to 0.$$

Hence $\|V\mu - \varphi_k V\mu\|_{L^2(B(0;1))} + \|\nabla V\mu - \nabla\varphi_k V\mu\|_{L^2(R^m)} \to 0$ and $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$.

Let now $\mu = \mu^+ - \mu^-$ be a real measure with compact support. Choose $\varphi \in \mathcal{C}^\infty(R^1)$, $\varphi \ge 0$, $\operatorname{spt}\varphi \subset (0,1)$ such that

$$\int_{R^m} \varphi(|y|)\, \mathrm{d}\mathcal{H}_m(y) = 1.$$

For $k \in N$ define $\varphi_k(x) = \varphi(kx)k^m$, $f_k^+ = \mu^+ * \varphi_k$, $f_k^- = \mu^- * \varphi_k$. Then $f_k^+, f_k^- \in \mathcal{C}_c^\infty(R^m)$ and $Vf_k^+ = \varphi_k * V\mu^+$, $Vf_k^- = \varphi_k * V\mu^-$ (see Sect. 1.20.) Since $V\mu^+$ is a superharmonic function (Proposition 3.11.2), Proposition 3.8.2 gives that $Vf_k^+ \uparrow V\mu^+$, $Vf_k^- \uparrow V\mu^-$. Lemma 3.39.4 gives that $\|\mu^+ - f_k^+\|_{\mathcal{E}} \to 0$, $\|\mu^- - f_k^-\|_{\mathcal{E}} \to 0$. Fix $r > 0$. Since $V\mu^+, V\mu^- \in L^1(B(0;r))$ by Proposition 3.3.4 and $0 \le Vf_k^+ \uparrow V\mu^+$, $0 \le Vf_k^- \uparrow V\mu^-$, Lebesgue lemma (Lemma 1.13.2) gives that $\|V\mu^+ - Vf_k^+\|_{L^1(B(0;r))} \to 0$, $\|V\mu^- - Vf_k^-\|_{L^1(B(0;r))} \to 0$ as $k \to \infty$. Put $f_k = f_k^+ - f_k^-$. Then $\|\mu - f_k\|_{\mathcal{E}} \le \|\mu^+ - f_k^+\|_{\mathcal{E}} + \|\mu^- - f_k^-\|_{\mathcal{E}} \to 0$, $\|V\mu - Vf_k\|_{L^1(B(0;r))} \to 0$. Lemma 3.39.5 gives that $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$ and $\|\mu\|_{\mathcal{E}} = \|\nabla V\mu\|_{L^2(R^m)}$.

Let now $\mu = \mu^+ - \mu^-$ be arbitrary. For $k \in N$ put $\mu_k^+ = \mu^+|B(0;k)$, $\mu_k^- = \mu^-|B(0;k)$, $\mu_k = \mu_k^+ - \mu_k^-$. Then $V\mu_k^+ \uparrow V\mu^+$, $V\mu_k^- \uparrow V\mu^-$. Lemma 3.39.4 gives that $\|\mu^+ - \mu_k^+\|_{\mathcal{E}} \to 0$, $\|\mu^- - \mu_k^-\|_{\mathcal{E}} \to 0$. Fix $r > 0$. Since $V\mu^+, V\mu^- \in L^1(B(0;r))$ and $0 \le V\mu_k^+ \uparrow V\mu^+$, $0 \le V\mu_k^- \uparrow V\mu^-$, Lebesgue lemma (Lemma 1.13.2) gives that $\|V\mu^+ - V\mu_k^+\|_{L^1(B(0;r)} \to 0$, $\|V\mu^- - V\mu_k^-\|_{L^1(B(0;r)} \to 0$ as $k \to \infty$. Thus $\|\mu - \mu_k\|_{\mathcal{E}} \to 0$, $\|V\mu - V\mu_k\|_{L^1(B(0;r))} \to 0$. Lemma 3.39.5 gives that $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$ and $\|\mu\|_{\mathcal{E}} = \|\nabla V\mu\|_{L^2(R^m)}$.

Corollary 3.39.7 *Let μ, ν be real measures with finite energy in R^m, $m > 2$. Then $\mu, \nu \in \mathcal{E}_2(R^m) = D^{-1,2}(R^m)$ and*

$$(\mu,\nu)_{\mathcal{E}} = \int_{R^m} \nabla V\mu \cdot \nabla V\nu\, \mathrm{d}\mathcal{H}_m.$$

Proof According to Theorems 3.39.3 and 3.39.6

$$(\mu,\nu)_{\mathcal{E}} = \frac{1}{4}[(\mu+\nu,\mu+\nu)_{\mathcal{E}} - (\mu-\nu,\mu-\nu)_{\mathcal{E}}]$$
$$= \frac{1}{4}\int_{R^m} [|\nabla V(\mu+\nu)|^2 - |\nabla V(\mu-\nu)|^2]\, \mathrm{d}\mathcal{H}_m = \int_{R^m} \nabla V\mu \cdot \nabla V\nu\, \mathrm{d}\mathcal{H}_m.$$

Since $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$ by Theorem 3.39.6, we have $\mu = -\Delta V\mu \in \mathcal{E}_2(R^m)$ by Theorems 3.9.2, 3.11.5 and Proposition 3.38.3. Since $2 > m/(m-1)$, Corollary 2.18.6 gives that $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$.

3.40 Distributions with Finite Energy

This section treats the space of distributions with finite energy $\mathcal{E}_2(R^m)$ for $m > 2$. For a closed set $F \subset R^m$, we denote by $\mathcal{E}_2(F)$ the space of distributions with finite energy supported on F. We discuss connections between these spaces and homogeneous Sobolev spaces. We will show that $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$ and both norms are equivalent. We will prove for an open set Ω with compact Lipschitz boundary that $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$ and both norms are equivalent.

Suppose that $m > 2$. Equip $\overset{\circ}{D}{}^{1,2}(R^m)$ with the norm $\|\nabla u\|_{L^2(R^m)}$. This norm is equivalent to the original norm by Proposition 1.25.8. The operator $-\Delta$ is an isomorphism of $\overset{\circ}{D}{}^{1,2}(R^m)$ onto $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$ and the volume potential V is the inverse of $-\Delta$ (see Theorem 3.38.10 and Corollary 2.18.6). Therefore $\|\nabla V\mu\|_{L^2(R^m)}$ is an equivalent norm on $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$. Elements of $\mathcal{E}_2(R^m)$ are called *distributions with finite energy*. Remark that the set of all real measures with finite energy is a dense subspace of $\mathcal{E}_2(R^m)$ (see Proposition 3.38.9 and Corollary 2.18.6).

Lemma 3.40.1 *Let $m > 2$. The space $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$ is a Hilbert space with the inner product*

$$(\mu, \nu)_{\mathcal{E}} = \langle \mu, V\nu \rangle = \int_{R^m} \nabla V\mu \cdot \nabla V\nu \, d\mathcal{H}_m.$$

If $\mu, \nu \in \mathcal{E}_2(R^m)$, $\mu \neq \nu$ and $0 < \lambda < 1$, then

$$\|\lambda\mu + (1-\lambda)\nu\|_{\mathcal{E}}^2 < \lambda\|\mu\|_{\mathcal{E}}^2 + (1-\lambda)\|\nu\|_{\mathcal{E}}^2.$$

Proof $\|\nabla V\mu\|_{L^2(R^m)}$ is an equivalent norm on $\mathcal{E}_2(R^m) = D^{-1,2}(R^m)$. If μ, ν are real measures with finite energy, we have by Corollary 3.39.7

$$\int_{R^m} \nabla V\mu \cdot \nabla V\nu \, d\mathcal{H}_m = (\mu, \nu)_{\mathcal{E}} = \int_{R^m} V\nu \, d\mu = \langle \mu, V\nu \rangle.$$

Since the space of real measures with finite energy is a dense subset of $\mathcal{E}_2(R^m)$, we obtain the first proposition of the lemma.

Let $\mu, \nu \in \mathcal{E}_2(R^m)$, $\mu \neq \nu$ and $0 < \lambda < 1$. Then

$$\lambda\|\mu\|_{\mathcal{E}}^2 + (1-\lambda)\|\nu\|_{\mathcal{E}}^2 - \|\lambda\mu + (1-\lambda)\nu\|_{\mathcal{E}}^2 = (\lambda - \lambda^2)\|\mu\|_{\mathcal{E}}^2 + [(1-\lambda) - (1-\lambda)^2]\|\nu\|_{\mathcal{E}}^2$$
$$-2\lambda(1-\lambda)(\mu, \nu)_{\mathcal{E}} = \lambda(1-\lambda)\|\mu - \nu\|_{\mathcal{E}}^2 > 0.$$

Proposition 3.40.2 *Let $F \subset R^m$ be closed, $m > 2$. Denote by $\mathcal{E}_2(F)$ the set of all $\mu \in \mathcal{E}_2(R^m)$ supported in F. Then $\mathcal{E}_2(F)$ is a closed subspace of $\mathcal{E}_2(R^m)$. Moreover, $\{V\mu; \mu \in \mathcal{E}_2(F)\} = \overset{\circ}{D}{}^{1,2}(R^m) \cap H(R^m \setminus F)$, $[\mathcal{E}_2(F)]^\perp = \{-\Delta\varphi; \varphi \in \overset{\circ}{D}{}^{1,2}(R^m \setminus F)\}$.*

Proof Let $\mu_n \in \mathcal{E}_2(F)$, $\mu_n \to \mu$ in $\mathcal{E}_2(R^m)$. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, $F \cap \operatorname{spt}\varphi = \emptyset$, then $\langle \mu, \varphi \rangle = \lim \langle \mu_n, \varphi \rangle = 0$. Therefore μ is supported on F.

Let now $\mu \in \mathcal{E}_2(F)$. Then $V\mu \in \overset{\circ}{D}{}^{1,2}(R^m)$ and $\mu = -\Delta V\mu$. If $\varphi \in \mathcal{C}_c^\infty(R^m \setminus F)$, then

$$0 = \langle \mu, \varphi \rangle = \int\limits_{R^m} \nabla\varphi \cdot \nabla V\mu \, d\mathcal{H}_m$$

by Theorem 3.19.3. Theorem 2.18.2 gives that $V\mu \in H(R^m \setminus F)$.

Let now $\mu \in \mathcal{E}_2(R^m) = D^{-1,2}(R^m)$ be such that $V\mu \in H(R^m \setminus F)$. If $\varphi \in \mathcal{C}_c^\infty(R^m \setminus F)$, Theorems 2.18.2 and 3.19.3 give

$$0 = \int\limits_{R^m} \nabla\varphi \cdot \nabla V\mu \, d\mathcal{H}_m = \langle \mu, \varphi \rangle.$$

Hence $\operatorname{spt}\mu \subset F$.

If $\mu \in \mathcal{E}_2(F)$ and $\varphi \in \mathcal{C}_c^\infty(R^m \setminus F)$ then

$$0 = \langle \mu, \varphi \rangle = \int\limits_{R^m} \nabla\varphi \cdot \nabla V\mu \, d\mathcal{H}_m = (\mu, -\Delta\varphi)_{\mathcal{E}}.$$

Since $[\mathcal{E}_2(F)]^\perp$ is a closed subspace of $\mathcal{E}_2(R^m)$, we have that $\{-\Delta\varphi; \varphi \in \overset{\circ}{D}{}^{1,2}(R^m \setminus F)\} \subset [\mathcal{E}_2(F)]^\perp$. Suppose now that $\{-\Delta\varphi; \varphi \in \overset{\circ}{D}{}^{1,2}(R^m \setminus F)\} \neq [\mathcal{E}_2(F)]^\perp$. Then there exists nontrivial $\nu \in \{-\Delta\varphi; \varphi \in [\overset{\circ}{D}{}^{1,2}(R^m \setminus F)\}]^\perp \cap [\mathcal{E}_2(F)]^\perp$. If $\varphi \in \mathcal{C}_c^\infty(R^m \setminus F)$, then $\varphi = V(-\Delta\varphi)$ by Corollary 2.2.7 and

$$0 = (\nu, -\Delta\varphi)_{\mathcal{E}} = \int\limits_{R^m} \nabla V\nu \cdot \nabla\varphi \, d\mathcal{H}_m.$$

So, $V\nu \in H(R^m \setminus F)$ by Theorem 2.18.2. We have proved that $\nu = -\Delta V\nu \in \mathcal{E}_2(F)$. That contradicts with $\nu \in [\mathcal{E}_2(F)]^\perp$.

Proposition 3.40.3 *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary,* $m > 2$*. Then* $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega) = W^{-1/2,2}(\partial\Omega)$ *and the corresponding norms are equivalent.*

Proof $H^{-1/2}(\partial\Omega) = F^{2,2}_{-1/2}(\partial\Omega) = B^{2,2}_{-1/2}(\partial\Omega) = W^{-1/2,2}(\partial\Omega)$ by Theorems 1.23.3 and 1.23.5 and Lemma 1.23.6.

By virtue of Theorem 1.24.2, we see that $W^{-1/2,2}(\partial\Omega)$ is the set of $f \in [W^{1,2}(R^m)]'$ supported on $\partial\Omega$. Since $\partial\Omega$ is compact, it is the set of $f \in [\overset{\circ}{D}{}^{1,2}(R^m)]' = D^{-1,2}(R^m)$ supported on $\partial\Omega$. Hence $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$.

Since $\mathcal{E}_2(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ are Banach spaces, $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$, the corresponding norms are equivalent by Proposition 1.3.6.

3.41 Energy in the Plane

In R^2 it is not possible to develop the theory of distributions with finite energy like in R^m with $m > 2$. So, we restrict ourselves to compactly supported real Radon measures. For compactly supported real measures μ with finite energy and $\mu(R^2) = 0$, we prove similar results like in the higher dimensional case.

Let μ be a compactly supported real Radon measure in R^2. We say that μ has finite energy if

$$\int_{R^2}\int_{R^2} |\ln(x-y)|\, \mathrm{d}|\mu|(y)\, \mathrm{d}|\mu|(x) < \infty.$$

Clearly, this condition is equivalent to $V|\mu| \in L^1(|\mu|)$ and to $\int V|\mu| \mathrm{d}|\mu| < \infty$. If μ, ν are compactly supported real Radon measures with finite energy in R^2, denote

$$(\mu,\nu)_{\mathcal{E}} = \int_{R^2}\int_{R^2} V\nu(x)\, \mathrm{d}\mu(x)$$

if this integral makes sense.

Lemma 3.41.1 *Let* $\mu, \nu \in \mathcal{C}_c^\infty(R^m)$*,* $\mu(R^2) = \nu(R^2) = 0$*. Then* $V\mu, V\nu \in D^{1,2}(R^2)$ *and*

$$(\mu,\nu)_{\mathcal{E}} = \int_{R^2} (\nabla V\mu)\cdot(\nabla V\nu)\, \mathrm{d}\mathcal{H}_2. \tag{3.48}$$

Proof $-\Delta V\mu = \mu$ by Theorem 3.10.3, $V\mu \in \mathcal{C}^\infty(R^2)$ according to Theorem 3.18.1, $V\mu(x) = O(|x|^{-1})$, $|\nabla V\mu(x)| = O(|x|^{-2})$ as $|x| \to \infty$ by

Lemma 3.12.2. Similarly for ν. By virtue of Gauss-Green theorem (Theorem 1.19.1)

$$0 = \lim_{r\to\infty} \int\limits_{\partial B(0;r)} (V\nu)\frac{\partial V\mu}{\partial n}\,\mathrm{d}\mathcal{H}_1 = \lim_{r\to\infty} \int\limits_{B(0;r)} [(\nabla V\mu)\cdot(\nabla V\nu)$$

$$+(V\nu)\Delta V\mu]\,\mathrm{d}\mathcal{H}_2 = \lim_{r\to\infty} \int\limits_{B(0;r)} (\nabla V\mu)\cdot(\nabla V\nu) - (\mu,\nu)_{\mathcal{E}}.$$

In particular, for $\nu = \mu$

$$\|\mu\|_{\mathcal{E}}^2 = \|\nabla V\mu\|_{L^2(R^2)}^2.$$

Thus $V\mu, V\nu \in D^{1,2}(R^2)$ and (3.48) holds.

Proposition 3.41.2 *Let μ be a compactly supported real Radon measure in R^2 with finite energy. Then $V\mu \in W^{1,2}_{loc}(R^2)$. If $\mu(R^2) = 0$ then $V\mu \in D^{1,2}(R^2)$.*

Proof Suppose first that $\mu(R^2) = 0$. Choose $\varphi \in \mathcal{C}^\infty(R^1)$, $\varphi \geq 0$, $\operatorname{spt}\varphi \subset (0,1)$ such that

$$\int\limits_{R^2} \varphi(|y|)\,\mathrm{d}\mathcal{H}_2(y) = 1.$$

For $k \in N$ define $\varphi_k(x) = \varphi(kx)k^m$, $\mu_k^+ = \mu^+ * \varphi_k$, $\mu_k^- = \mu^- * \varphi_k$, $\mu_k = \mu_k^+ - \mu_k^-$, where $\mu = \mu^+ - \mu^-$. Then $\mu_k^+, \mu_k^- \in \mathcal{C}_c^\infty(R^m)$ and $V\mu_k^+ = \varphi_k * V\mu^+$, $V\mu_k^- = \varphi_k * V\mu^-$ (see Sect. 1.20). Since $V\mu^+$ is a superharmonic function (Proposition 3.10.2), Proposition 3.8.2 gives that $V\mu_k^+ \uparrow V\mu^+$, $V\mu_k^- \uparrow V\mu^-$. Fix $R > 1$ such that $\operatorname{spt}\mu \subset B(0;R)$. According to Lemma 3.2.2, there exists $a > 0$ such that $V\mu_1^+ \geq -a$, $V\mu_1^- \geq -a$ on $B(0;2R)$. Hence $|V\mu_k| \leq |h_2| * |\mu| + a$ in $B(0;2R)$. Since $|h_2| * |\mu| \in L^1(|\mu|) \cap L^1(B(0;2R))$ (see Proposition 3.3.4 and Proposition 3.10.2), we have $V\mu_k \to V\mu$ in $L^1(|\mu|)$ and in $L^1(B(0;2R))$. (See Lemma 1.13.2.) We have $\mu_k \in \mathcal{C}_c^\infty(R^2)$. Since $V\mu$ is harmonic in $R^2 \setminus \overline{B(0;R)}$ by Proposition 3.10.2, Proposition 3.8.2 gives that $V\mu_k = V\mu$ on $R^2 \setminus B(0;2R)$. Thus $V\mu_k(x) \to 0$ as $|x| \to \infty$ by Lemma 3.12.2. Hence $\mu_k(R^2) = 0$ by Lemma 3.12.2. Lemma 3.41.1 gives that $V\mu_k \in D^{1,2}(R^2)$ and

$$\int\limits_{R^2} |\nabla V\mu_k|^2\,\mathrm{d}\mathcal{H}_2 = \int\limits_{R^2} V\mu_k\,\mathrm{d}\mu_k. \tag{3.49}$$

Denote

$$\tilde{V}\nu(x) = \int\limits_{R^2} \frac{1}{2\pi}\ln\frac{2R}{|x-y|}\,\mathrm{d}\nu(y).$$

Since $\mu(R^2) = 0$, spt $\mu \subset B(0; R)$, we have in spt μ_k

$$|V\mu_k| = |\varphi_k * V\mu| = |\varphi_k * \tilde{V}\mu| \leq \varphi_k * \tilde{V}|\mu| \leq \tilde{V}|\mu|$$

by Proposition 3.8.2. By virtue of (3.49)

$$\int\limits_{R^2} |\nabla V\mu_k|^2 \, \mathrm{d}\mathcal{H}_2 \leq \int\limits_{R^2} \tilde{V}|\mu| \, \mathrm{d}|\mu| < \infty.$$

Moreover, $V\mu_k \to V\mu$ in $L^1(B(0; 2R))$. Therefore, $V\mu_k$ is a bounded sequence in $D^{1,2}(R^2)$. The space $D^{1,2}(R^2)$ is reflexive by Lemma 1.25.3. According to Theorem 1.5.9, there exist a subsequence $V\mu_{k(n)}$ and $u \in D^{1,2}(R^2)$ such that $V\mu_{k(n)} \to u$ weakly in $D^{1,2}(R^2)$. Fix $\varphi \in \mathcal{C}_c^\infty(R^2)$. Then $\varphi \in D^{-1,2}(R^2)$ and

$$\int\limits_{R^2} \varphi V\mu_{k(n)} \, \mathrm{d}\mathcal{H}_2 = \langle \varphi, V\mu_{k(n)} \rangle \to \langle \varphi, u \rangle = \int\limits_{R^2} \varphi u \, \mathrm{d}\mathcal{H}_2.$$

On the other hand

$$\int\limits_{R^2} \varphi V\mu_{k(n)} \, \mathrm{d}\mathcal{H}_2 \to \int\limits_{R^2} \varphi V\mu \, \mathrm{d}\mathcal{H}_2.$$

Since $\langle V\mu, \varphi \rangle = \langle u, \varphi \rangle$ for each $\varphi \in \mathcal{C}_c^\infty(R^2)$, we infer that $V\mu = u \in D^{1,2}(R^2)$.

Let now $c = \mu(R^2) \neq 0$. Fix $r > 0$ such that spt $\mu \subset B(0; r)$. Choose a nonnegative $\varphi \in \mathcal{C}_c^\infty(B(0; r))$ such that $\int \varphi \, \mathrm{d}\mathcal{H}_2 = 1$. Set $\nu = \mu - c\varphi$. Easy calculation yields that $\int |h_2(x - y)| \varphi(y) \, \mathrm{d}\mathcal{H}_2(y) \leq C$ for all $x \in \overline{B(0; r)}$. By virtue of Fubini's theorem

$$\int\int |h_2(x-y)| \, \mathrm{d}|\nu|(y) \, \mathrm{d}|\nu|(x) \leq \int\int |h_2(x-y)| \, \mathrm{d}|\mu|(y) \, \mathrm{d}|\mu|(x)$$
$$+2|c| \int\int \varphi(y) |h_2(x-y)| \, \mathrm{d}\mathcal{H}_2|(y) \, \mathrm{d}|\nu|(x)$$
$$+c^2 \int\int \varphi(y)\varphi(x) |h_2(x-y)| \, \mathrm{d}\mathcal{H}_2|(y) \, \mathrm{d}\mathcal{H}_2|(x) < \infty.$$

Since ν is a real measure with finite energy and $\nu(R^2) = 0$, we have $V\nu \in D^{1,2}(R^2)$. Since $V\varphi \in W^{1,2}_{loc}(R^2)$ by Proposition 3.13.2, we conclude that $V\mu = V\nu + cV\varphi \in W^{1,2}_{loc}(R^2)$.

Theorem 3.41.3 *The set of real compactly supported measures with finite energy in R^2 is a linear space. If μ, ν are compactly supported real measures with finite energy in R^2, then $V\mu \in L^1(|\nu|)$ and $(\mu, \nu)_{\mathcal{E}} = (\nu, \mu)_{\mathcal{E}}$. If, moreover, $\mu(R^2) = \nu(R^2) = 0$*

then $V\mu, V\nu \in \overset{\circ}{D}{}^{1,2}(R^2)$ *and*

$$(\mu, \nu)_{\mathcal{E}} = \int\limits_{R^2} \nabla V\mu \cdot \nabla V\nu \, \mathrm{d}\mathcal{H}_2.$$

Proof Let μ, ν be compactly supported real measures with finite energy in R^2. Choose $\psi \in \mathcal{C}_c^\infty(B(0;2))$ such that $\psi = 1$ on $B(0;1)$, $0 \le \psi \le 1$. For $r > 0$ denote $\psi_r(x) = \psi(x/2r)$. Fix $r > 1$ such that $\operatorname{spt}\mu \cup \operatorname{spt}\nu \subset B(0;r)$. Denote

$$\tilde{V}\mu(x) = \int\limits_{R^2} \frac{1}{2\pi} \ln \frac{3r}{|x-y|} \, \mathrm{d}\mu(y).$$

Choose $\varphi \in \mathcal{C}^\infty(R^1)$, $\varphi \ge 0$, $\operatorname{spt}\varphi \subset (0,1)$ such that

$$\int\limits_{R^2} \varphi(|y|) \, \mathrm{d}\mathcal{H}_2(y) = 1.$$

For $k \in N$ define $\varphi_k(x) = \varphi(|kx|)k^m$. The function $\psi_r \tilde{V}|\mu| = \tilde{V}|\mu|$ is a nonnegative superharmonic function on $B(0;2r)$ (see Proposition 3.10.2). Proposition 3.8.2 gives that $0 \le \varphi_k * (\psi_r \tilde{V}|\mu|) \uparrow \tilde{V}|\mu|$ in $B(0;r)$. By virtue of the Theorem on monotone convergence (Theorem 1.13.3)

$$\int\limits_{R^2} \psi_r \tilde{V}|\mu| \, \mathrm{d}|\nu| = \lim_{k\to\infty} \int\limits_{R^2} \varphi_k * (\psi_r \tilde{V}|\mu|) \, \mathrm{d}|\nu|. \tag{3.50}$$

$|\nu| = -\Delta V|\nu|$ in the sense of distribution by Theorem 3.10.3. Proposition 3.41.2 gives $V|\nu| \in D^{1,2}(B(0;5r))$. Since $\varphi_k * (\psi_r \tilde{V}|\mu|) \in \mathcal{C}_c^\infty(B(0;5r))$, we have by Theorem 3.19.3

$$\int\limits_{R^2} \varphi_k * (\psi_r \tilde{V}|\mu|) \, \mathrm{d}|\nu| = \int\limits_{R^2} \nabla[\varphi_k * (\psi_r \tilde{V}|\mu|)] \cdot \nabla V|\nu| \, \mathrm{d}\mathcal{H}_2. \tag{3.51}$$

Since $\tilde{V}|\mu| \in W^{1,2}_{loc}(R^2)$ by Proposition 3.41.2, we have $\psi_r \tilde{V}|\mu| \in D^{1,2}(R^2)$. Thus $\varphi_k * \nabla[\psi_r \tilde{V}|\mu|] \to \nabla[\psi_r \tilde{V}|\mu|]$ in $L^2(R^2)$ (see Chap. 1). Moreover $\operatorname{spt}\varphi_k * \nabla[\psi_r \tilde{V}|\mu|] \subset B(0;3r)$. Since $V|\nu| \in W^{1,2}(B(0;3r))$ by Proposition 3.41.2, we have by (3.50) and (3.51)

$$\int\limits_{R^2} \psi_r \tilde{V}|\mu| \, \mathrm{d}|\nu| = \lim_{k\to\infty} \int\limits_{R^2} [\varphi_k * \nabla(\psi_r \tilde{V}|\mu|)] \cdot \nabla V|\nu| = \int\limits_{R^2} \nabla(\psi_r \tilde{V}|\mu|) \cdot \nabla V|\nu|. \tag{3.52}$$

Therefore

$$\int_{R^2} |V\mu| \, \mathrm{d}|\nu| \le \int_{R^2} \psi_r \left[\tilde{V}|\mu| + \frac{\ln 3r}{2\pi} |\mu(R^2)| \right] \mathrm{d}|\nu| < \infty.$$

Hence $(\mu, \nu)_{\mathcal{E}}$ is finite. Fubini's theorem gives that $(\mu, \nu)_{\mathcal{E}} = (\nu, \mu)_{\mathcal{E}}$. If α is a real number, then $\alpha\mu$ has finite energy.

$$\int_{R^2} |V(\mu+\nu)| \, \mathrm{d}|\mu+\nu| \le \int_{R^2} \psi_r \left[\tilde{V}|\mu+\nu| + \frac{\ln 3r}{2\pi} |\mu+\nu|(R^2) \right] \mathrm{d}|\mu+\nu|$$

$$\le \int_{R^2} |V\mu| \, \mathrm{d}|\mu| + \int_{R^2} |V\nu| \, \mathrm{d}|\nu| + 2\int_{R^2} |V\mu| \, \mathrm{d}|\nu| + \frac{\ln 3r}{2\pi} | \|\mu+\nu\|^2 < \infty.$$

Hence $\mu + \nu$ has finite energy.

Let now $\mu(R^2) = \nu(R^2) = 0$. Then $V\mu = \tilde{V}\mu$. Let $\mu = \mu^+ - \mu^-$, $\nu = \nu^+ - \nu^-$. Using (3.52) for μ^+ and ν^-, for μ^+ and ν^+, μ^- and ν^-, μ^- and ν^+, we obtain

$$\int_{R^2} \psi_r V\mu \, \mathrm{d}\nu = \int_{R^2} \nabla(\psi_r V\mu) \cdot \nabla V\nu \, \mathrm{d}\mathcal{H}_2. \tag{3.53}$$

Now we show that $\nabla(\psi_r V\mu) \to \nabla V\mu$ in $L^2(R^2)$ as $r \to \infty$. According to Lemma 3.12.2, there exist positive constants C_1, r_0 such that $|V\mu(x)| \le C_1/|x|$ for $|x| \ge r_0$. Fix C_2 such that $|\nabla\psi| \le C_2$. Then $|\nabla\psi_r| \le C_2/(2r)$. Since $V\nu \in D^{1,2}(R^2)$ by Proposition 3.41.2, $\nabla\psi_r = 0$ for $|x| \le 2r$ and for $|x| \ge 4r$, we have

$$\int_{R^2} |\nabla V\mu - \nabla(\psi_r V\mu)|^2 \, \mathrm{d}\mathcal{H}_2 \le \int_{R^2} (1-\psi_r)|\nabla V\mu|^2 \, \mathrm{d}\mathcal{H}_2 + \int_{R^2} |\nabla\psi_r|^2 |V\mu|^2 \, \mathrm{d}\mathcal{H}_2$$

$$\le \int_{R^2 \setminus B(0;2r)} |\nabla V\mu|^2 \, \mathrm{d}\mathcal{H}_2 + \int_{B(0;4r)\setminus B(0;2r)} \frac{C_1^2 C_2^2}{(2r)^4} \, \mathrm{d}\mathcal{H}_2 \to 0.$$

Thus $\psi_r V\mu \to V\mu$ in $D^{1,2}(R^2)$ and $V\mu \in \mathring{D}^{1,2}(R^2)$. (Since $\varphi_k * (\psi_r V\mu) \to \psi_r V\mu$ we have $\psi_r V\mu \in \mathring{D}^{1,2}(R^2)$.) From (3.53)

$$\int_{R^2} V\mu \, \mathrm{d}\nu = \int_{R^2} \psi_r V\mu \, \mathrm{d}\nu = \int_{R^2} \nabla(\psi_r V\mu) \cdot \nabla V\nu \, \mathrm{d}\mathcal{H}_2 \to \int_{R^2} \nabla V\mu \cdot \nabla V\nu \, \mathrm{d}\mathcal{H}_2.$$

Lemma 3.41.4 *Let μ be a compactly supported real measure with finite energy in R^2, $\mu(R^2) = 0$. If $(\mu, \mu)_{\mathcal{E}} = 0$ then $\mu = 0$.*

Proof $0 = (\mu,\mu)_{\mathcal{E}} = \|\nabla V\mu\|^2_{L^2(R^2)}$ by Theorem 3.41.3. Thus $\nabla V\mu \equiv 0$ and $V\mu$ is constant. So, $\mu = -\Delta V\mu = 0$ by Theorem 3.10.3.

Proposition 3.41.5 *Let μ be a compactly supported real measure with finite energy in R^2,* $\operatorname{spt}\mu \subset B(0;1)$. *Then* $(\mu,\mu)_{\mathcal{E}} \geq 0$. *If* $(\mu,\mu)_{\mathcal{E}} = 0$ *then* $\mu = 0$.

Proof Put $c = \mu(R^2)$, $\mathcal{H} = (2\pi)^{-1}\mathcal{H}_{m-1}|\partial B(0;1)$. Then $V\mathcal{H} = 0$ in $\overline{B(0;1)}$ by Lemma 2.5.1, $\mathcal{H}$ has finite energy, $(\mu - c\mathcal{H})(R^2) = 0$. By virtue of Fubini's theorem and Theorem 3.41.3

$$(\mu,\mu)_{\mathcal{E}} = \int V(\mu - c\mathcal{H})\,\mathrm{d}\mu = \int V\mu\,\mathrm{d}(\mu - c\mathcal{H}) = \int V(\mu - c\mathcal{H})\,\mathrm{d}(\mu - c\mathcal{H}) \geq 0.$$

If $(\mu,\mu)_{\mathcal{E}} = 0$ then $(\mu - c\mathcal{H}, \mu - c\mathcal{H})_{\mathcal{E}} = 0$ and $\mu - c\mathcal{H} = 0$ by Lemma 3.41.4. Since $\operatorname{spt}\mu \subset B(0;1)$, we deduce $\mu = 0$.

3.42 $D^{k,p}(R^m)$ and the Laplace Operator

In this section, we show that the Laplace operator maps $D^{k+2,p}(R^m)$ onto $D^{k,p}(R^m)$.

Theorem 3.42.1 *Denote* $\tilde{P}_{-1}(R^m) = \{0\}$, *for $l \in N_0$ denote by $\tilde{P}_l(R^m)$ the space of all polynomials of degree less or equal to l. Let $k \in N_0$,* $1 < p < \infty$.

- $D^{k,p}(R^m) = \tilde{W}^{k,p}(R^m) + \tilde{P}_{k-1}(R^m)$.
- *The Laplace operator Δ is a bounded linear operator from the space $D^{k+2,p}(R^m)$ onto $D^{k,p}(R^m)$.*

Proof Clearly, $\tilde{W}^{k,p}(R^m) + \tilde{P}_{k-1}(R^m) \subset D^{k,p}(R^m)$. By virtue of Proposition 1.14.7, we get $D^{0,p}(R^m) = L^p(R^m) = \tilde{W}^{0,p}(R^m)$. We have $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m) + \tilde{P}_0(R^m)$ by Proposition 3.38.4.

Suppose now that $D^{k,p}(R^m) = \tilde{W}^{k,p}(R^m) + \tilde{P}_{k-1}(R^m)$. The Laplace operator is a bounded linear operator from $D^{k+2,p}(R^m)$ into $D^{k,p}(R^m)$ by Theorem 3.17.2. Let $g \in D^{k,p}(R^m)$. Then $g = g_1 + g_2$ where $g_1 \in \tilde{W}^{k,p}(R^m)$ and $g_2 \in \tilde{P}_{k-1}(R^m)$. According to Theorem 3.17.2 and Lemma 3.17.1, there exist $f_1 \in \tilde{W}^{k+2,p}(R^m)$ and $f_2 \in \tilde{P}_{k+1}(R^m)$ such that $\Delta f_j = g_j$. So, $f_1 + f_2 \in \tilde{W}^{k+2,p}(R^m) + \tilde{P}_{k+1}(R^m) \subset D^{k+2,p}(R^m)$ and $\Delta(f_1 + f_2) = g$.

Suppose again that $D^{k,p}(R^m) = \tilde{W}^{k,p}(R^m) + \tilde{P}_{k-1}(R^m)$. Let $h \in D^{k+2,p}(R^m)$. Denote $g = \Delta h$. We have proved that there exists $f \in \tilde{W}^{k+2,p}(R^m) + \tilde{P}_{k+1}(R^m)$ such that $\Delta f = g$. We have $\Delta(h - f) = 0$. If α is a multiindex with $|\alpha| = k + 2$ then $\partial^\alpha(f - h) \in L^p(R^m)$. Since $\Delta\partial^\alpha(f - h) = \partial^\alpha\Delta(f - h) = 0$, Theorem 2.16.2 gives that $\partial^\alpha(f - h) \equiv 0$. Thus $f - h \in \tilde{P}_{k+1}(R^m)$. Hence $D^{k+2,p}(R^m) = \tilde{W}^{k+2,p}(R^m) + \tilde{P}_{k+1}(R^m)$.

We obtain the proposition of the theorem by the induction argument.

Chapter 4
PWB Solutions of the Dirichlet Problem

Abstract Chapter 4 is devoted to the Perron-Wiener-Brelot solution of the Dirichlet problem on bounded open sets.

Chapter 4 is devoted to the Perron-Wiener-Brelot solution of the Dirichlet problem on bounded open sets. Remark that u is a classical solution of the Laplace equation with the boundary condition f if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, $\Delta u = 0$ in Ω, and $u = f$ on $\partial\Omega$. Unfortunately, there exists a bounded domain $\Omega \subset R^m$ and a function $f \in \mathcal{C}^0(\partial\Omega)$ such that a classical solution of the Dirichlet problem for the Laplace equation on Ω with the boundary condition f does not exist. In this chapter, we construct a more general solution of the Dirichlet problem for some space of functions on the boundary that contains the space of continuous functions. Moreover, a classical solution of the Dirichlet problem is also a solution of the Dirichlet problem in this sense. In the next chapters, we shall study solutions of the Dirichlet problem for the Laplace equation in the sense of nontangential limit and also in the sense of traces in Sobolev and Besov spaces. We shall see that these solutions represent special classes of Perron-Wiener-Brelot solutions of the Dirichlet problem.

4.1 Upper Solutions of the Dirichlet Problem

In this section, we define the upper solution of the Dirichlet problem for the Laplace equation and the lower solution of the Dirichlet problem and study their fundamental properties.

Definition 4.1.1 Let $\omega \subset R^m$ be a domain, $m \geq 2$. An extended real-valued function u is hyperharmonic in ω if it is superharmonic in ω or $u \equiv +\infty$. If $\Omega \subset R^m$ is an open set, then an extended real-valued function u is hyperharmonic in Ω if it is a hyperharmonic in each component of Ω. An extended real-valued function u is hypoharmonic in Ω if $-u$ is hyperharmonic in Ω.

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Definition 4.1.2 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and f be an extended real-valued function on $\partial\Omega$. Denote $\mathcal{U}_f^\Omega = \{u; u$ hyperharmonic and bounded from below on Ω,

$$\liminf_{\Omega \ni y \to x} u(y) \geq f(x)$$

for all $x \in \partial\Omega\}$ the upper class of functions corresponding to f and $\mathcal{L}_f^\Omega = \{u; u$ hypoharmonic and bounded from above on Ω,

$$\limsup_{\Omega \ni y \to x} u(y) \leq f(x)$$

for all $x \in \partial\Omega\}$ the lower class of functions corresponding to f. The PWB (Perron-Wiener-Brelot) upper solution of the Dirichlet problem for the Laplace equation in Ω with the boundary condition f is

$$\overline{H}_f^\Omega = \inf\{u; u \in \mathcal{U}_f^\Omega\}.$$

The PWB lower solution of the Dirichlet problem for the Laplace equation in Ω with the boundary condition f is

$$\underline{H}_f^\Omega = \sup\{u; u \in \mathcal{L}_f^\Omega\}.$$

Lemma 4.1.3 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and f be an extended real-valued function on $\partial\Omega$. Then $\underline{H}_{(-f)}^\Omega = -\overline{H}_f^\Omega$. If ω is a component of Ω, then $\overline{H}_f^\Omega = \overline{H}_f^\omega$, and $\underline{H}_f^\omega = \underline{H}_f^\Omega$ in ω.*

Proof Clearly, $\mathcal{L}_{-f}^\Omega = \{-v; v \in \mathcal{U}_f^\Omega\}$. This gives

$$\underline{H}_f^\Omega = \sup\{-v; v \in \mathcal{U}_f^\Omega\} = -\inf\{v; v \in \mathcal{U}_f^\Omega\} = -\overline{H}_f^\Omega.$$

Since $\partial\omega \subset \partial\Omega$, the extended real-valued function f is defined on $\partial\omega$. If $u \in \mathcal{U}_f^\Omega$, then $u \in \mathcal{U}_f^\omega$, and thus $\overline{H}_f^\omega \leq \overline{H}_f^\Omega$ in ω. If $u \in \mathcal{U}_f^\omega$, define $u = +\infty$ on $\Omega \setminus \omega$. Then $u \in \mathcal{U}_f^\Omega$. Hence, $\overline{H}_f^\Omega \leq \overline{H}_f^\omega$ in ω. Therefore, $\overline{H}_f^\Omega = \overline{H}_f^\omega$ in ω. Moreover, $\underline{H}_f^\omega = -\overline{H}_{-f}^\omega = -\overline{H}_{-f}^\Omega = \underline{H}_f^\Omega$.

Proposition 4.1.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and f be an extended real-valued function on $\partial\Omega$. Then $\underline{H}_f^\Omega \leq \overline{H}_f^\Omega$.*

Proof According to Lemma 4.1.3, we can suppose that Ω is connected. Fix $u \in \mathcal{L}_f^\Omega$, $v \in \mathcal{U}_f^\Omega$. We show that $u \leq v$. It is clear if $u \equiv -\infty$ or $v \equiv +\infty$. Suppose now that

u is a subharmonic function and v is a superharmonic function. Then

$$\liminf_{\substack{y\to x\\ y\in\Omega}}(v(y)-u(y)) \geq \liminf_{\substack{y\to x\\ y\in\Omega}} v(y) - \limsup_{\substack{y\to x\\ y\in\Omega}} u(y) \geq f(x) - f(x) = 0$$

for each $x \in \partial\Omega$. Proposition 3.4.1 gives that $u \leq v$ in Ω. Thus,

$$u \leq \inf\{w; w \in \mathcal{U}_f^\Omega\} = \overline{H}_f^\Omega$$

and

$$\underline{H}_f^\Omega = \sup\{w; w \in \mathcal{L}_f^\Omega\} \leq \overline{H}_f^\Omega.$$

Lemma 4.1.5 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and f and g be extended real-valued functions on $\partial\Omega$. If c is a positive constant, then $\overline{H}_{cf}^\Omega = c\overline{H}_f^\Omega$, and $\underline{H}_{cf}^\Omega = c\underline{H}_f^\Omega$. If $f+g$ makes sense, then $\overline{H}_{f+g}^\Omega \leq \overline{H}_f^\Omega + \overline{H}_g^\Omega$ provided it makes a sense, and $\underline{H}_{f+g}^\Omega \geq \underline{H}_f^\Omega + \underline{H}_g^\Omega$ provided it makes a sense. If $f \leq g$, then $\overline{H}_f^\Omega \leq \overline{H}_g^\Omega$, $\underline{H}_f^\Omega \leq \underline{H}_g^\Omega$.*

Proof Let now $u \in \mathcal{U}_f^\Omega$ and $v \in \mathcal{U}_g^\Omega$. Then $u+v \in \mathcal{U}_{f+g}^\Omega$. Thus, $\overline{H}_{f+g}^\Omega \leq u+v$. Since $u \in \mathcal{U}_f^\Omega$ was arbitrary, we have $\overline{H}_{f+g}^\Omega \leq \overline{H}_f^\Omega + v$. Since $v \in \mathcal{U}_g^\Omega$ was arbitrary, we deduce $\overline{H}_{f+g}^\Omega \leq \overline{H}_f^\Omega + \overline{H}_g^\Omega$. Moreover, $\underline{H}_{f+g}^\Omega = -\overline{H}_{-f-g}^\Omega \geq -\overline{H}_{-f}^\Omega - \overline{H}_{-g}^\Omega = \underline{H}_f^\Omega + \underline{H}_g^\Omega$ by Lemma 4.1.3.

The rest is clear.

Theorem 4.1.6 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, ω be a component of Ω and f be an extended real-valued function on $\partial\Omega$. Then $\overline{H}_f^\Omega = -\infty$ on ω or $\overline{H}_f^\Omega = +\infty$ on ω or $\overline{H}_f^\Omega \in H(\omega)$. The same is true for $\underline{H}_f^\Omega$.*

Proof According to Lemma 4.1.3, we can suppose that Ω is connected. If $\mathcal{U}_f^\Omega = \{+\infty\}$, then $\overline{H}_f^\Omega \equiv \infty$. Suppose now that there is a superharmonic function in $\mathcal{U}_f^\Omega$. If $u, v \in \mathcal{U}_f^\Omega$, then $\min(u,v) \in \mathcal{U}_f^\Omega$ by Lemma 3.3.3. If $u, v \in \mathcal{U}_f^\Omega$ and $B = B(z;\rho)$ is a ball with $\overline{B} \subset \Omega$, denote

$$u_B(x) = \begin{cases} u(x), & x \in \Omega \setminus B, \\ \displaystyle\int_{\partial B(z;\rho)} P(x,y)u(y)\,\mathrm{d}\mathcal{H}_{m-1}(y), & x \in B, \end{cases} \tag{4.1}$$

where $P(x,y)$ is the Poisson kernel corresponding to the ball B. Then $u_B \in \mathcal{U}_f^\Omega$ by Proposition 3.6.2. Lemma 3.6.3 forces that $\overline{H}_f^\Omega \equiv -\infty$ or $\overline{H}_f^\Omega \in H(\Omega)$.

Since $\underline{H}_f^\Omega = -\overline{H}_f^\Omega$ by Lemma 4.1.5, we obtain the same proposition for $\underline{H}_f^\Omega$.

4.2 Resolutive Functions

u is a classical solution of the Laplace equation with boundary condition f if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, $\Delta u = 0$ in Ω, and $u = f$ on $\partial\Omega$. In this section, we generalize a solution of the Dirichlet problem in the following way: If $\overline{H}_f^\Omega = \underline{H}_f^\Omega$ is a harmonic function, we say that it is a solution of the Dirichlet problem with boundary condition f. We show that a classical solution of the Dirichlet problem is also a solution of the Dirichlet problem in this sense. We prove that for every continuous function on the boundary, there exists a solution of the Dirichlet problem in this generalized sense.

Definition 4.2.1 Let $\Omega \subset R^m$, $m \geq 2$, be a bounded open set. An extended real-valued function f on $\partial\Omega$ is called resolutive if $\underline{H}_f^\Omega = \overline{H}_f^\Omega$ is harmonic. In this case, $H_f^\Omega = \overline{H}_f^\Omega = \underline{H}_f^\Omega$ is called the Perron-Wiener-Brelot solution of the Dirichlet problem with the boundary condition f. We write shortly PWB-solution of the Dirichlet problem.

Proposition 4.2.2 *Let $\Omega \subset R^m$, $m \geq 2$, be a bounded open set. If $f \in \mathcal{C}^0(\partial\Omega)$ and there is a classical solution u of the Dirichlet problem for the Laplace equation on Ω with the boundary condition f, then f is resolutive and $H_f^\Omega = u$.*

Proof Clearly $u \in \mathcal{U}_f^\Omega \cap \mathcal{L}_f^\Omega$. According to Proposition 4.1.4, we obtain $u \leq \underline{H}_f^\Omega \leq \overline{H}_f^\Omega \leq u$. This gives $\overline{H}_f^\Omega = \underline{H}_f^\Omega = u \in H(\Omega)$.

Proposition 4.2.3 *Let $\Omega \subset R^m$, $m \geq 2$, be a bounded open set. The set of resolutive extended real-valued functions on $\partial\Omega$ forms a vector space. If f and g are resolutive functions on $\partial\Omega$ and c is a real number, then $H_{cf}^\Omega = cH_f^\Omega$ and $H_{f+g}^\Omega = H_f^\Omega + H_g^\Omega$. If f_j is a sequence of resolutive functions on $\partial\Omega$ and f_j converges uniformly to f on $\partial\Omega$, then f is resolutive and $H_{f_j}^\Omega$ converges uniformly to H_f^Ω.*

Proof Suppose that f and g are resolutive functions on $\partial\Omega$ and c is a real number. According to Lemmas 4.1.5 and Lemma 4.1.3, we have $H_{cf}^\Omega = cH_f^\Omega$ and $H_f^\Omega + H_g^\Omega \leq \underline{H}_{f+g}^\Omega \leq \overline{H}_{f+g}^\Omega \leq H_f^\Omega + H_g^\Omega$.

Let now f_j be a sequence of resolutive functions on $\partial\Omega$ that converges uniformly to f on $\partial\Omega$. Fix $\epsilon > 0$. Then there is $n \in N$ such that for each $j \geq n$, we have $|f - f_j| < \epsilon$. According to Lemma 4.1.5 and Proposition 4.2.2,

$$H_{f_j}^\Omega - \epsilon = H_{f_j-\epsilon}^\Omega \leq \underline{H}_f^\Omega \leq \overline{H}_f^\Omega \leq H_{f_j+\epsilon}^\Omega = H_{f_j}^\Omega + \epsilon.$$

Thus, $H_{f_j}^\Omega$ converges uniformly to $\underline{H}_f^\Omega = \overline{H}_f^\Omega$. Proposition 2.11.1 gives that $\underline{H}_f^\Omega = \overline{H}_f^\Omega$ is a harmonic function.

Proposition 4.2.4 *Let* $\Omega \subset R^m$ *be a bounded open set,* $m \geq 2$, $f_j \nearrow f$, *and* $g_j \searrow g$ *on* $\partial\Omega$. *If* $\overline{H}_{f_1}^{\Omega} > -\infty$, $\underline{H}_{g_1}^{\Omega} < \infty$ *on* Ω *then*

$$\overline{H}_f^{\Omega} = \lim_{j\to\infty} \overline{H}_{f_j}^{\Omega}, \quad \underline{H}_g^{\Omega} = \lim_{j\to\infty} \overline{H}_{g_j}^{\Omega}.$$

If, in addition,

$$\overline{H}_{f_j}^{\Omega} = \underline{H}_{f_j}^{\Omega}, \quad \overline{H}_{g_j}^{\Omega} = \underline{H}_{g_j}^{\Omega} \tag{4.2}$$

then $\overline{H}_f^{\Omega} = \underline{H}_f^{\Omega}$, $\overline{H}_g^{\Omega} = \underline{H}_g^{\Omega}$.

Proof According to Lemma 4.1.3, we can suppose that Ω is connected. Since

$$h := \lim_{j\to\infty} \overline{H}_{f_j}^{\Omega} \leq \overline{H}_f^{\Omega}$$

by Lemma 4.1.5, we can suppose that there is $x_0 \in \Omega$ such that $h(x_0) < \infty$. Since $-\infty < \overline{H}_{f_1}^{\Omega}(x_0) \leq \overline{H}_{f_j}^{\Omega}(x_0) \leq h(x_0) < \infty$, Theorem 4.1.6 gives that $\overline{H}_{f_j}^{\Omega} \in H(\Omega)$ for each j. Since h is a limit of an increasing sequence of harmonic functions, we have $h \in H(\Omega)$ by Proposition 2.11.3.

Fix $\epsilon > 0$, $z \in \Omega$. For each $j \in N$, choose $v_j \in \mathcal{U}_{f_j}^{\Omega}$ such that

$$v_j(z) < \overline{H}_{f_j}^{\Omega}(z) + \epsilon 2^{-j}$$

and define

$$v = h + \sum_{j=1}^{\infty}(v_j - \overline{H}_{f_j}^{\Omega}), \quad w_k = h + \sum_{j=1}^{k}(v_j - \overline{H}_{f_j}^{\Omega}).$$

Then w_k is an increasing sequence of superharmonic functions because $v_j \geq \overline{H}_{f_j}^{\Omega}$. Since $w_k \nearrow v$, Proposition 3.25.1 gives that $v \equiv \infty$ or v is a superharmonic function in Ω. Since

$$v(z) \leq h(z) + \sum_{j=1}^{\infty} \epsilon 2^{-j} = h(z) + \epsilon < \infty$$

we infer that v is a superharmonic function in Ω. Since $h \geq \overline{H}_{f_j}^{\Omega}$ and $v_j \geq \overline{H}_{f_j}^{\Omega}$, we obtain

$$v \geq w_k \geq h + (v_k - \overline{H}_{f_k}^{\Omega}) \geq v_k.$$

If $x \in \partial\Omega$, then

$$\liminf_{\Omega \ni y \to x} v(y) \geq \liminf_{\Omega \ni y \to x} v_k(y) \geq f_k(x).$$

Hence,

$$f(x) = \lim_{k \to \infty} f_k(x) \leq \liminf_{\Omega \ni y \to x} v(y)$$

and $v \in \mathcal{U}_f^\Omega$. This forces

$$h(z) \leq \overline{H}_f^\Omega(z) \leq v(z) \leq h(z) + \epsilon.$$

Since ϵ is arbitrary, we obtain $h(z) = \overline{H}_f^\Omega(z)$.

Suppose now (4.2). According to Lemma 4.1.5, we have

$$\underline{H}_f^\Omega \geq \underline{H}_{f_j}^\Omega = \overline{H}_{f_j}^\Omega.$$

Therefore,

$$\underline{H}_f^\Omega \geq \lim_{j \to \infty} \overline{H}_{f_j}^\Omega = h = \overline{H}_f^\Omega.$$

Since $\underline{H}_f^\Omega \leq \overline{H}_f^\Omega$ by Proposition 4.1.4, we deduce $\underline{H}_f^\Omega = \overline{H}_f^\Omega$.

The rest is a consequence of Lemma 4.1.3.

Lemma 4.2.5 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$, and u be a superharmonic function in Ω. If there is a finite limit*

$$f(x) = \lim_{\Omega \ni y \to x} u(x)$$

at each $x \in \partial\Omega$, then f is a resolutive function.

Proof Put $v(x) = u(x)$ for $x \in \Omega$ and $v(x) = f(x)$ for $x \in \partial\Omega$. Then v is a lower semicontinuous function on $\overline{\Omega}$. Since $\overline{\Omega}$ is a compact set and $v(x) > -\infty$, Lemma 3.2.2 gives that there is a real constant c such that $v \geq c$. Since $u \in \mathcal{U}_f^\Omega$, Theorem 4.1.6 forces that $\overline{H}_f^\Omega \in H(\Omega)$ or $\overline{H}_f^\Omega \equiv -\infty$. Since $f \geq c$, we have $\overline{H}_f^\Omega \geq \overline{H}_c^\Omega = c$ by Lemma 4.1.5 and Proposition 4.2.2. Therefore, $\overline{H}_f^\Omega \in H(\Omega)$. If $x \in \partial\Omega$, then

$$\limsup_{\Omega \ni y \to x} \overline{H}_f(y) \leq \limsup_{\Omega \ni y \to x} u(y) = f(x).$$

Hence, $\overline{H}_f^\Omega \in \mathcal{L}_f^\Omega$ and thus $\overline{H}_f^\Omega \le \underline{H}_f^\Omega$. Since $\underline{H}_f^\Omega \le \overline{H}_f^\Omega$ by Proposition 4.1.4, we infer $\underline{H}_f^\Omega = \overline{H}_f^\Omega$.

Lemma 4.2.6 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$, and $K \subset \Omega$ be a compact set. If $f \in \mathcal{C}^0(K)$ and $\epsilon > 0$, then there are continuous superharmonic functions u, v on Ω such that $|f - (u - v)| < \epsilon$ on K.*

Proof According to the Stone-Weierstrass theorem (Theorem 1.16.2), there exists a polynomial P such that $|P - f| < \epsilon$ on K. Since $\overline{\Omega}$ is a compact set, there is a constant M such that $|\Delta P| \le M$ on $\overline{\Omega}$ (see Lemma 1.2.13). Put $v(x) = -M|x|^2/(2m)$ and $u(x) = P(x) + v(x)$. Then $\Delta v = -M \le 0$ and $\Delta u(x) = \Delta P(x) - M \le 0$. So, u and v are superharmonic functions in Ω by Theorem 3.19.4 and $|f - (u - v)| = |f - P| < \epsilon$ on K.

Theorem 4.2.7 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$. If $f \in \mathcal{C}^0(\partial\Omega)$, then f is resolutive.*

Proof Choose $r > 0$ such that $\overline{\Omega} \subset B(0; r)$. For each $n \in N$, there are continuous superharmonic functions u_n and v_n in $B(0; r)$ such that we have $|f - (u_n - v_n)| < 1/n$ on $\partial\Omega$ (see Lemma 4.2.6). Since u_n and v_n are resolutive by Lemma 4.2.5, the function $u_n - v_n$ is resolutive by Proposition 4.2.3. Since the resolutive functions $u_n - v_n$ converge uniformly to f on $\partial\Omega$, Proposition 4.2.3 forces that f is resolutive.

Proposition 4.2.8 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$. Suppose that f and g are extended real-valued functions on $\partial\Omega$ such that $f = g$ on $\partial\Omega$ except possibly on a polar subset of $\partial\Omega$. Then $\overline{H}_f^\Omega = \overline{H}_g^\Omega$ and $\underline{H}_f^\Omega = \underline{H}_g^\Omega$. If, in addition, f is resolutive, then g is also resolutive and $H_f^\Omega = H_g^\Omega$ in Ω.*

Proof Let $z \in \Omega$ and $\overline{H}_f^\Omega(z) < \beta$. Choose $u \in \mathcal{U}_f^\Omega$ such that $u(z) < \beta$. Put $A = \{x \in \partial\Omega; f(x) \ne g(x)\}$. According to Theorem 3.26.3, there exists a superharmonic function v in R^m such that $v = \infty$ on A and $v(z) \in R$. Since v is bounded from below on $\overline{\Omega}$ (see Lemma 3.2.2), there exists a constant C such that $v + C \ge 0$ on $\overline{\Omega}$. Choose $\epsilon > 0$ such that $u(z) + \epsilon(v(z) + C) < \beta$. Then $u + \epsilon(v + C) \in \mathcal{U}_g^\Omega$ and thus $\overline{H}_g^\Omega(z) < \beta$. We have proved $\overline{H}_g^\Omega \le \overline{H}_f^\Omega$. Similarly, $\overline{H}_f^\Omega \le \overline{H}_g^\Omega$. Lemma 4.1.3 gives $\underline{H}_g^\Omega = -\overline{H}_{-g}^\Omega = -\overline{H}_{-f}^\Omega = \underline{H}_f^\Omega$.

Theorem 4.2.9 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$. Suppose that f is an extended real-valued function on $\partial\Omega$. Let F be defined on $\overline{\Omega}$ by $F = f$ on $\partial\Omega$ and $F = \overline{H}_f^\Omega$ on Ω. If ω is an open subset of Ω, then $\overline{H}_F^\omega = F = \overline{H}_f^\Omega$ in ω. If f is resolutive, then F is resolutive (with respect to ω) and $H_F^\omega = H_f^\Omega$ in ω.*

Proof We can suppose that Ω and ω are connected (see Lemma 4.1.3). Theorem 4.1.6 gives that $\overline{H}_F^\Omega \equiv -\infty$ or $\overline{H}_F^\Omega \equiv \infty$ or $\overline{H}_F^\Omega \in H(\Omega)$. If $u \in \mathcal{U}_f^\Omega$, then clearly $u \in \mathcal{U}_F^\omega$. Hence, $\overline{H}_F^\omega \le \overline{H}_f^\Omega$ on ω. If $\overline{H}_f^\Omega \equiv -\infty$, then $\overline{H}_F^\omega \equiv -\infty$. If $\overline{H}_f^\Omega \equiv \infty$ then $\mathcal{U}_F^\omega = \{F\}$ and thus $\overline{H}_F^\omega = F = \infty$ in ω. Let now $F = \overline{H}_f^\Omega \in H(\Omega)$. Let $u \in \mathcal{U}_F^\omega$ be superharmonic. Define

$$\tilde{u} = \begin{cases} \min(u, F), & \text{on } \omega, \\ F, & \text{on } \Omega \setminus \omega \end{cases}$$

Then $\tilde{u}$ is a superharmonic function in Ω by Corollary 3.3.10. Let $v \in \mathcal{U}_f^\Omega$ be superharmonic. Define $w = \tilde{u} + v - \overline{H}_f^\Omega$. Then w is superharmonic by Lemma 3.3.3. If $x \in \Omega \setminus \omega$ or $u(x) \geq \overline{H}_f^\Omega(x)$, then $w(x) = v(x)$. If $x \in \omega$ and $u(x) < \overline{H}_f^\Omega(x)$, then $w(x) \geq u(x)$. So, w is bounded below. If $z \in \partial\Omega \setminus \partial\omega$, then

$$\liminf_{(\Omega\setminus\omega)\ni x\to z} w(x) = \liminf_{(\Omega\setminus\omega)\ni x\to z} v(x) \geq f(z).$$

If $z \in \partial\Omega \cap \partial\omega$, then

$$\liminf_{(\Omega\setminus\omega)\ni x\to z} w(x) \geq \min(\liminf_{\Omega\ni x\to z} v(x), \liminf_{\omega\ni x\to z} u(x)) \geq \min(f(z), F(z)) = f(z).$$

It follows that $w \in \mathcal{U}_f^\Omega$. So, $\tilde{u} + v = w + \overline{H}_f^\Omega \geq 2\overline{H}_f^\Omega$ on Ω. Since this holds for all $v \in \mathcal{U}_f^\Omega$, we see that $\tilde{u} \geq \overline{H}_f^\Omega$ on Ω. In particular, $u \geq \overline{H}_f^\Omega$ on ω. Since u is an arbitrary element of $\mathcal{U}_F^\omega$, we have $\overline{H}_F^\omega \geq \overline{H}_f^\Omega$ on ω, as required.

Let now f be resolutive. Then $-f$ is resolutive by Proposition 4.2.3. By virtue of Lemma 4.1.3,

$$\overline{H}_F^\omega = H_f^\Omega = -\overline{H}_{-f}^\Omega = -\overline{H}_{-F}^\omega = \underline{H}_F^\omega$$

and F is resolutive (with respect to ω).

4.3 Approximation of Domains

Theorem 4.3.1 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $f \in C^0(\overline{\Omega})$. If Ω_n is an increasing sequence of open sets in R^m such that $\Omega = \cup\{\Omega_n; n \in N\}$ then*

$$H_f^\Omega(x) = \lim_{n\to\infty} H_f^{\Omega_n}(x), \quad x \in \Omega.$$

Proof Fix $x \in \Omega$, $\epsilon > 0$. For $u \in \mathcal{U}_f^\Omega$, put $v_u(y) = u(y) - f(y)$ on Ω and

$$v_u(y) = \liminf_{\Omega\ni z\to y} v_u(z)$$

for $y \in \partial\Omega$. Since v_u is lower semicontinuous on $\overline{\Omega}$ and $v_u \geq 0$ on $\partial\Omega$, the set $K_u = \{y \in \overline{\Omega}; v_u(y) \leq -\epsilon\}$ is a closed subset of Ω. The set $\{x\} \cup K_u$ is compact, so there are $n(1), \dots, n(k) \in N$ such that

$$\{x\} \cup K_u \subset \bigcup_{j=1}^{k} \Omega_{n(j)}.$$

If $n \geq n_0 = \max(n(1), \dots, n(k))$, then $\{x\} \cup K_u \subset \Omega_n$. This means that

$$\liminf_{\Omega_n \ni z \to y} u(z) \geq f(y) - \epsilon$$

for each $y \in \partial\Omega_n$ and $u + \epsilon \in \mathcal{U}_f^{\Omega_n}$. Hence, $u(x) + \epsilon \geq \overline{H}_f^{\Omega_n}(x)$. Since f is resolutive in Ω and in Ω_n by Theorem 4.2.7 and since $u \in \mathcal{U}_f^{\Omega}$ is arbitrary, we have

$$H_f^{\Omega}(x) \geq H_f^{\Omega_n}(x), \qquad H_f^{\Omega}(x) \geq \limsup_{n\to\infty} H_f^{\Omega_n}(x).$$

According to Lemma 4.1.3,

$$H_f^{\Omega}(x) = -H_{-f}^{\Omega}(x) \leq -\limsup_{n\to\infty} H_{-f}^{\Omega_n}(x) = \liminf_{n\to\infty} H_f^{\Omega_n}(x).$$

4.4 Harmonic Measures

In this section, we prove that for each $x \in \Omega$, there exists a probabilistic measure μ_x^{Ω} supported on $\partial\Omega$ (so-called harmonic measure) that f is resolutive if and only if $f \in L^1(\mu_x^{\Omega})$ and $u(x) = \int f \, \mathrm{d}\mu_x^{\Omega}$ gives the solution of the Dirichlet problem with the boundary condition f.

Theorem 4.4.1 *Let $\Omega \subset R^m$, $m \geq 2$, be a bounded open set. If $x \in \Omega$, then there exists unique Radon measure μ_x^{Ω} on $\partial\Omega$ such that*

$$H_f^{\Omega}(x) = \int_{\partial\Omega} f \, \mathrm{d}\mu_x^{\Omega} \tag{4.3}$$

for each $f \in \mathcal{C}^0(\partial\Omega)$. This measure satisfies $\mu_x^{\Omega}(\partial\Omega) = 1$.

Proof Fix $x \in \Omega$. Then $f \mapsto H_f^{\Omega}(x)$ represents a linear functional on $\mathcal{C}^0(\partial\Omega)$ by Theorem 4.2.7 and Proposition 4.2.3. If $f \geq 0$, then $H_f^{\Omega}(x) \geq 0$ by Proposition 4.2.2 and Lemma 4.1.5. According to the Riesz representation theorem (Theorem 1.13.8), there is unique Radon measure μ_x^{Ω} on $\partial\Omega$ such that (4.3) holds. By virtue of Proposition 4.2.2, we have $\mu_x^{\Omega}(\partial\Omega) = H_1^{\Omega}(x) = 1$.

Definition 4.4.2 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $x \in \Omega$. The Radon measure μ_x^{Ω} from Theorem 4.4.1 is called harmonic measure relative to Ω and x.

Corollary 4.4.3 *Let $\Omega \subset R^m$ be an open set, $m \geq 2$. If $f \in \mathcal{C}^0(\partial\Omega)$, then f is resolutive and*

$$\min_{y\in\partial\Omega} f(y) \leq H_f^\Omega(x) \leq \max_{y\in\partial\Omega} f(y), \qquad x \in \Omega. \tag{4.4}$$

Proof f is resolutive by Theorem 4.2.7. Fix $x \in \Omega$. Let μ_x^Ω be the harmonic measure relative to Ω and x. Then $H_f^\Omega(x)$ is given by (4.3). Since $\mu_x^\Omega \geq 0$ and $\mu_x^\Omega(\partial\Omega) = 1$, we obtain (4.4).

Lemma 4.4.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $f : \partial\Omega \to (-\infty, \infty\rangle$ be a lower semicontinuous extended real-valued function. If $x \in \Omega$, then*

$$\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) = \int_{\partial\Omega} f \, \mathrm{d}\mu_x^\Omega.$$

Proof The function f is bounded from below by Lemma 3.2.2. According to Lemma 3.2.4, there exists an increasing sequence f_k of continuous functions on $\partial\Omega$ such that $f_k \nearrow f$. The functions f_k are resolutive by Theorem 4.2.7. Proposition 4.2.4, Theorem 4.4.1, and Corollary 1.13.4 give

$$\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) = \lim_{k\to\infty} H_{f_k}^\Omega(x) = \lim_{k\to\infty} \int_{\partial\Omega} f_k \, \mathrm{d}\mu_x^\Omega = \int_{\partial\Omega} f \, \mathrm{d}\mu_x^\Omega.$$

Theorem 4.4.5 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $f : \partial\Omega \to \langle -\infty, \infty\rangle$ be an extended real-valued function, $x \in \Omega$.*

- *If $f \in L^1(\mu_x^\Omega)$, then*

$$\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) = \int_{\partial\Omega} f \, \mathrm{d}\mu_x^\Omega \in R^1. \tag{4.5}$$

- *If $\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) \in R^1$, then $f \in L^1(\mu_x^\Omega)$.*

Proof Suppose first that $f = \chi_M$ is the characteristic function of $M \subset \partial\Omega$. If M is closed or open in the metric space $\partial\Omega$, then $-f$ or f is a lower semicontinuous function and Lemma 4.4.4 and Lemma 4.1.3 give (4.5). Let now M be a μ_x^Ω-measurable set. Fix $\epsilon > 0$. Since μ_x^Ω is a Radon measure, there are an open set G and a closed set K such that $K \subset M \subset G$ and $\mu_x^\Omega(G) - \epsilon \leq \mu_x^\Omega(M) \leq \mu_x^\Omega(K) + \epsilon$. According to Lemma 4.1.5, we obtain

$$\overline{H}_f^\Omega(x) \leq \overline{H}_{\chi_G}^\Omega(x) = \mu_x^\Omega(G) \leq \mu_x^\Omega(M) + \epsilon,$$

$$\underline{H}_f^\Omega(x) \geq \underline{H}_{\chi_K}^\Omega(x) = \mu_x^\Omega(K) \geq \mu_x^\Omega(M) - \epsilon.$$

Since ϵ is arbitrary, we infer $\overline{H}_f^\Omega(x) \le \mu_x^\Omega(x) \le \underline{H}_f^\Omega(x)$. Since $\underline{H}_f^\Omega(x) \le \overline{H}_f^\Omega(x)$ by Lemma 4.1.4, we obtain (4.5).

If f is simple (i.e., a linear combination of characteristic functions of μ_x^Ω-measurable sets), then we obtain (4.5) using Lemma 4.1.3 and Lemma 4.1.5.

If $f \in L^1(\mu_x^\Omega)$ and $f \ge 0$, then there is an increasing sequence of simple functions f_j such that $f_j \nearrow f$ as $j \to \infty$. According to Proposition 4.2.4 and Theorem 1.13.3,

$$\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) = \lim_{j\to\infty} \overline{H}_{f_j}^\Omega(x) = \lim_{j\to\infty} \int\limits_{\partial\Omega} f_j \, \mathrm{d}\mu_x^\Omega = \int\limits_{\partial\Omega} f \, \mathrm{d}\mu_x^\Omega .$$

If $f \in L^1(\mu_x^\Omega)$, put $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then $f^+, f^- \in L^1(\mu_x^\Omega)$. According to Lemma 4.1.5 and Lemma 4.1.3,

$$\begin{aligned}\overline{H}_f^\Omega(x) &\le \overline{H}_{f^+}^\Omega(x) + \overline{H}_{(-f^-)}^\Omega(x) = \int\limits_{\partial\Omega} f^+ \, \mathrm{d}\mu_x^\Omega - \int\limits_{\partial\Omega} f^- \, \mathrm{d}\mu_x^\Omega \\ &= \underline{H}_{f^+}^\Omega(x) + \underline{H}_{(-f^-)}^\Omega(x) \le \underline{H}_f^\Omega(x).\end{aligned}$$

Since $\underline{H}_{f^+}^\Omega(x) \le \overline{H}_{f^+}^\Omega(x)$ by Proposition 4.1.4, we obtain (4.5).

Suppose now that $\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) \in R^1$. Fix a sequence $u_n \in \mathcal{U}_f^\Omega$ such that $u_n(x) \le \overline{H}_f^\Omega(x) + 1/n$. Since $\min(u, v) \in \mathcal{U}_f^\Omega$ for each $u, v \in \mathcal{U}_f^\Omega$ (see Lemma 3.3.3), we can suppose that u_n is a decreasing sequence. Similarly, we can find an increasing sequence $v_n \in \mathcal{L}_f^\Omega$ such that $v_n(x) \ge \underline{H}_f^\Omega(x) - 1/n$. Define for $n \in N$ and $z \in \partial\Omega$

$$h_n(z) = \liminf_{\Omega\ni y\to z} u_n(y), \quad w_n(z) = \limsup_{\Omega\ni y\to z} v_n(y).$$

Then h_n is a decreasing sequence of lower semicontinuous functions on $\partial\Omega$, w_n is an increasing sequence of upper semicontinuous functions on $\partial\Omega$, and $w_n \le f \le h_n$. According to Lemma 4.4.4 and Lemma 4.1.3,

$$\begin{aligned}\int_{\partial\Omega} w_n \, \mathrm{d}\mu_x^\Omega &\le \int_{\partial\Omega} h_n \, \mathrm{d}\mu_x^\Omega = \overline{H}_{h_n}^\Omega(x) \le u_n(x) \le \overline{H}_f^\Omega(x) + \frac{1}{n}, \\ \int_{\partial\Omega} w_n \, \mathrm{d}\mu_x^\Omega &= \underline{H}_{w_n}^\Omega(x) \ge v_n(x) \ge \underline{H}_f^\Omega(x) - \frac{1}{n} = \overline{H}_f^\Omega(x) - \frac{1}{n}.\end{aligned}$$

Thus, $h_n, w_n \in L^1(\mu_x^\Omega)$ and

$$\int |h_n - w_n| \, \mathrm{d}\mu_x^\Omega = \int (h_n - w_n) \, \mathrm{d}\mu_x^\Omega \le 2/n.$$

Put

$$h(z) = \lim_{n\to\infty} h_n(z), \quad w(z) = \lim_{n\to\infty} w_n(z).$$

Then h and w are Borel measurable functions and $w \leq f \leq h$. Since $w_1 \leq w \leq h \leq h_1$ and $h_1, w_1 \in L^1(\mu_x^\Omega)$, we infer that $w, h \in L^1(\mu_x^\Omega)$. Corollary 1.13.4 gives

$$\int_{\partial\Omega} (h - w)\, \mathrm{d}\mu_x^\Omega = \lim_{n\to\infty} \int_{\partial\Omega} (h_n - w_n)\, \mathrm{d}\mu_x^\Omega = 0.$$

$w \leq h$ forces $\mu_x^\Omega(\{z; h(z) \neq w(z)\}) = 0$. Since $w \leq f \leq h$ and $h \in L^1(\mu_x^\Omega)$, we deduce $w \in L^1(\mu_x^\Omega)$.

Corollary 4.4.6 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $f : \partial\Omega \to \langle -\infty, +\infty\rangle$ be a Borel measurable extended real-valued function. If $-\infty < \underline{H}_f^\Omega \leq \overline{H}_f^\Omega < \infty$ in Ω, then f is resolutive and*

$$H_f^\Omega(x) = \int_{\partial\Omega} f\, \mathrm{d}\mu_x^\Omega \qquad \forall x \in \Omega.$$

Proof Put $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, and $f_n^+ = \min(f^+, n)$ for $n \in N$. Then f_n^+ is Borel measurable. If $x \in \Omega$, then $f_n^+ \in L^1(\mu_x^\Omega)$, because $0 \leq f_n^+ \leq n$ and $0, n \in L^1(\mu_x^\Omega)$. Since $f_n^+ \geq 0$, we obtain by Theorem 4.4.5

$$0 \leq \underline{H}_{f_n^+}^\Omega = \overline{H}_{f_n^+}^\Omega.$$

Since $f_n^+ \nearrow f^+$, Proposition 4.2.4 gives that

$$0 \leq \underline{H}_{f^+}^\Omega = \overline{H}_{f^+}^\Omega.$$

Fix $x \in \Omega$. Since $\overline{H}_f^\Omega(x) < \infty$, there is $u \in \mathcal{U}_f^\Omega$ such that $u(x) < \infty$. Since u is bounded from below, there is a constant $c \geq 0$ such that $u + c \geq 0$ in Ω. Since $u + c \in \mathcal{U}_{f^+}^\Omega$, we have

$$\overline{H}_{f^+}^\Omega(x) \leq u(x) + c < \infty.$$

This forces that f^+ is resolutive (compare Theorem 4.1.6). Similarly, f^- is resolutive. Proposition 4.2.3 gives that $f = f^+ - f^-$ is resolutive. The rest is a consequence of Theorem 4.4.5.

Corollary 4.4.7 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$, and $f : \partial\Omega \to \langle -\infty, +\infty \rangle$ be an extended real-valued function. Then the following are equivalent:*

1. *f is resolutive.*
2. *$f \in L^1(\mu_x^\Omega)$ for all $x \in \Omega$.*
3. *$f \in L^1(\mu_x^\Omega)$ for some $x \in \Omega$.*

Proof 1) $\Rightarrow$ 2): This implication follows from Theorem 4.4.5.

2 $\Rightarrow$ 3): This implication is trivial.

3) $\Rightarrow$ 1): Theorem 4.4.5 gives that $\underline{H}_f^\Omega(x) = \overline{H}_f^\Omega(x) \in R$. Thus, $\underline{H}_f^\Omega \in H(\Omega)$ and $\overline{H}_f^\Omega \in H(\Omega)$ by Theorem 4.1.6. Since $\underline{H}_f^\Omega \leq \overline{H}_f^\Omega$ (see Proposition 4.1.4), $\overline{H}_f^\Omega - \underline{H}_f^\Omega$ is a nonnegative harmonic function in Ω. Since $\overline{H}_f^\Omega(x) - \underline{H}_f^\Omega(x) = 0$, Theorem 2.4.2 gives that $\overline{H}_f^\Omega - \underline{H}_f^\Omega = 0$ in Ω.

Proposition 4.4.8 *Let $\omega \subset \Omega$ be bounded open subsets of R^m, $m \geq 2$, and $x \in \omega$. If $B \subset \partial\omega \cap \partial\Omega$ is a Borel set, then $\mu_x^\omega(B) \leq \mu_x^\Omega(B)$.*

Proof Put $f = \chi_B$. Fix $u \in \mathcal{U}_f^\Omega$. Since $0 \in \mathcal{L}_f^\Omega$, Proposition 4.1.4 gives that $0 \leq u$. Thus, $u \in \mathcal{U}_f^\omega$ and $\overline{H}_f^\omega(x) \leq \overline{H}_f^\Omega(x)$. Theorem 4.4.5 gives $\mu_x^\omega(B) = \overline{H}_f^\omega(x) \leq \overline{H}_f^\Omega(x) = \mu_x^\Omega(B)$.

Proposition 4.4.9 *Let ω be a component of a bounded open set $\Omega \subset R^m$, $m \geq 2$. If $x \in \omega$, then $\mu_x^\Omega(\partial\Omega \setminus \partial\omega) = 0$ and $\mu_x^\Omega = \mu_x^\omega$.*

Proof Let $B \subset \partial\omega$ be a Borel set. Then $\mu_x^\omega(B) \leq \mu_x^\Omega(B)$ by Proposition 4.4.8. On the other hand, according to Theorem 4.4.1 and Proposition 4.4.8, we have

$$\mu_x^\Omega(B) = 1 - \mu_x^\Omega(\partial\Omega \setminus B) \leq 1 - \mu_x^\Omega(\partial\omega \setminus B) \leq 1 - \mu_x^\omega(\partial\omega \setminus B) = \mu_x^\omega(B).$$

Thus, $\mu_x^\Omega(B) = \mu_x^\omega(B)$ and $\mu_x^\Omega(\partial\Omega \setminus \partial\omega) = 1 - \mu_x^\Omega(\partial\omega) = 1 - \mu_x^\omega(\partial\omega) = 0$.

Theorem 4.4.10 *Let ω be a component of a bounded open set $\Omega \subset R^m$, $m \geq 2$, and $B \subset \partial\Omega$ be a Borel set. If $x, y \in \omega$, then $\mu_x^\Omega(B) = 0$ if and only if $\mu_y^\Omega(B) = 0$. Moreover, there exists a nonnegative bounded measurable function g on $\partial\Omega$ such that $\mu_y^\Omega = g\mu_x^\Omega$.*

Proof Let $f = \chi_B$ be a characteristic function of B. Then f is resolutive by Corollary 4.4.6. Since $0 \in \mathcal{L}_f^\Omega$, the function H_f^Ω is nonnegative. If $H_f^\Omega(x) = \mu_x^\Omega(B) = 0$ (compare Corollary 4.4.6), then $H_f^\Omega \equiv 0$ on ω by Theorem 2.4.2. According to Corollary 4.4.6, we have $0 = H_f^\Omega(y) = \mu_y^\Omega(B)$. According to Theorem 1.13.10, there exists $g \in L^1(\mu_x^\Omega)$ such that $\mu_y^\Omega = g\mu_x^\Omega$. Since $\mu_y^\Omega(\partial\Omega \setminus \partial\omega) = 0 = \mu_x^\Omega(\partial\Omega \setminus \partial\omega)$ by Proposition 4.4.9, we can suppose that $g = 0$ on $\partial\Omega \setminus \partial\omega$. Since $\mu_y^\Omega \geq 0$ we deduce that $g \geq 0$. According to Theorem 2.8.2 there exists a positive constant C_1 such that $h(y) \leq C_1 h(x)$ for each nonnegative harmonic function h on ω. Fix $C_2 > C_1$, and set $M = \{x \in \partial\omega; g(x) > C_2\}$. Then $f = \chi_M$

is a resolutive function by Corollary 4.4.7. Put $h = H_f^\Omega$. Since h is a nonnegative harmonic function, we have

$$C_1\mu_x^\Omega(M) = C_1 h(x) \geq h(y) = \mu_y^\Omega(M) = \int\limits_M g \,\mathrm{d}\mu_x^\Omega \geq \int\limits_M C_2 \,\mathrm{d}\mu_x^\Omega = C_2\mu_x^\Omega(M).$$

This gives that $\mu_x^\Omega(M) = 0$.

4.5 Regular Sets

In this section, we characterize regular boundary points. In particular, we obtain that for Ω with Lipschitz boundary and f continuous on $\partial\Omega$, there exists a classical solution of the Dirichlet problem with the boundary condition f.

Definition 4.5.1 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. A point $y \in \partial\Omega$ is called regular (for Ω) if

$$\lim_{\Omega \ni x \to y} H_f^\Omega(x) = f(y)$$

for each $f \in \mathcal{C}^0(\partial\Omega)$. Otherwise, y is called irregular. We say that the set Ω is regular if every point of $\partial\Omega$ is regular.

Definition 4.5.2 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. A function u is called a barrier (for Ω) at $y \in \partial\Omega$ if u is positive and superharmonic on $\Omega \cap U$ for some open neighborhood U of y and

$$\lim_{\Omega \ni x \to y} u(x) = 0. \tag{4.6}$$

Lemma 4.5.3 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $y \in \partial\Omega$. If there is a barrier for Ω at y, then there is $u \in H(\Omega)$ such that $|x-y|^2 \leq u(x) \leq (\operatorname{diam}\Omega)^2$ and (4.6) hold.*

Proof Put $f(x) = |x-y|^2$. Since f is resolutive by Theorem 4.2.7, the function $u(x) := H_f^\Omega(x)$ is harmonic in Ω. Since $\Delta f = 2m > 0$, the function f is subharmonic in Ω by Theorem 3.19.4. Since $f \in \mathcal{L}_f^\Omega$, we have $u(x) = H_f^\Omega(x) \geq f(x) = |x-y|^2$. Since $M := (\operatorname{diam} G)^2 \in \mathcal{U}_f^\Omega$, we have $u = H_f^\Omega \leq M$.

Fix $\epsilon > 0$. We shall prove that there is $r > 0$ such that $u(x) \leq \epsilon$ on $\Omega \cap B(y;r)$. Let v be a barrier for Ω at y and $r_0 \in (0, \sqrt{\epsilon/3})$ be such that v is a positive superharmonic function on $\Omega \cap B(y;r_0)$. Remark that $f(x) \leq \epsilon/3$ on $\partial\Omega \cap \overline{B(y;r_0)}$. Since $y \in \partial\Omega$, we can choose $r_1 \in (0, r_0)$ such that $\Omega \cap \partial B(y, r_1) \neq \emptyset$. Put $U = \Omega \cap B(y;r_1)$. Choose a compact set $F \subset \Omega \cap \partial B(y;r_1)$ such that $\mathcal{H}_{m-1}(\partial U \setminus \partial\Omega \setminus F) < \epsilon/(3M)$. Put $g(x) = M$ for $x \in (\partial U \setminus \partial\Omega \setminus F)$ and $g(x) = 0$ elsewhere. Denote by Pg

the Poisson integral corresponding to the ball $B(y; r_1)$ and g. Since $0 \leq Pg, Pg(y) < \epsilon/3$, and $Pg \in H(B(y; r_1))$ (see Theorem 2.5.4), there is $\rho \in (0; r_1)$ such that $Pg \leq \epsilon/3$ on $B(y; \rho)$. Since v is a positive superharmonic function on $\Omega \cap B(y; r_0)$ and F is a compact subset of $\Omega \cap B(y; r_0)$, there is a positive constant c such that $v \geq c$ on F (see Lemma 3.2.2). Since $v(x) \to 0$ as $x \to y$, there is $r \in (0, \rho)$ such that $Mv(x)/c \leq \epsilon/3$ on $\Omega \cap B(y; r)$. The function $w(x) = Mv(x)/c + Pg(x) + \epsilon/3$ is superharmonic in U and $w(x) \leq \epsilon$ in $\Omega \cap B(y; r)$. Fix $\varphi \in \mathcal{L}_f^\Omega$. If $z \in \partial U \cap \partial\Omega$, then

$$\liminf_{U \ni x \to z} [w(x) - \varphi(x)] \geq \epsilon/3 - \limsup_{\Omega \ni x \to z} \varphi(x) \geq \epsilon/3 - f(z) \geq 0.$$

If $z \in F$, then

$$\liminf_{U \ni x \to z} [w(x) - \varphi(x)] \geq \liminf_{U \ni x \to z} [Mv(x)/c - \varphi(x)]$$
$$\geq \frac{M}{c} c - \limsup_{\Omega \ni x \to z} \varphi(x) \geq M - \limsup_{\Omega \ni x \to z} H_f^\Omega(x) \geq M - M \geq 0.$$

If $z \in \partial U \setminus \partial\Omega \setminus F$, then

$$\liminf_{U \ni x \to z} [w(x) - \varphi(x)] \geq \liminf_{U \ni x \to z} [Pg(x) - \varphi(x)]$$
$$= M - \limsup_{\Omega \ni x \to z} \varphi(x) \geq M - \limsup_{\Omega \ni x \to z} H_f^\Omega(x) \geq M - M \geq 0$$

by Theorem on the Poisson integral (Theorem 2.5.4). According to Proposition 3.4.1, we obtain $\varphi \leq w$ in U. Since $w \leq \epsilon$ on $B(y, r) \cap \Omega$, we infer that $\varphi \leq \epsilon$ on $B(y, r) \cap \Omega$. Since $\varphi \in \mathcal{L}_f^\Omega$ was arbitrary, we deduce that $u = H_f^\Omega \leq \epsilon$ on $B(y, r) \cap \Omega$.

Proposition 4.5.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $y \in \partial\Omega$. Suppose that f and g are extended real-valued functions on $\partial\Omega$, f is bounded from above, and g is bounded from below. If there is a barrier for Ω at y, then*

$$\limsup_{\Omega \ni x \to y} \overline{H}_f^\Omega(x) \leq \limsup_{\partial\Omega \ni x \to y} f(x), \qquad \liminf_{\Omega \ni x \to y} \underline{H}_g^\Omega(x) \geq \liminf_{\partial\Omega \ni x \to y} g(x). \tag{4.7}$$

Proof Suppose

$$\limsup_{\partial\Omega \ni x \to y} f(x) < M < \infty.$$

Define $\tilde{f} = f$ on $\partial\Omega \setminus \{y\}$ and $\tilde{f}(y) = M$. Since $\{y\}$ is a polar set by Lemma 3.26.2, Proposition 4.2.8 gives that $\overline{H}_{\tilde{f}}^\Omega = \overline{H}_f^\Omega$. There exists $r > 0$ such that $\tilde{f} \leq M$ on $B(y; r) \cap \partial\Omega$. Choose a constant L such that $\tilde{f} \leq L$ on $\partial\Omega$. According to Lemma 4.5.3, there exists $u \in H(\Omega)$ such that $|x - y|^2 \leq u(x) \leq (\operatorname{diam}\Omega)^2$ and

$u(x) \to 0$ as $x \to y$. Put $v(x) = M + u(x)L/r^2$. Then $v \in H(\Omega)$. If $z \in \partial\Omega \setminus B(y; r)$, then

$$\liminf_{\Omega \ni x \to z} v(x) \geq \liminf_{\Omega \ni x \to z} |x - y|^2 L/r^2 = |z - y|^2 L/r^2 \geq L \geq \tilde{f}(z).$$

If $z \in B(y; r) \cap \partial\Omega$, then

$$\liminf_{\Omega \ni x \to z} v(x) \geq M \geq \tilde{f}(z).$$

Since $v \in \mathcal{U}_{\tilde{f}}^{\Omega}$, we have

$$\limsup_{\Omega \ni x \to y} \overline{H}_f^{\Omega}(x) = \limsup_{\Omega \ni x \to y} \overline{H}_{\tilde{f}}^{\Omega}(x) \leq \limsup_{\Omega \ni x \to y} v(x) = M.$$

Since M was arbitrary, we obtain the first part of (4.7). Since $-g$ is bounded from above, Lemma 4.1.3 gives

$$\liminf_{\Omega \ni x \to y} \underline{H}_g^{\Omega}(x) = -\limsup_{\Omega \ni x \to y} \overline{H}_{(-g)}^{\Omega}(x) \geq -\limsup_{\partial\Omega \ni x \to y} (-g(x)) = \liminf_{\partial\Omega \ni x \to y} g(x).$$

Theorem 4.5.5 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. A point $y \in \partial\Omega$ is regular if and only if there exists a barrier for Ω at y.*

Proof If there exists a barrier at y, then y is regular by Proposition 4.5.4.

Conversely, suppose that y is regular. Put $f(x) = |x - y|^2$. Since f is resolutive by Theorem 4.2.7, the function $H_f^{\Omega}(x)$ is harmonic in Ω. Since $\Delta f = 2m > 0$, the function f is subharmonic in Ω by Theorem 3.19.4. Since $f \in \mathcal{L}_f^{\Omega}$, we have $H_f^{\Omega}(x) \geq f(x) = |x - y|^2$. Since y is regular, we infer that $H_f^{\Omega}(x) \to 0$ as $x \to y$ in Ω. Thus, H_f^{Ω} is a barrier for Ω at y.

Theorem 4.5.6 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. A point $y \in \partial\Omega$ is irregular for Ω if and only if there is some component ω of Ω such that $y \in \partial\omega$ and y is irregular for ω. In particular, y is regular for Ω if y is not in the boundary of any component of Ω.*

Proof Fix $y \in \partial\Omega$. Suppose first that y is a regular point for Ω. According to Theorem 4.5.5, there is a barrier v for Ω at y. If ω is a component of Ω such that $y \in \partial\omega$ then v is a barrier for ω at y. Theorem 4.5.5 gives that y is a regular point for ω.

Suppose now that y is a regular point for each component ω of Ω for which $y \in \partial\omega$. Let $\{\omega_j; j \in J\}$ be all component of Ω. The set J is finite or denumerable. Denote $J_1 = \{j; y \notin \partial\omega_j\}$ and $J_2 = \{j; y \in \partial\omega_j\}$. For $j \in J_1$, put $v(x) = \operatorname{dist}(y, \overline{\omega_j})$ for $x \in \omega_j$. Let now $j \in J_2$. According to Theorem 4.5.5, there is a barrier for ω_j at y. Lemma 4.5.3 gives that there is $v_j \in H(\omega_j)$ such that $v_j > 0$ and

$$\lim_{\omega_j \ni x \to y} v_j(x) = 0.$$

Put $v(x) = \min(v_j(x), 1/j)$ for $x \in \omega_j$. Then v is a superharmonic function in ω_j by Lemma 3.3.3. Since v is superharmonic positive function in Ω and

$$\lim_{\Omega \ni x \to y} v(x) = 0,$$

v is a barrier for Ω at y. Theorem 4.5.5 gives that y is a regular point for Ω.

Theorem 4.5.7 *Let $\Omega \subset R^m$ be a bounded open set, $y \in \partial\Omega$. If there exist a coordinate system centered at y and $\rho > 0$ such that $\Omega \cap \{x \in B(0;\rho); x_1 \leq 0, x_2 = 0\} = \emptyset$, then y is regular for Ω. In particular, if $\Omega \subset R^2$ and y is an end point of some line segment lying in $R^2 \setminus \Omega$, then y is regular.*

Proof Express the first two coordinates in the polar coordinates, i.e., $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$. Define $u(x) = \sqrt{r}\cos(\varphi/2)$ for $x \in \Omega \cap B(0;\rho)$. Since $-\pi < \varphi < \pi$, we have $u > 0$. By virtue of Lemma 2.1.6,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = -\frac{1}{4}r^{-3/2}\cos(\varphi/2)$$
$$+\frac{1}{2}r^{-3/2}\cos(\varphi/2) - \frac{1}{4}r^{-3/2}\cos(\varphi/2) = 0.$$

Since $u(x) \to 0$ as $x \to y$, the function u is a barrier. The point y is regular by Theorem 4.5.5.

Theorem 4.5.8 *Let $\Omega \subset R^m$ be a bounded open set, $m > 2$ and $y \in \partial\Omega$. Suppose that there exist a circular cone Γ of vertex y and an $(m-1)$-dimensional hyperplane M containing the axis of Γ such that $\Gamma \cap M \cap B(y;\delta) \subset R^m \setminus \Omega$ for some $\delta > 0$. Then y is regular.*

Proof We may suppose that $y = 0$. Define $\omega = B(0;1) \setminus \overline{(\Gamma \cap M)}$. Then every point of $\partial\omega \setminus \{0\}$ is a regular point of ω by Theorem 4.5.7. Put $f(x) = |x|^2$. The function f is resolutive by Theorem 4.2.7. Define $F = f$ on $\partial\omega$ and $F = H_f^\omega$ on ω. Then F is continuous on $\overline{\omega} \setminus \{0\}$. We show that F is a barrier at y for ω. The function F is harmonic in ω by Theorem 4.1.6. Since $-\Delta f \leq 0$, the function f is subharmonic by Corollary 3.19.3. Thus, $f \in \mathcal{L}_f^\omega$, and therefore $0 < f \leq F$ in ω. We now show that $F(x) \to 0$ as $|x| \to 0$. Since $f \leq 1$ on $\partial\omega$, we infer that $1 \in \mathcal{U}_f^\omega$ and thus $F \leq 1$. Theorem 2.4.2 gives that $F \equiv 1$ or $F < 1$ in ω. If $z \in \partial\omega$ and $0 < |z| < 1$, then $F(x) \to f(z) = |z|^2 < 1$ as $x \to z$. So, $F \not\equiv 1$, and therefore $F < 1$ in ω. Define $G = B(0;1/2) \setminus \overline{(\Gamma \cap M)}$. Then $H_F^G = F$ by Theorem 4.2.9. Define

$$\alpha = \sup_{x \in \overline{G}} F(x).$$

Theorem 4.4.1 gives

$$\alpha \leq \sup_{x \in \partial G} F(x) < 1.$$

On the other hand,

$$\alpha \geq \sup_{x \in \partial G \cap \partial \omega} F(x) = \frac{1}{4}.$$

Define $u(x) = F(x) - \alpha F(2x)$. Then u is a bounded harmonic function in G. If $z \in \partial\omega \cap \partial G \setminus \{0\}$, then

$$\lim_{G \ni x \to z} u(x) = |z|^2 - \alpha 4|z|^2 \leq 0.$$

If $z \in \partial G \setminus \partial\omega$, then

$$\lim_{G \ni x \to z} u(x) \leq \alpha - \alpha \cdot 1 = 0.$$

Corollary 2.4.4 gives that $u \leq 0$ in G. We know

$$\beta := \limsup_{G \ni x \to 0} F(x) \geq \liminf_{G \ni x \to 0} F(x) \geq 0.$$

Since $F(x) \leq \alpha F(2x)$, we have $\beta \leq \alpha\beta$. Since $0 < \alpha < 1$ and $\beta \geq 0$, we deduce that $\beta = 0$.

Since F is a barrier at y for ω, it is a barrier at y for Ω. Theorem 4.5.5 gives that y is a regular point.

4.6 Regularity and the Green Function

As a consequence of results about PWB-solutions of the Dirichlet problem, we are able to study limits of the Green function and Green potentials at boundary points.

Proposition 4.6.1 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, and $G(x, y) = h_m(x - y) - g_\Omega(x, y)$ be the Green function of Ω. For a fixed $y \in \Omega$, put $f(x) = h_m(x - y)$ on $\partial\Omega$. Then $g_\Omega(\cdot, y) = H_f^\Omega$.*

Proof $f \in \mathcal{C}^0(\partial G)$ is resolutive by Theorem 4.2.7; hence, $H_f^\Omega \in H(\Omega)$. Since $h_m(\cdot - y)$ is a superharmonic function (see Example 3.3.9), we have $h_m(\cdot - y) \in \mathcal{U}_f^\Omega$ and thus $H_f^\Omega(x) \leq h_m(x - y)$. According to the definition of Green's function, we have $H_f^\Omega(x) \leq g_\Omega(x, y)$. If $z \in \partial\Omega$, then

$$\limsup_{\Omega \ni x \to z} g_\Omega(x, y) \leq \limsup_{\Omega \ni x \to z} h_m(x - y) = f(z)$$

and thus $g_\Omega(\cdot, y) \in \mathcal{L}_f^\Omega$ because $g_\Omega(\cdot, y) \in H(\Omega)$. Hence, $g_\Omega(x, y) \leq H^\Omega f$.

Theorem 4.6.2 *Let* $\Omega \subset R^m$ *be a bounded open set with the Green function* G_Ω*,* $m \geq 2$*, and* $y \in \partial\Omega$*. The following are equivalent:*

1. *y is regular for Ω.*
2. *For every $z \in \Omega$,*

$$\lim_{\Omega \ni x \to y} G_\Omega(x, z) = 0. \tag{4.8}$$

3. *Every component of Ω contains a point z for which (4.8) holds.*

Proof 1) $\Rightarrow$ 2): Fix $z \in \Omega$. Put $f(x) = h_m(x - z)$. Since y is regular for Ω, we have

$$\lim_{\Omega \ni x \to y} H_f^\Omega(x) = f(y) = h_m(y - z).$$

Proposition 4.6.1 gives $G_\Omega(x, z) = h_m(x - z) - H_f^\Omega(x)$. This forces (4.8).

3) $\Rightarrow$ 1): Let ω be a component of Ω such that $y \in \partial\omega$. Fix $z \in \omega$ such that (4.8) holds. Then $G_\Omega(\cdot, z)$ is a barrier for ω at y by Proposition 3.9.7. According to Theorem 4.5.5, the point y is regular for ω. Theorem 4.5.6 gives that y is regular for Ω.

Theorem 4.6.3 *Let* $\Omega \subset R^m$ *be a bounded open set with the Green function* G_Ω*,* $m \geq 2$*, and* μ *be a finite real Borel measure with compact support in* Ω*,* $f \in L^p(\Omega)$*, and* $p > m/2$*. If* $z \in \partial\Omega$ *is regular for* Ω*, then*

$$\lim_{\Omega \ni x \to z} G_\Omega \mu(x) = 0, \qquad \lim_{\Omega \ni x \to z} G_\Omega f(x) = 0.$$

Proof We can suppose that $p < \infty$. Then $p' = p/(p-1) < m/(m-2)$. There exists $C \geq 0$ such that $0 \leq G_\Omega(x, y) \leq h_m(x - y) + C$ on $\Omega \times \Omega$ (see Proposition 3.9.14 and Proposition 3.9.15). If $y \in \Omega$, then $G_\Omega(x, y) \to 0$ as $x \to z$ by Theorem 4.6.2. Lebesgue lemma (Lemma 1.13.2) gives

$$\lim_{\Omega \ni x \to z} G_\Omega \mu(x) = 0.$$

Fix $\epsilon > 0$. Since $h_m(x) + C \in L_{loc}^{p'}(R^m)$, there exists $\rho > 0$ such that

$$\int_{B(0;\rho)} |h_m(x) + C|^{p'} \, d\mathcal{H}_m \leq \epsilon^{p'}.$$

Put $f_1 = f\chi_{B(z;\rho/2)}$ and $f_2 = f - f_1$. If $x \in \Omega \cap B(z; \rho/2)$ then we obtain by Hölder's inequality (Proposition 1.14.1)

$$|G_\Omega f_1(x)| \leq \|f_1\|_{L^p(\Omega)} \left[\int_{B(x;\rho)} |h_m(x - y) + C|^{p'} \, d\mathcal{H}_m(y) \right]^{1/p'} \leq \epsilon \|f\|_{L^p(\Omega)}.$$

We obtain by virtue of Lebesgue lemma (Lemma 1.13.2)

$$\lim_{\Omega \ni x \to z} G_\Omega f_2(x) = 0.$$

Thus, there is $0 < r < \rho/4$ such that $|G_\Omega f_2(x)| \le \epsilon$ for $x \in B(z; r) \cap \Omega$. Hence, $|G_\Omega f(x)| \le |G_\Omega f_1(x)| + |G_\Omega f_2(x)| \le \epsilon[\|f\|_{L^p(\Omega)} + 1]$ for $x \in B(z; r) \cap \Omega$.

Proposition 4.6.4 *Let $\Omega \subset R^m$ be an open set with the Green function G_Ω, $m \ge 2$, and $z \in \Omega$. We extend $G_\Omega(\cdot, z)$ by 0 on $R^m \setminus \overline{\Omega}$ and by*

$$G_\Omega(x, z) = \limsup_{\Omega \ni y \to x} G_\Omega(y, z), \qquad x \in \partial\Omega .$$

Then $G_\Omega(\cdot, z)$ is a subharmonic function in $R^m \setminus \{z\}$. There exists a polar subset K of $\partial\Omega$ such that $G_\Omega(\cdot, z)$ is continuous in all points of $R^m \setminus K$ and $G_\Omega(x, z) = 0$ for $x \in \partial\Omega \setminus K$.

Proof $G_\Omega(\cdot, z)$ is continuous on $R^m \setminus \partial\Omega$. Fix $x \in \partial\Omega$. Let $c > G_\Omega(x, z)$. Then there exists $\delta > 0$ such that $G_\Omega(y, z) < c$ for $y \in B(x; \delta) \cap \Omega$. If $y \in B(x; \delta)$, then $G_\Omega(y, z) \le c$. So,

$$G_\Omega(x, z) = \limsup_{y \to x} G_\Omega(y, z). \tag{4.9}$$

According to Theorem 3.37.1, there exists a polar set $K \subset \partial\Omega$ such that $G_\Omega(\cdot, z) = 0$ on $\partial\Omega \setminus K$. Since $G_\Omega \ge 0$, the relation (4.9) gives that $G_\Omega(\cdot, z)$ is continuous on $R^m \setminus K$.

Suppose now that Ω is bounded. Fix $r > 0$ such that $\overline{\Omega} \subset B(z; r)$. Fix a constant c such that $f(x) = h_m(x - z) + c \ge 0$ in $B(0; 2r)$. According to Corollary 3.9.6 and Proposition 3.9.8, there exists the Green function of $B(z; 2r)$. The function f is superharmonic in R^m by Example 3.3.9. Set $E = B(z; 2r) \setminus \Omega$. Let $\tilde{R}^E_f$ be the regularized reduced function of f relative to E in $B(z; 2r)$. Then $\tilde{R}^E_f$ is a nonnegative superharmonic function in $B(z; 2r)$ by Theorem 3.27.3. Theorem 3.28.1 gives that $\tilde{R}^E_f = R^E_f \in H(\Omega)$. Since $\tilde{R}^E_f(x) - c \le h_m(x - z)$, we have $R^E_f(x) - c \le g_\Omega(x, z)$ in Ω by the definition of the Green function. If h is a nonnegative superharmonic function on $B(z; 2r)$ such that $h \ge f$ on E, then $h \in \mathcal{U}^\Omega_f$. Hence, $R^E_f \ge \overline{H}^\Omega_f$ in Ω. Since $H^\Omega_f(x) = g_\Omega(x, z) + c$ by Proposition 4.6.1, we have $g_\Omega(x, z) = R^E_f(x) - c = \tilde{R}^E_f(x) - c$ in Ω. The function $f - \tilde{R}^E_f$ is subharmonic in $B(z; 2r) \setminus \{z\}$ by Remark 2.1.3 and Lemma 3.3.3. We have proved that $f(x) - \tilde{R}^E_f(x) = G_\Omega(x, z)$ in Ω. Theorem 3.28.1 gives that $f(x) - \tilde{R}^E_f(x) = 0 = G_\Omega(x, z)$ on $B(z; 2r) \setminus \overline{\Omega} \subset E$. According to Corollary 3.36.2, there exists a polar set $L \subset \partial\Omega$ such that $f - \tilde{R}^E_f = 0$ on $\partial\Omega \setminus L$. Thus, $f - \tilde{R}^E_f(x) = G_\Omega(x, z)$ for $x \in \partial\Omega \setminus (L \cup K)$. Since $\mathcal{H}_m(L \cup K) = 0$ by Lemma 3.26.2, we have $G_\Omega(x, z) = f(x) - \tilde{R}^E_f(x)$ almost everywhere on $B(z; 2r)$. Fix $x \in K \cup L$. Since

$$\limsup_{y \to x} [f(y) - \tilde{R}^E_f(y)] = f(x) - \tilde{R}^E_f(x)$$

by Proposition 3.3.6, we have $f(x) - \tilde{R}_f^E(x) \geq G_\Omega(x, z)$. If $0 < \rho < r$, then

$$G_\Omega(x,z) \leq f(x) - \tilde{R}_f^E(x) \leq \oint_{B(x;\rho)} [f - \tilde{R}_f^E] \,\mathrm{d}\mathcal{H}_m = \oint_{B(x;\rho)} G_\Omega(y,z) \,\mathrm{d}\mathcal{H}_m(y).$$

Thus, $G_\Omega(\cdot, z)$ is a subharmonic function in $R^m \setminus \{z\}$ by Proposition 3.3.7.

Let now Ω be unbounded. According to Corollary 3.9.6 and Proposition 3.9.8, there exists the Green function of $\Omega \cap B(0; n)$. Denote by $G_n(x, z)$ the extended Green function of $\Omega \cap B(0; n)$. Proposition 3.9.5 gives that G_n is an increasing sequence and $G_n(x, z) \leq G_\Omega(x, z)$. Denote

$$G = \lim_{n\to\infty} G_n.$$

Since $G_\Omega(\cdot, z)$ is locally bounded on $R^m \setminus \{z\}$ by Proposition 3.9.16, Proposition 3.25.1 gives that $G(\cdot, z)$ is a subharmonic function on $R^m \setminus \{z\}$. Clearly $G(x, z) \leq G_\Omega(x, z)$. We have $G(x, z) = G_\Omega(x, z)$ on $R^m \setminus \partial\Omega$ by Theorem 3.9.13. If $x \in \partial\Omega$, then

$$G_\Omega(x,z) = \limsup_{\Omega \ni y \to x} G_\Omega(y,z) = \limsup_{\Omega \ni y \to x} G(y,z) \leq \limsup_{y \to x} G(y,z) = G(x,z)$$

by Proposition 3.3.6. Thus, $G_\Omega(x, z) = G(x, z)$.

Corollary 4.6.5 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. Let G_Ω be the Green function corresponding to Ω. Fix $z \in \Omega$. Define $G_\Omega(\cdot, z) = 0$ on $R^m \setminus \Omega$. Then $G_\Omega(\cdot, z)$ is a finite continuous subharmonic function on $R^m \setminus \{z\}$.*

Proof Clearly, $G_\Omega(\cdot, z)$ is finite and continuous on $R^m \setminus \partial\Omega \setminus \{z\}$. The set Ω is regular by Theorem 4.5.7. So, $G_\Omega(\cdot, z)$ is a finite continuous function on $R^m \setminus \{z\}$ by Theorem 4.6.2. Proposition 4.6.4 gives that $G_\Omega(\cdot, z)$ is subharmonic on $R^m \setminus \{z\}$.

Chapter 5
L^p-Solutions of Boundary Value Problems

Abstract Chapter 5 studies boundary value problems for the Laplace equation on bounded and unbounded domains with compact Lipschitz boundary, where boundary conditions are from Lebesgue's spaces.

Chapter 5 is devoted to so-called L^p-solutions of boundary value problems for the Laplace equation on bounded and unbounded open sets with compact Lipschitz boundary: the Dirichlet problem, the Neumann problem, the Robin problem, the derivative oblique problem, the transmission problem, and the jump problem on cracks. Boundary conditions are p-integrable on the boundary, and they are fulfilled in the sense of a nontangential limit. The maximal function of a solution is p-integrable on the boundary for the Dirichlet problem. The maximal functions of a solution and its derivatives are p-integrable on the boundary in other cases. If u is a solution of the Dirichlet problem with a boundary condition from $W^{1,p}(\partial\Omega)$, then we show that the maximal function of ∇u is p-integrable on the boundary. (Then $u \in W^{1,p}(\Omega)$ for Ω bounded.) We also study very weak L^p-solutions of the Neumann problem and the Robin problem. Here boundary conditions are from $W^{-1,p}(\partial\Omega)$. The maximal function of a solution u is p-integrable on the boundary, there exists a nontangential limit of u at almost all points of the boundary, and the boundary condition is fulfilled in the sense of distributions. We are also interested about very weak solutions of the transmission problem and the jump problem, where the jump of traces is in $L^p(\partial\Omega)$ and the jump of Neumann conditions is from $W^{-1,p}(\partial\Omega)$.

We study also the relations between L^p-solutions of the Dirichlet problem and PWB-solutions of the Dirichlet problem. We show that an L^p-solution of the Dirichlet problem is a Perron-Wiener-Brelot solution of the Dirichlet problem. As a consequence we obtain that the harmonic measure and the surface measure on the boundary are mutually absolutely continuous.

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Solutions of boundary value problems are looked for in the form of an appropriate linear combination of a single layer potential and a double layer potential. For this reason we define a single layer potential $\mathcal{S}^\Omega f$ and a double layer potential $\mathcal{D}^\Omega f$ with density $f \in L^p(\partial\Omega)$ and study their properties. We show that $\mathcal{S}^\Omega f$ is a harmonic function in Ω, $\mathcal{S}^\Omega f$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all points of $\partial\Omega$, the nontangential maximal functions of $\mathcal{S}^\Omega f$ and $\nabla\mathcal{S}^\Omega f$ are p-integrable on the boundary, $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$, $\mathcal{S}^\Omega : B^{p,q}_s(\partial\Omega) \to B^{p,q}_{s+1}(\partial\Omega)$ are bounded linear operators for $-1 < s < 0$ and $1 < q < \infty$. We prove that $\mathcal{D}^\Omega f$ has a nontangential limit at almost all points of the boundary and the nontangential maximal function of $\mathcal{D}^\Omega f$ is p-integrable on the boundary. For $f \in W^{1,p}(\partial\Omega)$, we obtain that there exists a nontangential limit of $\nabla\mathcal{D}^\Omega f$ at almost all points of $\partial\Omega$ and the nontangential maximal function of $\nabla\mathcal{D}^\Omega f$ is p-integrable on the boundary.

5.1 The Formulation of Problems

Let $\Omega \subset R^m$ be a domain with Lipschitz boundary. If $x \in \partial\Omega$, $a > 0$ denote the nontangential approach regions of opening a at the point x by

$$\Gamma_a(x) = \Gamma_a^\Omega(x) = \{y \in \Omega; |x-y| < (1+a)\,\mathrm{dist}(y,\partial\Omega)\}.$$

We fix $a > 0$ large enough such that $x \in \overline{\Gamma_a(x)}$ for every $x \in \partial\Omega$. If now v is a function defined in Ω, we denote the nontangential maximal function of v on $\partial\Omega$ by

$$M_a^\Omega(v)(x) = \sup\{|v(y)|; y \in \Gamma_a(x)\}.$$

If $h > 0$, denote

$$M_{a,h}^\Omega(v)(x) = \sup\{|v(y)|; y \in \Gamma_a(x) \cap B(x;h)\}.$$

If $x \in \partial\Omega$, then

$$v(x) = \lim_{\Gamma_a(x) \ni y \to x} v(y)$$

is the nontangential limit of v with respect to Ω at x. If $a > 0$ is fixed, we shall write $\Gamma(x)$ instead of $\Gamma_a(x)$.

If Ω is fixed and v is defined on $R^m \setminus \partial\Omega$, we shall write by v_+ the nontangential limit of v with respect to Ω and by v_- the nontangential limit of v with respect to $\omega = R^m \setminus \overline{\Omega}$.

Definition 5.1.1 Let $\Omega \subset R^m$ be an open set with Lipschitz boundary, $g \in L^p(\partial\Omega)$, $1 \le p \le \infty$. We say that u is an L^p-solution of the first (or Dirichlet) problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

if $u \in H(\Omega)$, $M_a^\Omega u \in L^p(\partial\Omega)$ for all $a > 0$ and $g(x)$ is the nontangential limit of u at almost all $x \in \partial\Omega$.

Definition 5.1.2 Let $\Omega \subset R^m$ be an open set with Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $g \in L^p(\partial\Omega)$, $1 < p < \infty$. We say that u is an L^p-solution of the third (or Robin) problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \quad \text{on } \partial\Omega$$

if $u \in H(\Omega)$, $M_a^\Omega u + M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ for all $a > 0$, at almost all $x \in \partial\Omega$ there exist nontangential limits of u and ∇u, and $n(x) \cdot \nabla u(x) + h(x)u(x) = g(x)$. If $h \equiv 0$, we say about the second (or Neumann) problem.

Definition 5.1.3 Let $\Omega \subset R^m$ be an open set with Lipschitz boundary, $\Theta \in L^\infty(\partial\Omega, R^m)$, $h \in L^\infty(\partial\Omega)$, $g \in L^p(\partial\Omega)$, $1 < p < \infty$. We say that u is an L^p-solution of the derivative oblique problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \Theta \cdot \nabla u + hu = g \quad \text{on } \partial\Omega$$

if $u \in H(\Omega)$, $M_a^\Omega u + M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ for all $a > 0$, at almost all $x \in \partial\Omega$ there exist nontangential limits of u and ∇u, and $\Theta(x) \cdot \nabla u(x) + h(x)u(x) = g(x)$.

Definition 5.1.4 Let $\Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega_+)$, $g \in W^{1,p}(\partial\Omega_+)$, $f \in L^p(\partial\Omega_+)$, $1 < p < \infty$. We say that $u_\pm$ is an L^p-solution of the transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$a_+ u_+ - a_- u_- = g, \quad b_+ \frac{\partial u_+}{\partial n} - b_- \frac{\partial u_-}{\partial n} + hu_+ = f \quad \text{on } \partial\Omega_+$$

if $u_\pm \in H(\Omega_\pm)$, $M_a^{\Omega_\pm} u_\pm + M_a^{\Omega_\pm}(\nabla u_\pm) \in L^p(\partial\Omega_\pm)$ for all $a > 0$, at almost all $x \in \partial\Omega_+$ there exist nontangential limits of $u_\pm$ and $\nabla u_\pm$, and the boundary conditions on $\partial\Omega_+$ are fulfilled in the sense of the nontangential limit.

5.2 The Single Layer Potential

We shall look for solutions of boundary value problems for the Laplace equation in the form of a single layer potential or a double layer potential. In this section, we study properties of a single layer potential. If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary and $f \in L^p(\partial\Omega)$ with $1 < p < \infty$, we show that $\mathcal{S}^\Omega f$ is a harmonic function in Ω, $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\Omega)$ is a bounded linear operator, $\|M_a^\Omega(\mathcal{S}^\Omega f) + M_a^\Omega(\nabla\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$, $\mathcal{S}^\Omega f$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all points of $\partial\Omega$, and $\mathcal{S}^\Omega$ is a compact operator on $L^p(\partial\Omega)$.

Remember that

$$h_m(x) = \begin{cases} -\frac{1}{2\pi}\ln|x|, & m = 2, \\ \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial B(0;1))}|x|^{2-m}, & m > 2, \end{cases}$$

is a fundamental solution of the Laplace equation in R^m. If $\Omega \subset R^m$ is an open set with compact Lipschitz boundary or a graph Lipschitz domain and $f \in L^p(\partial\Omega)$, $1 \le p \le \infty$, we define

$$\mathcal{S}^\Omega f(x) = \int_{\partial\Omega} f(y) h_m(x-y)\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

the single layer potential with density f. If Ω has compact boundary, then $\mathcal{S}^\Omega f \in H(R^m \setminus \partial\Omega)$ by Propositions 3.10.2, 3.9.2, and 3.11.2. If Ω is a graph Lipschitz domain, then $\mathcal{S}^\Omega f \in H(R^m \setminus \partial\Omega)$ if and only if $\mathcal{S}^\Omega f(x)$ is finite for some x. To overcome the problem that $\mathcal{S}f$ is not defined as a harmonic function on Ω, we define a modified single layer potential. Fix $z \in R^m \setminus \partial\Omega$. For $f \in L^p(\partial\Omega)$, define

$$\tilde{\mathcal{S}}_z^\Omega f(x) = \int_{\partial\Omega} f(y)[h_m(x-y) - h_m(z-y)]\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

If $\mathcal{S}^\Omega f(x)$ is finite for some x, then $\mathcal{S}^\Omega f(z)$ is finite and $\tilde{\mathcal{S}}_z^\Omega f(y) = \mathcal{S}^\Omega f(y) - \mathcal{S}^\Omega f(z) \in H(R^m \setminus \partial\Omega)$, $\nabla\tilde{\mathcal{S}}_z^\Omega f(y) = \nabla\mathcal{S}^\Omega f(y)$.

If $\Omega \subset R^m$ is an open set with compact Lipschitz boundary or a graph Lipschitz domain and $f \in L^1_{loc}(\partial\Omega)$, we define the maximal function of f

$$M^{\partial\Omega} f(x) = \sup_{r>0} \fint_{B(x;r)\cap\partial\Omega} |f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y), \quad x \in \partial\Omega. \tag{5.1}$$

If $1 < p < \infty$, then there exists a constant C dependent only on Ω and p such that

$$\|M^{\partial\Omega} f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$$

(see Lemma 1.14.11).

Lemma 5.2.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary or graph Lipschitz domain, $m \geq 2$, $1 \leq p < \infty$, $z \in R^m \setminus \partial\Omega$. If $f \in L^p(\partial\Omega)$, then $\tilde{\mathcal{S}}^{\Omega}_z f \in H(R^m \setminus \partial\Omega)$.*

Proof Fix $r > 0$. Put $f_1 = f$ on $\partial\Omega \cap B(z; 2r)$, $f_1 = 0$ elsewhere, $f_2 = f - f_1$. Then $\mathcal{S}^{\Omega} f_1 \in H(R^m \setminus \partial\Omega)$ by Propositions 3.10.2, 3.9.2, and 3.11.2. Hence $\tilde{\mathcal{S}}^{\Omega}_z f_1 \in H(R^m \setminus \partial\Omega)$. If $x \in B(z; r)$, $y \in \partial\Omega \setminus B(z; 2r)$, then $|h_m(x-y) - h_m(z-y)| \leq C|x-z||z-y|^{1-m}$ by Lemma 2.9.7. Hölder's inequality (Proposition 1.14.1) yields that $\tilde{\mathcal{S}}^{\Omega}_z f_2(x)$ is finite. Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives that $\tilde{\mathcal{S}}^{\Omega}_z f_2 \in H(R^m \setminus \partial\Omega)$.

Proposition 5.2.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. Then $\mathcal{S}^{\Omega}$ is a compact bounded linear operator on $L^p(\partial\Omega)$. If $f \in L^p(\partial\Omega)$, then $\mathcal{S}^{\Omega} f(z)$ is the nontangential limit of $\mathcal{S}^{\Omega} f$ for almost all $z \in \partial\Omega$. If $a > 0$, then there exists a constant C dependent on Ω, p, and a such that*

$$\|M^{\Omega}_a(\mathcal{S}^{\Omega} f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \tag{5.2}$$

Proof There exists a constant C_1 such that $|h_m(x-y)| \leq L(x,y) := C_1|x-y|^{3/2-m}$ for $x, y \in \overline{\Omega}$. There are $z^1, \ldots, z^k \in \partial\Omega$ and $\rho > 0$ such that $\partial\Omega \subset B(z^1; \rho) \cup \cdots \cup B(z^k; \rho)$, and for each $j \in \{1, \ldots, k\}$, there is a coordinate system centered at z^j and a Lipschitz continuous function φ^j such that $B(0; 2\rho) \cap \Omega = \{[x', x_m] \in B(0; 2\rho); x_m > \varphi^j(x')\}$. According to Lemma 1.18.1, there exist $\alpha_j \in \mathcal{C}^{\infty}_c(B(z^j; \rho))$ such that $\alpha_1 + \cdots + \alpha_k = 1$ on $\partial\Omega$. Choose $\psi \in \mathcal{C}^{\infty}_c(B(0; \rho))$ such that $\psi = 1$ on a neighborhood of 0. Denote

$$K_0 f(x) = \int_{\partial\Omega} [1 - \psi(x-y)] h_m(x-y) f(y) \, \mathrm{d}\mathcal{H}_{m-1}(y),$$

$$K_j f(x) = \int_{\partial\Omega} \psi(x-y) h_m(x-y) \alpha_j(y) f(y) \, \mathrm{d}\mathcal{H}_{m-1}(y), \quad j = 1, \ldots, k.$$

Then K_0 is a compact bounded linear operator on $L^p(\partial\Omega)$ by Proposition 1.26.2. If $j \in \{1, \ldots, k\}$, then K_j can be studied as an operator on R^{m-1}. Proposition 1.26.6 gives that K_j is a compact bounded linear operator on $L^p(\partial\Omega)$. Hence $\mathcal{S}^{\Omega} = K_0 + K_1 + \cdots + K_k$ is a compact bounded linear operator on $L^p(\partial\Omega)$ (compare Theorem 1.7.4). Denote

$$Lf(x) = \int_{\partial\Omega} L(x,y) f(y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

By the same way, we prove that L is a bounded linear operator on $L^p(\Omega)$, i.e., there exists a constant C_2 such that $\|Lf\|_{L^p(\partial\Omega)} \leq C_2\|f\|_{L^p(\partial\Omega)}$.

Let $z \in \partial\Omega$, $x \in \Gamma_a(z)$. Then

$$\left| \int\limits_{B(z;2|x-z|)\cap\partial\Omega} h_m(x-y)f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y) \right|$$

$$\leq \int\limits_{B(z;2|x-z|)\cap\partial\Omega} C_1 \left(\frac{1+a}{|x-z|} \right)^{m-3/2} |f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

According to Corollary 1.17.6, there exists a constant C_3 dependent only on Ω and a such that

$$\left| \int\limits_{B(z;2|x-z|)\cap\partial\Omega} h_m(x-y)f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y) \right| \leq C_3\sqrt{|x-z|}M^{\partial\Omega}f(z). \tag{5.3}$$

If $y \in \partial\Omega \setminus B(x;2|x-z|)$, then $|z-y| \leq |z-x| + |y-x| \leq 2|x-y|$ and thus $|h_m(x-y)| \leq L(x,y) \leq 2^m L(z,y)$. This forces

$$\left| \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} h_m(x-y)f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y) \right| \leq 2^m L|f|(z). \tag{5.4}$$

Equations (5.3) and (5.4) give

$$M_a^\Omega(\mathcal{S}^\Omega f) \leq C_3\sqrt{\operatorname{diam}\Omega}[M^{\partial\Omega}f(z)] + 2^m L|f|(z).$$

According to Lemma 1.14.11, there exists a constant C_4 depending only on Ω and p such that

$$\|M^{\partial\Omega}f\|_{L^p(\partial\Omega)} \leq C_4\|f\|_{L^p(\partial\Omega)}.$$

Thus we obtain (5.2) with $C = C_4C_3\sqrt{\operatorname{diam}\Omega} + C_2 2^m$.

$L|f|(z)$, $M^{\partial\Omega}f(z)$ are finite at almost all $z \in \partial\Omega$. Fix such z. Fix $\epsilon > 0$. For $r > 0$, put $f_r = f$ on $B(z;r) \cap \partial\Omega$, $f_r = 0$ elsewhere. Choose $r > 0$ such that $L|f_r|(z) < 6^{-m}\epsilon$. Theorem 1.13.6 gives that $\mathcal{S}^\Omega(f-f_r)$ is continuous in z. So, there exists $0 < \delta < \epsilon^2(1+3C_3M^{\partial\Omega}f(z))^{-2}$ such that $|\mathcal{S}^\Omega(f-f_r)(x) - \mathcal{S}^\Omega(f-f_r)(z)| < \epsilon/3$ for $x \in B(x;\delta)$. If $x \in \Gamma_a(z) \cap B(z;\delta)$, then (5.3) and (5.4) give

$$\begin{aligned} |\mathcal{S}^\Omega f(x) - \mathcal{S}^\Omega f(z)| &\leq |\mathcal{S}^\Omega(f-f_r)(x) - \mathcal{S}^\Omega(f-f_r)(z)| + |\mathcal{S}^\Omega f_r(z)| + |\mathcal{S}^\Omega f_r(x)| \\ &\leq \frac{\epsilon}{3} + L|f_r|(z) + 2^m L|f_r|(z) + C_3\sqrt{\delta}M^{\partial\Omega}f(z) < \epsilon. \end{aligned}$$

Proposition 5.2.3 *Let* $\Omega \subset R^m$ *be an unbounded open set with compact Lipschitz boundary,* $1 < p < \infty$, $a > 0$, $f \in L^p(\partial\Omega)$. *Then* $\mathcal{S}^\Omega f(z)$ *is the nontangential limit of* $\mathcal{S}^\Omega f$ *at almost all* $z \in \partial\Omega$. *If* $h > 0$, *then* $M^\Omega_{a,h}(\mathcal{S}^\Omega f) \in L^p(\partial\Omega)$. *Set* $X = \{f \in L^p(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0\}$ *for* $m = 2$ *and* $X = L^p(\partial\Omega)$ *for* $m > 2$. *Then* $M^\Omega_a(\mathcal{S}^\Omega f) \in L^p(\partial\Omega)$ *if and only if* $f \in X$. *Moreover, there exists a constant* C *dependent only on* Ω, p, *and* a *such that*

$$\|M_a(\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)} \qquad \forall f \in X. \tag{5.5}$$

Proof If $f \in L^p(\partial\Omega) \setminus X$, then Lemma 3.12.2 gives $|\mathcal{S}^\Omega f(x)| \to \infty$ as $|x| \to \infty$. Hence $M^\Omega_a f \equiv \infty$.

Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Denote $\omega = \Omega \cap B(0; r + 2h)$. Put $f = 0$ on $\partial\omega \setminus \partial\Omega$. Then $\mathcal{S}^\Omega f = \mathcal{S}^\omega f$. If $z \in \partial\Omega$, $x \in \Gamma^\Omega_a(z) \cap B(z; h)$, $y \in \partial\omega \setminus \partial\Omega$, then $(1 + a)|x - y| > |y| - |x| > (r + 2h) - (r + h) = h > |x - z|$. Thus $\Gamma^\Omega_a(z) \cap B(z; h) \subset \Gamma^\omega_a(z)$. Proposition 5.2.2 gives that $\mathcal{S}^\Omega f(z)$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all $z \in \partial\Omega$. Moreover,

$$\|M^\Omega_{a,h} f\|_{L^p(\partial\Omega)} \le \|M^\omega_a f\|_{L^p(\partial\omega)} \le C_1\|f\|_{L^p(\partial\Omega)}$$

where C_1 depends only on ω, p, and a (see Proposition 5.2.2).

According to Hölder's inequality (Proposition 1.14.1), there exists a constant C_2 dependent only on Ω and p such that

$$\|f\|_{L^1(\partial\Omega)} \le C_2\|f\|_{L^p(\partial\Omega)}.$$

We can suppose that $h > 5r$. If $x \in \Gamma^\Omega_a(z) \setminus B(z; h)$ then $\mathrm{dist}(x, \partial\Omega) > 3r$. If $m > 2$, then

$$|\mathcal{S}^\Omega f(x)| \le C_2 r^{2-m}\|f\|_{L^p(\partial\Omega)}.$$

So, (5.5) holds with $C = C_1 + C_2 r^{2-m}\mathcal{H}_{m-1}(\partial\Omega)^{1/p}$. Let now $m = 2$ and $f \in X$. According to Lemma 2.9.7, there exists a constant C_3 such that $|h_2(y - x) - h_2(0 - x)| \le C_3|y||x|^{-1} \le C_3$ for $2|y| < |x|$. Then

$$|\mathcal{S}^\Omega f(x)| = \left| \int_{\partial\Omega} f(y)[h_2(y - x) - h_2(0 - x)] \, \mathrm{d}\mathcal{H}_{m-1}(y) \right| \le C_2 C_3\|f\|_{L^p(\partial\Omega)}.$$

So, (5.5) holds with $C = C_1 + C_2 C_3 \mathcal{H}_{m-1}(\partial\Omega)^{1/p}$.

Corollary 5.2.4 *Let* $\Omega \subset R^m$ *be a graph Lipschitz domain,* $z \in R^m \setminus \partial\Omega$, $1 < p < \infty$, $f \in L^p(\partial\Omega)$. *Then* $\tilde{\mathcal{S}}^\Omega_z f(x)$ *is the nontangential limit of* $\tilde{\mathcal{S}}^\Omega_z f$ *at almost all* $x \in \partial\Omega$.

Proof Fix $r > 0$. Define $f_1 = f$ on $B(z; 2r) \cap \partial\Omega$, $f_1 = 0$ elsewhere, $f_2 = f$ on $\partial\Omega \setminus B(z; 2r)$, $f_2 = 0$ elsewhere. Then $\tilde{\mathcal{S}}^\Omega_z f_2(x)$ is the limit of $\tilde{\mathcal{S}}^\Omega_z f_2$ at all $x \in B(z; r) \cap \partial\Omega$

by the theorem on continuity of parametrized integrals (Theorem 1.13.6). Choose a bounded domain ω with Lipschitz boundary such that $B(z;2r)\cap\partial\Omega = B(z;2r)\cap\partial\omega$. Then $\tilde{\mathcal{S}}_z^\Omega f(x) = \tilde{\mathcal{S}}_z^\omega f(x)$ is the nontangential limit of $\tilde{\mathcal{S}}_z^\Omega f = \tilde{\mathcal{S}}_z^\omega f$ at almost all $x \in B(z;r)\cap\partial\Omega$ by Proposition 5.2.2.

Lemma 5.2.5 *Let ϕ be a Lipschitz function on R^{m-1}, $\Omega = \{[x',x_m]; x' \in R^{m-1}, x_m > \phi(x')\}$, $m \geq 2$, $1 < p < \infty$. For $f \in L^p(\partial\Omega)$, $j \in \{1,\dots,m\}$, $\epsilon > 0$, and $x = [x',x_m] \in R^m$, define*

$$L_{j,\epsilon}^\Omega f(x) = \int\limits_{\partial\Omega\setminus B(x;\epsilon)} \frac{x_j - y_j}{|x-y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y), \tag{5.6}$$

$$\tilde{L}_{j,\epsilon}^\Omega f(x) = \int\limits_{\{[y',y_m]\in\partial\Omega; |x'-y'|>\epsilon\}} \frac{x_j - y_j}{|x-y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y), \tag{5.7}$$

$$\tilde{L}_j^\Omega f(x) = \lim_{\epsilon\downarrow 0} \tilde{L}_{j,\epsilon}^\Omega f(x), \tag{5.8}$$

$$L_*^\Omega f(x) = \sup_{\epsilon>0} \sum_{j=1}^m |L_{j,\epsilon}^\Omega f(x)|, \quad \tilde{L}_*^\Omega f(x) = \sup_{\epsilon>0} \sum_{j=1}^m |\tilde{L}_{j,\epsilon}^\Omega f(x)|. \tag{5.9}$$

Then the limit $\tilde{L}_j^\Omega f(x)$ exists for almost all $x \in \partial\Omega$, $\tilde{L}_j^\Omega f \in L^p(\partial\Omega)$, and there exists a constant C depending only on Ω and p such that

$$\sum_{j=1}^m \|\tilde{L}_j^\Omega f\|_{L^p(\partial\Omega)} \leq \|\tilde{L}_*^\Omega f\|_{L^p(\partial\Omega)} + \|L_*^\Omega f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}.$$

Proof Fix a constant M such that $|\phi(x') - \phi(y')| \leq M|x'-y'|$, $|\nabla\phi| \leq M$. Denote $\Phi(x') = [\Phi_1(x'),\dots,\Phi_m(x')] = [x',\phi(x')]$. Define for $g \in L^p(R^{m-1})$

$$\tilde{K}_j(x',y') = \frac{\Phi_j(x') - \Phi_j(y')}{|\Phi(x') - \Phi(y')|^m}, \qquad x',y' \in R^{m-1},$$

$$\tilde{K}_{j,\epsilon} g(x') = \int\limits_{R^{m-1}\setminus B(x';\epsilon)} \tilde{K}_j(x',y') g(y')\, \mathrm{d}\mathcal{H}_{m-1}(y'),$$

$$\tilde{K}_j g(x') = \lim_{\epsilon\downarrow 0} \tilde{K}_{j,\epsilon} g(x'), \qquad \tilde{K}_j^* g(x') = \sup_{\epsilon>0} |\tilde{K}_{j,\epsilon} g(x')|.$$

Choose $F_j \in \mathcal{C}^\infty(R^m)$ such that $F_j(y) = y_j|y|^{-m}$ for $|y| \geq 1$. We have $|\Phi(x') - \Phi(y')|/|x'-y'| \geq 1$. Theorem 1.27.3 gives that

$$\tilde{K}_j(x',y') = \frac{1}{|x'-y'|^{m-1}} F((\Phi(x') - \Phi(y'))/|x'-y'|)$$

is a Calderón-Zygmund kernel. According to Propositions 1.27.4, 1.27.5, and 1.27.6 and Lemma 1.14.11, the limit $\tilde{K}_j g(x')$ exists for almost all $x' \in R^{m-1}$, $\tilde{K}_j g \in L^p(R^{m-1})$, and there exists a constant C_1 depending only on ϕ and p such that

$$\|\tilde{K}_j g\|_{L^p(R^{m-1})} \leq \|\tilde{K}_j^* g\|_{L^p(R^{m-1})} \leq C_1 \|g\|_{L^p(R^{m-1})}.$$

For $f \in L^p(\partial\Omega)$, denote

$$g_f(y') = f(y', \phi(y'))\sqrt{1 + |\nabla\phi(y')|^2}.$$

If $x = [x', x_m] \in \partial\Omega$, then $\tilde{L}_{j,\epsilon} f(x) = \tilde{K}_{j,\epsilon} g_f(x')$. Therefore the limit $\tilde{L}_j^\Omega f(x)$ exists for almost all $x \in \partial\Omega$, $\tilde{L}_j^\Omega f \in L^p(\partial\Omega)$, and

$$\sum_{j=1}^m \|\tilde{L}_j^\Omega f\|_{L^p(R^{m-1})} \leq \|\tilde{L}_*^\Omega f\|_{L^p(R^{m-1})} \leq mC_1(M+1)\|f\|_{L^p(R^{m-1})}.$$

For $x = [x', x_m] \in \partial\Omega$ and $\epsilon > 0$, denote $\mathcal{M}(x', \epsilon) = \{y'; |y' - x'| < \epsilon, |[y' - x', \phi(y') - \phi(x')]| > \epsilon\}$. Then

$$\begin{aligned} |L_{j,\epsilon}^\Omega f(x) - \tilde{L}_{j,\epsilon}^\Omega f(x)| &\leq \int\limits_{\mathcal{M}(x',\epsilon)} |\tilde{K}_j(x', y') g_f(y')| \, \mathrm{d}\mathcal{H}_{m-1}(y') \\ &\leq \int\limits_{\{\epsilon/(M+1) < |y'-x'| < \epsilon\}} |x' - y'|^{1-m} |g_f(y')| \, \mathrm{d}\mathcal{H}_{m-1}(y') \\ &\leq \left(\frac{M+1}{\epsilon}\right)^{m-1} \int\limits_{\{|x'| < \epsilon\}} |g_f(y')| \, \mathrm{d}\mathcal{H}_{m-1}(y'). \end{aligned}$$

According to Corollary 1.17.6, there exists a constant C_2 such that

$$|L_{j,\epsilon}^\Omega f(x) - \tilde{L}_{j,\epsilon} g_f(x')| \leq C_2 M^{\partial\Omega} f(x).$$

Hence

$$L_*^\Omega f(x) \leq \tilde{L}_*^\Omega f(x) + mC_2 M^{\partial\Omega} f(x).$$

According to Lemma 1.14.11, there exists a constant C_3 dependent only on Ω and p such that

$$\|M^{\partial\Omega} f\|_{L^p(\partial\Omega)} \leq C_3 \|f\|_{L^p(\partial\Omega)}.$$

Thus

$$\|L_*^\Omega f\|_{L^p(\partial\Omega)} \le [mC_1(M+1) + mC_2C_3]\|f\|_{L^p(\partial\Omega)}.$$

Lemma 5.2.6 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \ge 2$, $1 < p < \infty$. For $f \in L^p(\partial\Omega)$, $j \in \{1,\dots,m\}$, $\epsilon > 0$, and $x \in \partial\Omega$, define $L_{j,\epsilon}^\Omega f(x)$ and $L_*^\Omega f(x)$ by (5.6) and (5.9). Then there exists a constant C depending only on Ω and p such that*

$$\|L_*^\Omega f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}.$$

Proof There exist $z^1,\dots,z^k$ and $\delta > 0$ such that $\partial\Omega \subset B(z^1;\delta) \cup B(z^2;\delta) \cup \dots \cup B(z^k;\delta)$, and for each $j \in \{1,\dots,m\}$, there exist a coordinate system and a graph Lipschitz domain Ω_j with $\Omega \cap B(z^j;2\delta) = \Omega_j \cap B(z^j;2\delta)$. According to Lemma 1.18.1, there exist $\alpha_j \in \mathcal{C}_c^\infty(B(z^j;\delta))$ with $0 \le \alpha_j \le 1$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$ on $\partial\Omega$. According to Lemma 5.2.5, there exist constants $C_1,\dots,C_k$ depending on Ω and p such that

$$\|L_*^\Omega(\alpha_j f)\|_{L^p(B(z^j;2\delta)\cap\partial\Omega)} \le C_j\|\alpha_j f\|_{L^p(\partial\Omega)}.$$

Denote $c = \delta^{1-m}\mathcal{H}_{m-1}(\partial\Omega)^{p/(p-1)}$. If $x \in \partial\Omega \setminus B(z^j;2\delta)$, then

$$|L_{j,\epsilon}^\Omega(\alpha_j f)(x)| \le \int\limits_{\partial\Omega} \delta^{1-m}|f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y) \le c\|f\|_{L^p(\partial\Omega)}$$

by Hölder's inequality (Proposition 1.14.1). Therefore

$$\|L_*^\Omega(\alpha_j f)\|_{L^p(\partial\Omega)} \le (C_j + mc)\|f\|_{L^p(\partial\Omega)},$$

$$\|L_*^\Omega f\|_{L^p(\partial\Omega)} \le \sum_{j=1}^k \|L_*^\Omega(\alpha_j f)\|_{L^p(\partial\Omega)} \le \sum_{j=1}^k (C_j + mc)\|f\|_{L^p(\partial\Omega)}.$$

Lemma 5.2.7 *Let $\Omega \subset R^m$ be open, $m \ge 2$, $1 < p < \infty$, $a > 0$, $j \in \{1,\dots,m\}$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. For $f \in L^p(\partial\Omega)$, $x \in \Omega$, denote*

$$L_j^\Omega f(x) = \int\limits_{\partial\Omega} \frac{x_j - y_j}{|x-y|^m} f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y). \tag{5.10}$$

Then there exists a constant C depending on Ω, p and a such that

$$\|M_a^\Omega(L_j^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}.$$

Proof We use the notation from Lemma 5.2.5. Lemmas 5.2.5 and 5.2.6 give that there exists a constant C_1 such that

$$\|L_*^\Omega f\|_{L^p(\partial\Omega)} \le C_1 \|f\|_{L^p(\partial\Omega)}.$$

Fix $z \in \partial\Omega$, $x \in \Gamma_a^\Omega$. Denoting $L_j(x,y) = (x_j - y_j)|x-y|^{-m}$,

$$\int\limits_{B(z;2|x-z|)\cap\partial\Omega} |L_j(x,y)||f(y)| \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le \int\limits_{B(z;2|x-z|)\cap\partial\Omega} \left(\frac{1+a}{|x-z|}\right)^{m-1} |f(y)| \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

There exists a constant C_2 such that $\mathcal{H}_{m-1}(B(y;r)\cap\partial\Omega) \le C_2 r^{m-1}$ for all $y \in R^m$ and $r > 0$ (see Corollary 1.17.6). Thus

$$\int\limits_{B(z;2|x-z|)\cap\partial\Omega} |L_j(x,y)||f(y)| \, \mathrm{d}\mathcal{H}_{m-1}(y) \le C_2(1+a)^m M^{\partial\Omega} f(z). \tag{5.11}$$

Since $L_j(y,\tilde{y})$ is a multiple of $\partial_j h_m(y-\tilde{y})$, Lemma 2.9.7 gives that there exists a constant C_3 dependent only on m such that if $|y-z| > 2|x-y|$, then $|L_j(x,y) - L_j(z,y)| \le C_3|x-z||z-y|^{-m}$. Denote $\mu = |f|\mathcal{H}_{m-1}|\partial\Omega$. Since $\mu(B(z;r)) \le C_2 r^{m-1} M^{\partial\Omega} f(z)$, Lemma 1.26.1 gives

$$\left|\int\limits_{\partial\Omega\setminus B(z;2|x-z|)} L_j(x;y) f(y) \, \mathrm{d}\mathcal{H}_{m-1}\right| \le \left|\int\limits_{\partial\Omega\setminus B(z;2|x-z|)} L_j(z;y) f(y) \, \mathrm{d}\mathcal{H}_{m-1}\right|$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |L_j(x;y) - L_j(z,y)||f(y)| \, \mathrm{d}\mathcal{H}_{m-1}(y) \le L_*^\Omega f(z)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} C_3|x-z||z-y|^{-m} \, \mathrm{d}\mu(y) \le L_*^\Omega f(z)$$

$$+ C_3|x-z|C_2[M^{\partial\Omega} f(z)](m-1)(2|x-z|)^{-1}.$$

This and (5.11) give that $|L_j^\Omega f(x)| \le L_*^\Omega f(z) + C_2[(1+a)^m + C_3 m] M^{\partial\Omega} f(z)$. Since there exists a constant C_4 dependent only on Ω such that

$$\|M^{\partial\Omega} f\|_{L^p(\partial\Omega)} \le C_4 \|f\|_{L^p(\partial\Omega)}$$

(see Lemma 1.14.11), we have

$$\|M_a^\Omega(L_j f)\|_{L^p(\partial\Omega)} \le C_1\|f\|_{L^p(\partial\Omega)} + C_4C_2[(1+a)^m + C_3 m]\|f\|_{L^p(\partial\Omega)}.$$

Proposition 5.2.8 *Let $\Omega \subset R^m$ be open, $m \ge 2$, $1 < p < \infty$, $a > 0$, $z \in R^m \setminus \partial\Omega$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. If $f \in L^p(\partial\Omega)$, then*

$$\nabla\tilde{\mathcal{S}}_z^\Omega f(x) = \int\limits_{\partial\Omega} \frac{(y-x)f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}\,\mathrm{d}\mathcal{H}_{m-1}(y), \quad x \in R^m \setminus \partial\Omega,$$

$$\|M_a^\Omega(\nabla\tilde{\mathcal{S}}_z^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$$

with a constant C depending on Ω, p, and a.

Proof Fix $x \in \Omega$ and $r > 0$. Put $f_1 = f\chi_{B(x;r)}$, $f_2 = f - f_1$. Then $\nabla\tilde{\mathcal{S}}_z^\Omega f_j(x) = \mathcal{H}_{m-1}(B(0;1))^{-1}[L_1^\Omega f_j(x), \dots, L_m^\Omega f_j(x)]$, where $L_k^\Omega f$ is given by (5.10). (See Proposition 3.13.1 for f_1 and Theorem 1.13.7 for f_2.) Thus $\nabla\tilde{\mathcal{S}}_z^\Omega f(x) = \mathcal{H}_{m-1}(B(0;1))^{-1}[L_1^\Omega f(x), \dots, L_m^\Omega f(x)]$. The rest is a consequence of Lemma 5.2.7.

Corollary 5.2.9 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$, $m \ge 2$. Then $\mathcal{S}^\Omega$ is a bounded linear operator from $L^p(\partial\Omega)$ to $W^{1,p}(\Omega)$.*

Proof If $f \in L^p(\partial\Omega)$, then $M_a(\mathcal{S}^\Omega f), M_a(\nabla\mathcal{S}^\Omega f) \in L^p(\partial\Omega)$ by Propositions 5.2.2 and 5.2.8. Thus $\mathcal{S}^\Omega$ is a linear operator from $L^p(\partial\Omega)$ to $W^{1,p}(\Omega)$ by Lemma 1.28.1. If $f_k \to f$ in $L^p(\partial\Omega)$ and $x \in \Omega$, then $\mathcal{S}^\Omega f_k(x) \to \mathcal{S}^\Omega f(x)$ by Hölder's inequality (Proposition 1.14.1). So, $\mathcal{S}^\Omega$ is a closed linear operator from $L^p(\partial\Omega)$ to $W^{1,p}(\Omega)$. The closed-graph theorem (Theorem 1.4.15) gives that $\mathcal{S}^\Omega$ is a bounded linear operator from $L^p(\partial\Omega)$ to $W^{1,p}(\Omega)$.

5.3 The Double Layer Potential

In this section, we study properties of a double layer potential. If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary and $f \in L^p(\partial\Omega)$ with $1 < p < \infty$, we show that $\|M_a^\Omega\mathcal{D}^\Omega f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$ and that $\mathcal{D}^\Omega f$ has a nontangential limit $\frac{1}{2}f + K_\Omega f$ at almost all points of the boundary, where K_Ω is a bounded linear operator on $L^p(\partial\Omega)$.

Let $\Omega \subset R^m$ be open, $m \ge 2$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. For $y \in \partial\Omega$ such that $n^\Omega(y)$ exists, denote for $x \in R^m \setminus \{y\}$

$$K_\Omega(x,y) = -n^\Omega(y)\cdot\nabla_y h_m(x-y) = \frac{n(y)\cdot(y-x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}.$$

$n^\Omega(y)$ exists at almost all $y \in \partial\Omega$ by Remark 1.17.5, and $K_\Omega(x, y)$ is defined for almost all $y \in \partial\Omega$. If $f \in L^p(\partial\Omega)$, we define the double layer potential with density f

$$\mathcal{D}^\Omega f(x) = \int_{\partial\Omega} K_\Omega(x, y) f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y), \qquad x \in R^m \setminus \partial\Omega\,.$$

Then $\mathcal{D}^\Omega f \in H(R^m \setminus \partial\Omega)$ by theorem on differentiability of parametrized integral (Theorem 1.13.7).

For $x \in \partial\Omega$, $\epsilon > 0$ define

$$K_{\Omega,\epsilon} f(x) = \int_{\partial\Omega \setminus B(x;\epsilon)} K_\Omega(x, y) f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Hölder's inequality (Proposition 1.14.1) gives that $K_{\Omega,\epsilon} f(x)$ is well-defined and finite. Set

$$K_\Omega f(x) = \lim_{\epsilon \downarrow 0} K_{\Omega,\epsilon} f(x)$$

whenever this limit exists.

Proposition 5.3.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$. Then*

$$\mathcal{D}^\Omega 1 = \begin{cases} 1 \text{ in } \Omega\,, \\ 0 \text{ in } R^m \setminus \overline{\Omega}. \end{cases}$$

Proof In Ω, we have $1 = \mathcal{D}1$ by theorem on three potentials (Theorem 2.2.4). Fix $x \in R^m \setminus \overline{\Omega}$. Put $v(y) = h_m(y - x)$. Then, v is a harmonic function on a neighborhood of $\overline{\Omega}$, and Corollary 2.2.3 gives

$$\mathcal{D}^\Omega 1(x) = -\int_{\partial\Omega} \frac{\partial v}{\partial n}\, d\mathcal{H}_{m-1} = 0.$$

Proposition 5.3.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2, f \equiv 1$. If $x \in \partial\Omega$ is such that there exists $n^\Omega(x)$, then $K_\Omega f(x) = 1/2$.*

Proof If $r > 0$ is small enough, then $U(r) = \Omega \setminus \overline{B(x; r)}$ is a domain with Lipschitz boundary. According to Proposition 5.3.1,

$$0 = \mathcal{D}^{U(r)} f(x) = K_{\Omega,r} f(x) - \int_{\Omega \cap \partial B(x;r)} \frac{n(y) \cdot (y - x)}{\mathcal{H}_{m-1}(\partial B(0; 1))|x - y|^m}\, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus

$$K_{\Omega,r}f(x) = \int_{\Omega\cap\partial B(x;r)} \frac{n(y)\cdot(y-x)\mathrm{d}\mathcal{H}_{m-1}}{\mathcal{H}_{m-1}(\partial B(0;1))r^m} = \int_{\Omega\cap\partial B(x;r)} \frac{\mathrm{d}\mathcal{H}_{m-1}}{\mathcal{H}_{m-1}(B(0;r))}\,.$$

Since there exists $n^\Omega(x)$, we have

$$K_\Omega f(x) = \lim_{r\downarrow 0} K_{\Omega,r}f(x) = \lim_{r\downarrow 0} \int_{\Omega\cap\partial B(x;r)} \frac{1}{\mathcal{H}_{m-1}(B(0;r))}\,d\mathcal{H}_{m-1} = \frac{1}{2}.$$

Proposition 5.3.3 *Let $\Omega \subset R^m$ be open, $m \ge 2$, $1 < p < \infty$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. If $f \in L^p(\partial\Omega)$, then there exists the limit $K_\Omega f(x)$ at almost all $x \in \partial\Omega$. If we denote*

$$K_{\Omega,*}f(x) = \sup_{\epsilon>0} |K_{\Omega,\epsilon}f(x)|,$$

then

$$\|K_\Omega f\|_{L^p(\partial\Omega)} \le \|K_{\Omega,*}f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$$

where a constant C depends only on Ω and p.

Proof We use the notation of Lemma 5.2.5. If $\epsilon > 0$, then

$$K_{\Omega,\epsilon}f(x) = \frac{1}{\mathcal{H}_{m-1}(\partial B(0;1))} \sum_{j=1}^{m} L^\Omega_{j,\epsilon}(n_j f)(x).$$

Lemmas 5.2.5 and 5.2.6 give that there exists a constant C dependent only on Ω and p such that

$$\|K_{\Omega,*}f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}.$$

We now show that there exists the limit $K_\Omega f(x)$ at almost all $x \in \partial\Omega$. Suppose first that Ω is bounded. Then there exist $z^1,\dots,z^k \in \partial\Omega$ and $r > 0$ such that $\partial\Omega \subset B(z^1;r)\cup B(z^2;r)\cup\dots\cup B(z^k;r)$, and for each $j \in \{1,\dots,k\}$, there exist a coordinate system and a Lipschitz function φ_j on R^{m-1} such that $\Omega\cap B(z^j;2r) = \omega(j)\cap B(z^j;2r)$ with $\omega(j) = \{[x',x_m]; x' \in R^{m-1}, x_m > \varphi_j(x')\}$. Fix $j \in \{1,\dots,k\}$. Choose a constant c such that $|\varphi_j(x') - \varphi_j(y')| \le c|x'-y'|$ and $\mathcal{H}_{m-1}(B(x;\rho)\cap\partial\omega(j)) \le c\rho^{m-1}$ for all $x \in R^m$ and $\rho > 0$ (see Corollary 1.17.6). Define $\tilde f = f$ on $\partial\Omega\cap B(z^j;2r)$, $\tilde f = 0$ elsewhere. Fix $x \in \partial\Omega\cap B(z^j;r)$ such that $f(x)$ is finite,

$$\lim_{\rho\downarrow 0} \oint_{\partial\Omega\cap B(x;\rho)} |f(y) - f(x)|\,\mathrm{d}\mathcal{H}_{m-1}(y) = 0,$$

and there exist $n^{\Omega}(x)$ and the limits $\tilde{L}_l^{\omega(j)}(n_l\tilde{f})(x)$, $\tilde{L}_l^{\omega(j)}[n_l\chi_{B(z^j,2r)}](x)$ for $l = 1,\dots,m$. (These conditions are fulfilled for almost all $x \in \partial\Omega \cap B(z^j;r)$ by Remark 1.17.5, Proposition 1.13.9 and Lemma 5.2.5.) Denote $f_1 = f - f(x)$ on $\partial\Omega \cap B(z^j;2r)$, $f_1 = 0$ elsewhere; $f_2 = f - f(x)$ on $\partial\Omega \setminus B(z^j;2r)$, $f_1 = 0$ elsewhere; $f_3 = f(x)$. Fix $\delta > 0$. Then there exists $\rho_0 \in (0, r/(1+c))$ such that for all $\rho \in (0,\rho_0)$

$$\oint_{\partial\Omega\cap B(x;\rho)} |f(y) - f(x)| \, \mathrm{d}\mathcal{H}_{m-1}(y) < \delta.$$

If $0 < \epsilon < \rho_0/(1+c)$, denote $D(x,\epsilon) = \partial\Omega \cap B(x;(1+c)\epsilon) \setminus B(x;\epsilon)$. Since $K_{\Omega,\epsilon}f_1(x) = K_{\omega(j),\epsilon}f_1(x)$, we have

$$\left| K_{\Omega,\epsilon}f_1(x) - \sum_{l=1}^{m} \frac{\tilde{L}_{l,\epsilon}^{\Omega}(n_lf_1)(x)}{\mathcal{H}_{m-1}(\partial B(0;1))} \right|$$

$$\le \int_{D(x,\epsilon)} |K_{\Omega}(x,y)||f(y) - f(x)| \, \mathrm{d}\mathcal{H}_{m-1}(y) \le \int_{D(x,\epsilon)} \frac{|f(y) - f(x)|}{\epsilon^{m-1}} \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le (c+1)^m \oint_{\partial\Omega\cap B(x;(1+c)\epsilon)} |f(y) - f(x)| \, \mathrm{d}\mathcal{H}_{m-1}(y) \le (c+1)^m\delta.$$

Since δ was arbitrary and $\tilde{L}_{l,\epsilon}^{\Omega}(n_lf_1)(x) \to \tilde{L}_l^{\Omega}(n_lf_1)(x)$, we deduce that there exists the limit $K_{\Omega}f_1(x)$. Since there exists the limit $K_{\Omega}f_3(x)$ by Proposition 5.3.2 and $K_{\Omega,\epsilon(1)}f_2(x) = K_{\Omega,\epsilon(2)}f_2(x)$ for $0 < \epsilon(1) < \epsilon(2) < r$, we infer that there exists the limit $K_{\Omega}f(x) = K_{\Omega}f_1(x) + K_{\Omega}f_2(x) + K_{\Omega}f_3(x)$.

Let now $\partial\Omega$ be compact and Ω be unbounded. Put $\omega = R^m \setminus \overline{\Omega}$. We have proved that there exists the limit $K_{\omega}f(x)$ at almost all $x \in \partial\omega = \partial\Omega$. Since $K_{\Omega}(x,y) = -K_{\omega}(x,y)$, we deduce that there exists the limit $K_{\Omega}f(x) = -K_{\omega}f(x)$ at almost all $x \in \partial\Omega$.

Let now Ω be a graph Lipschitz domain. For $k \in N$, define $f_k = f$ on $\partial\Omega \cap B(0;2k)$, $f_k = 0$ elsewhere. There exists a bounded domain $\omega(k)$ with Lipschitz boundary such that $B(0;3k) \cap \Omega = B(0;3k) \cap \omega(k)$. We have proved that there exist the limits $K_{\omega(k)}f_k(x)$ for $x \in \partial\omega(k) \setminus M_k$ with $\mathcal{H}_{m-1}(M_k) = 0$. If $x \in \partial\Omega \cap B(0;k)$, $0 < \delta < \epsilon < 1$, we have $K_{\Omega,\epsilon}f_k(x) = K_{\omega(k),\epsilon}f_k(x)$, $K_{\Omega,\epsilon}[f - f_k](x) = K_{\Omega,\delta}[f - f_k](x)$. Therefore there exists the limit $K_{\Omega}f(x) = K_{\Omega}f_k(x) + K_{\Omega}[f - f_k](x)$ at each $x \in \partial\Omega \cap B(0;k) \setminus M_k$. Since $\mathcal{H}_{m-1}(\cup M_k) = 0$, the limit $K_{\Omega}f(x)$ exists at almost all $x \in \partial\Omega$.

Lemma 5.3.4 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary or Ω be a graph Lipschitz domain, $\tau \in L^\infty(\partial\Omega, R^m)$, $1 < p < \infty$, $f \in L^p(\partial\Omega)$. For $x \in R^m$, $\epsilon > 0$ define*

$$T_\epsilon f(x) = \int\limits_{\partial\Omega\setminus B(x;\epsilon)} \frac{\tau(y)\cdot(y-x)}{|x-y|^m} f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y),$$

$$Tf(x) = \lim_{\epsilon\downarrow 0} T_\epsilon f(x).$$

If $z \in \partial\Omega$ is a Lebesgue's point of f and $f(z) = 0$, then the following assertions are equivalent:

1. *There exists the finite limit $Tf(z)$.*
2. *There exists a finite nontangential limit of Tf at z.*
3. *$Tf(z)$ is the finite nontangential limit of Tf at z.*

Proof Let $\delta, a > 0$. Fix a constant C_1 such that $\mathcal{H}_{m-1}(B(x;r) \cap \partial\Omega) \le C_1 r^{m-1}$ for all $x \in R^m$, $r > 0$ (compare Corollary 1.17.6). There exists a constant C_2 such that $|(x-y)|x-y|^{-m} - (z-y)|z-y|^{-m}| \le C_2|x-z||z-y|^{-m}$ for $|y-z| \ge 2|z-x|$. (See Lemma 2.9.7.) Let $|\tau| \le C_3$. Fix $r_1 > 0$ such that

$$\oint\limits_{B(z;r)\cap\partial\Omega} |f|\,\mathrm{d}\mathcal{H}_{m-1} \le C_4\delta,$$

where $C_4 = [6m(2^m C_1 + 1)(C_2 + 1)(C_3 + 1)]^{-1}$. Put $\mu = |f|\mathcal{H}_{m-1}|[B(z;r_1) \cap \partial\Omega]$. Then $\mu(B(z;r)) \le C_4\delta C_1 r^{m-1}$ for each $r > 0$. Fix $r_2 \in (0; r_1)$ such that

$$\int\limits_{\partial\Omega\setminus B(z;r_1)} \left| \frac{\tau(y)\cdot(y-x)}{|x-y|^m} - \frac{\tau(y)\cdot(y-z)}{|z-y|^m} \right| |f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y) < \frac{\delta}{3}$$

for $x \in B(z;r_2)$. Let $0 < \epsilon < r_2$. If $x \in \Gamma_a^\Omega(z) \cap \partial B(z;\epsilon/2)$, then

$$\begin{aligned}
|T_\epsilon f(z) - Tf(x)| \le{}& \left| \int\limits_{B(z;\epsilon)\cap\partial\Omega} \frac{\tau(y)\cdot(y-x)}{|x-y|^m} f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y) \right| \\
&+ \int\limits_{\partial\Omega\setminus B(z;r_1)} \left| \frac{\tau(y)\cdot(y-x)}{|x-y|^m} - \frac{\tau(y)\cdot(y-z)}{|z-y|^m} \right| |f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y) \\
&+ \int\limits_{\partial\Omega\cap B(z;r_1)\setminus B(z;\epsilon)} \left| \frac{\tau(y)\cdot(y-x)}{|x-y|^m} - \frac{\tau(y)\cdot(y-z)}{|z-y|^m} \right| |f(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)
\end{aligned}$$

$$\le 2^{m-1}C_1 \oint\limits_{B(z;\epsilon)\cap\partial\Omega} |f|\,\mathrm{d}\mathcal{H}_{m-1} + \frac{\delta}{3} + \int\limits_{R^m\setminus B(z;\epsilon)} \frac{C_2C_3|x-z|}{|z-y|^m}\,\mathrm{d}\mu(y)$$

$$\le \frac{2\delta}{3} + C_2C_3|x-z|C_1C_4\delta\frac{m}{\epsilon} \le \delta$$

by Lemma 1.26.1. The rest is trivial.

Theorem 5.3.5 *Let $\Omega \subset R^m$ be open, $m \ge 2$, $1 < p < \infty$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. Fix $a > 0$. Let $f \in L^p(\partial\Omega)$. If $x \in \partial\Omega$, denote by $[\mathcal{D}^\Omega f(x)]_+$ the nontangential limit of $\mathcal{D}^\Omega f$ at x with respect to Ω and by $[\mathcal{D}^\Omega f(x)]_-$ the nontangential limit of $\mathcal{D}^\Omega f$ at x with respect to $\omega = R^m \setminus \overline{\Omega}$. Then*

$$[\mathcal{D}^\Omega f(x)]_\pm = \pm\frac{1}{2}f(x) + K_\Omega f(x) \tag{5.12}$$

at almost all $x \in \partial\Omega$. Moreover,

$$\|M_a^\Omega(\mathcal{D}^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \tag{5.13}$$

where C depends only on Ω, p and a.

Proof We use the notation from Lemmas 5.2.5 and 5.2.7. If $f \in L^p(\partial\Omega)$, then

$$\mathcal{D}^\Omega f = -\sum_{j=1}^m L_j^\Omega(n_j^\Omega f).$$

According to Lemma 5.2.7, there exists a constant C such that (5.13) holds.

Now we prove (5.12). Suppose first that Ω is bounded. At almost all $z \in \partial\Omega$, there exists $n^\Omega(z)$ (see Remark 1.17.5), z is a Lebesgue's point of f (see Proposition 1.13.9), and there exists finite $K_\Omega f(z)$ (see Proposition 5.3.3). Fix such z. Denote $f_1(x) = f(z)$, $f_2(x) = f(x) - f(z)$. Then

$$[\mathcal{D}^\Omega f_1(z)]_\pm = \pm\frac{1}{2}f_1(z) + K_\Omega f_1(z)$$

by Propositions 5.3.1 and 5.3.2. Lemma 5.3.4 gives

$$[\mathcal{D}^\Omega f_2(z)]_\pm = \frac{1}{2}f_2(z) + K_\Omega f_2(z).$$

Adding, we obtain (5.12).

Let now $\partial\Omega$ be compact, Ω be unbounded. We get (5.12) from this relation for ω and the fact that $\mathcal{D}^\Omega f = -\mathcal{D}^\omega f$, $K_\Omega f = -K_\omega f$.

Let now Ω be a graph Lipschitz domain. For $k \in N$, define $f_k = f$ on $\partial\Omega \cap B(0;2k)$, $f_k = 0$ elsewhere. Then $K_\Omega(f - f_k)(z)$ is the nontangential limits of $\mathcal{D}^\Omega(f - f_k)$ for all $z \in \partial\Omega \cap B(0;k)$ (see Theorem 1.13.6). We can find a bounded open set $\Omega(k)$ with Lipschitz boundary such that $\partial\Omega \cap B(0;2k) = \partial\Omega(k) \cap B(0;2k)$. We have proved that $f_k(z)/2 + K_{\Omega(k)}f_k(z)$ is the nontangential limit of $\mathcal{D}^{\Omega(k)}f_k$ for $z \in \partial\Omega(k) \cap B(0;k) \setminus M_k$, where $\mathcal{H}_{m-1}(M_k) = 0$. Since $\mathcal{D}^\Omega f_k = \mathcal{D}^{\Omega(k)}f_k$, $K_\Omega f_k = K_{\Omega(k)}f_k$, we obtain that $f(z)/2 + K_\Omega f(z)$ is the nontangential limit of $\mathcal{D}^\Omega f$ for $z \in \partial\Omega \cap B(0;k) \setminus M_k$. Since $\mathcal{H}_{m-1}(\cup M_k) = 0$, we have this relation at almost all $z \in \partial\Omega$. The second relation from (5.12) follows from $K_\Omega(x,y) = -K_\omega(x,y)$, $K_\Omega f = -K_\omega f$, $\mathcal{D}^\Omega f = -\mathcal{D}^\omega f$.

5.4 The Trace of Derivatives of a Single Layer Potential

If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary and $f \in L^p(\partial\Omega)$ with $1 < p < \infty$, we prove that there exists a finite nontangential limit of $\nabla \mathcal{S}^\Omega f$ at almost all points of the boundary. Moreover, the normal derivative of $\mathcal{S}^\Omega f$ is equal to $\frac{1}{2}f - K'_\Omega f$. Further we study behavior of the single layer potential on the boundary. We show that $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$, $\mathcal{S}^\Omega : B_s^{p,q}(\partial\Omega) \to B_{s+1}^{p,q}(\partial\Omega)$ are bounded linear operators for $-1 < s < 0$ and $1 < q < \infty$.

Let $\Omega \subset R^m$ be open, $m \geq 2$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. For $x, y \in \partial\Omega$ such that $n^\Omega(x)$ exists, denote

$$K'_\Omega(x,y) = K_\Omega(y,x) = \frac{n(x)\cdot(x-y)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}.$$

$n^\Omega(x)$ exists at almost all $y \in \partial\Omega$ by Remark 1.17.5, and $K'_\Omega(x,y)$ is defined for almost all $x \in \partial\Omega$. For $f \in L^p(\partial\Omega)$, $1 < p < \infty$, $\epsilon > 0$, define

$$K'_{\Omega,\epsilon}f(x) = \int\limits_{\partial\Omega\setminus B(x;\epsilon)} K'_\Omega(x,y)f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

Hölder's inequality (Proposition 1.14.1) gives that $K'_{\Omega,\epsilon}f(x)$ is well-defined and finite for almost all $x \in \partial\Omega$. Set

$$K'_\Omega f(x) = \lim_{\epsilon\downarrow 0} K'_{\Omega,\epsilon}f(x) \tag{5.14}$$

whenever this limit exists.

Lemma 5.4.1 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. For $j,k \in \{1,\dots,m\}$, denote $\tau_{jk} = [\delta_{1j}n_k - \delta_{1k}n_j, \dots, \delta_{mj}n_k - \delta_{mk}n_j]$. If $x \in \overline{\Omega}$, $\epsilon > 0$, then*

$$\int\limits_{\partial\Omega\setminus B(x;\epsilon)} \tau_{jk}(y) \cdot \nabla h_m(y-x) \,\mathrm{d}\mathcal{H}_{m-1}(y) = 0.$$

Proof Easy calculation yields that $\tau_{jk}(y) \cdot \nabla h_m(y-x) = 0$ on $\partial B(x;\epsilon)$. By virtue of the Gauss-Green theorem (Theorem 1.19.1),

$$\int\limits_{\partial\Omega\setminus B(x;\epsilon)} \tau_{jk}(y) \cdot \nabla h_m(y-x) \,\mathrm{d}\mathcal{H}_{m-1}(y) = \int\limits_{\partial[\Omega\setminus B(x;\epsilon)]} \tau_{jk}(y) \cdot \nabla h_m(y-x)$$

$$= \int\limits_{\Omega\setminus B(x;\epsilon)} [\partial_j\partial_k h_m(y-x) - \partial_k\partial_j h_m(y-x)] \,\mathrm{d}\mathcal{H}_m(y) = 0.$$

Lemma 5.4.2 *Let ϕ be a Lipschitz function on R^{m-1}, $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \phi(x')\}$, $m \geq 2$, $1 < p < \infty$. For $j \in \{1,\dots,m-1\}$ define the vector function $v_j(y', y_m) = [\delta_{1j}, \delta_{2j}, \dots, \delta_{(m-1)j}, \partial_j\phi(y')]/\sqrt{1 + |\nabla\phi(y')|^2}$ on $\partial\Omega$. For $\epsilon > 0$, $f \in L^p(\partial\Omega)$ and $x \in R^m$ define*

$$Z^{\Omega}_{j,\epsilon}f(x) = \int\limits_{\partial\Omega\setminus B(x;\epsilon)} f(y) \frac{v_j(y)\cdot(y-x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} \,\mathrm{d}\mathcal{H}_{m-1}(y),$$

$$Z^{\Omega}_j f(x) = \lim_{\epsilon\downarrow 0} Z^{\Omega}_{j,\epsilon}f(x).$$

For almost all $z \in \partial\Omega$, there exists the finite limit $Z^{\Omega}_j f(z)$, and it is the nontangential limit of $Z^{\Omega}_j f$ at z.

Proof Suppose first that $g(y') = f(y', \phi(y')) \in \mathcal{C}^\infty_c(R^{m-1})$. Put $u_j(y', y_m) = [\partial_j g(y')]/\sqrt{1 + |\nabla\phi(y')|^2}$. For $x \in R^m \setminus \partial\Omega$, set $w_x(y') = h_m(x - [y', \phi(y')])$. We obtain by integration per partes

$$Z^{\Omega}_j f(x) = -\int\limits_{R^{m-1}} g(y')\partial_j w_x(y') \,\mathrm{d}\mathcal{H}_{m-1}(y') = \int\limits_{R^{m-1}} [\partial_j g(y')] w_x(y')$$

$$= \int\limits_{\partial\Omega} u_j(y) h_m(x-y) \,\mathrm{d}\mathcal{H}_{m-1}(y) = \mathcal{S}^{\Omega} u_j(x).$$

The nontangential limit $[Z^{\Omega}_j f(z)]_+ = [\mathcal{S}^{\Omega} u_j(z)]_+ = \mathcal{S}^{\Omega} u_j(z)$ at almost all $z \in \partial\Omega$. (See Proposition 5.2.3.)

Let now f be general. We use the notation from Lemma 5.2.7. Clearly

$$Z_j^\Omega f = L_j^\Omega \frac{-f}{\mathcal{H}_{m-1}(\partial B(0;1))\sqrt{1+|\nabla\phi|^2}} + L_m^\Omega \frac{-f\partial_j\phi}{\mathcal{H}_{m-1}(\partial B(0;1))\sqrt{1+|\nabla\phi|^2}}.$$

According to Lemma 5.2.7, there exists a positive constant C_1 depending on Ω and p such that

$$\|M_a^\Omega(Z_j^\Omega f)\|_{L^p(\partial\Omega)} \leq C_1\|f\|_{L^p(\partial\Omega)}.$$

We can choose $f_k \in \mathcal{C}_c^\infty(\partial\Omega)$ such that $f_k \to f$ in $L^p(\partial\Omega)$ (see Propositions 1.14.6 and 1.14.7). Then

$$\|M_a^\Omega(Z_j^\Omega(f-f_k))\|_{L^p(\partial\Omega)} \leq C_1\|f-f_k\|_{L^p(\partial\Omega)} \to 0.$$

According to Proposition 1.14.8, we may suppose that $M_a^\Omega(Z_j^\Omega(f-f_k))(z) \to 0$ for $z \in \partial\Omega \setminus A_0$ with $\mathcal{H}_{m-1}(A_0) = 0$. Moreover, we can suppose that each $z \in \partial\Omega \setminus A_0$ is a Lebesgue point of f (see Proposition 1.13.9). We have proved that $[Z_j^\Omega f_k(z)]_+$ is finite for $z \in \partial\Omega \setminus A_k$ with $\mathcal{H}_{m-1}(A_k) = 0$. Let $z \in \partial\Omega \setminus \cup A_k$. Fix $\epsilon > 0$. Fix $k \in N$ such that $|M_a^\Omega(Z_j^\Omega(f-f_k))(z)| < \epsilon/3$. Fix $\delta > 0$ such that $|[Z_j^\Omega f_k(z)]_+ - Z_j^\Omega f_k(x)| < \epsilon/3$ for $x \in \Gamma_a^\Omega(z) \cap B(z;\delta)$. If $x, y \in \Gamma_a^\Omega(z) \cap B(z;\delta)$, then

$$\begin{aligned}|Z_j^\Omega f(x) - Z_j^\Omega f(y)| \leq |M_a^\Omega(Z_j^\Omega(f-f_k))(z)| + |[Z_j^\Omega f_k(z)]_+ - Z_j^\Omega f_k(x)|\\ +|[Z_j^\Omega f_k(z)]_+ - Z_j^\Omega f_k(y)| < \epsilon.\end{aligned}$$

Therefore there exists finite $[Z_j^\Omega f(z)]_+$, the nontangential limits of $Z_j^\Omega f$ at z. Fix $r > 0$. Define $g_1 = f$ on $\partial\Omega \setminus B(z;r)$, $g_1 = 0$ on $\partial\Omega \cap B(z;r)$, $g_2 = f(z)$ on $\partial\Omega \cap B(z;r)$, $g_2 = 0$ on $\partial\Omega \setminus B(z;r)$, $g_3 = f - g_1 - g_2$. Then there exists finite $Z_j g_1(z)$ and it is the nontangential limit of $Z_j g_1$ at z (see Lemma 5.3.4). Choose a bounded domain ω with Lipschitz boundary such that $B(z;r) \cap \omega = B(z;r) \cap \Omega$. Denote $\tau = [-\delta_{1j}n_m^\omega, \ldots, -\delta_{m-1j}n_m^\omega, n_j^\omega]$. Then $\tau = v_j$ on $\partial\Omega \cap B(z;r)$. According to Lemma 5.4.1,

$$Z_{j,\epsilon}^\Omega g_2(x) = f(z) \int\limits_{\partial\omega\setminus B(x;\epsilon)} \tau \cdot \nabla h_m(x-y)\, d\mathcal{H}_{m-1}(y)$$

$$-f(z) \int\limits_{\partial\omega\setminus B(x;r)} \tau \cdot \nabla h_m(x-y) = -f(z) \int\limits_{\partial\omega\setminus B(x;r)} \tau \cdot \nabla h_m(x-y)\, d\mathcal{H}_{m-1}.$$

Hence $Z_j^\Omega g_2(z)$ is finite, and it is the limit of $Z_j^\Omega g_2$ at z. (Compare Theorem 1.13.6.) Since there exist the nontangential limits of $Z_j^\Omega f$, $Z_j^\Omega g_1$ and $Z_j^\Omega g_2$ at z and they are finite, there exists the nontangential limit of $Z_j^\Omega g_3$ at z, and it is finite. According to

Lemma 5.3.4, there exists finite $Z_j^\Omega g_3(z)$, and it is the nontangential limit of $Z_j^\Omega g_3$ at z. So, $Z_j^\Omega f(z)$ is finite, and it is the nontangential limit of $Z_j^\Omega f$ at z.

Lemma 5.4.3 *Suppose that* $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \phi(x')\}$ *where* ϕ *is a Lipschitz function on* R^{m-1}. *Define the normal vector*

$$v_m(y', y_m) = [\partial_1\phi(y), \dots, \partial_{m-1}\phi(y'), -1]/\sqrt{1 + |\nabla\phi(y')|^2}.$$

For $j \in \{1, \dots, m-1\}$ *the tangential vectors* v_j *are defined in Lemma 5.4.2. Fix a vector* $\tau \in R^m$. *Then there exist* $g_1, \dots, g_m \in L^\infty(\partial\Omega)$ *such that* $\tau = g_1v_1 + \cdots + g_mv_m$. *If* τ *is a tangential vector to* $\partial\Omega$ *at some* $y \in \partial\Omega$, *then* $g_m(y) = 0$.

Proof $v_1(y), \dots, v_m(y)$ forms a basis of R^m. Denote by $A(y) = (a_{ij}(y))$ the matrix formed by the vectors $v_1(y), \dots, v_m(y)$. For $k = 1, \dots, m-1$, denote $B^k = (b_{ij}^k)$ the matrix of the type $(m-k+1) \times (m-k+1)$ with $b_{ij}^k = a_{i+k-1,j+k-1}$ and by c_k its determinant. Clearly $c_{m-1} = -[1 + (\partial_{m-1}\phi)^2]/([1 + |\nabla\phi|^2]$. Since

$$c_k = c_{k+1}/\sqrt{1 + |\nabla\phi|^2} - (\partial_k\phi)^2/(\sqrt{1 + |\nabla\phi|^2})^k,$$

the induction argument gives that $c_1 = -(1 + |\nabla\phi|^2)^{2-(m/2)}$. Fix a constant C_1 such that $|\nabla\phi| \le C_1$. We have $(1 + C_1)^{-m} \le |c_1(y)| \le 1$. Remember that $c_1(y)$ is the determinant of $A(y)$. The formula for the calculation of A^{-1} gives that all entries of the inverse matrix $A(y)^{-1}$ are from $L^\infty(\partial\Omega)$. So, there exist $g_1, \dots, g_m \in L^\infty(\partial\Omega)$ such that $\tau = g_1v_1 + \cdots + g_mv_m$. If τ is a tangential vector to $\partial\Omega$ at some $y \in \partial\Omega$, then $g_m(y) = 0$.

Proposition 5.4.4 *Let* Ω *be an open set with compact Lipschitz boundary or a graph Lipschitz domain,* $m \ge 2$, $1 < p < \infty$, $f \in L^p(\partial\Omega)$, $k \in \{1, \dots, m\}$. *Define*

$$L_{k,\epsilon}^\Omega f(x) = \int\limits_{\partial\Omega \setminus B(x;\epsilon)} \frac{x_k - y_k}{|x - y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y),$$

$$L_k^\Omega f(x) = \lim_{\epsilon \downarrow 0} L_{k,\epsilon}^\Omega f(x).$$

Then the limit $L_k^\Omega f(x)$ *exists for almost all* $x \in \partial\Omega$ *and* $L_k^\Omega f \in L^p(\partial\Omega)$.

Proof Suppose first that $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \phi(x')\}$ where ϕ is a Lipschitz function on R^{m-1}. Let v_j, $Z_{j,\epsilon}^\Omega g$, $Z_j^\Omega g$ have meaning from Lemmas 5.4.2 and 5.4.3. Denote $\tau = (\delta_{1k}, \dots, \delta_{mk})$. According to Lemma 5.4.3, there exist $g_1, \dots, g_m \in L^\infty(\partial\Omega)$ such that $\tau = g_1v_1 + \cdots + g_mv_m$. If $x \in \partial\Omega$, then

$$L_{k,\epsilon}^\Omega f(x) = \sum_{j=1}^{m-1} \mathcal{H}_{m-1}(\partial B(0;1)) Z_{j,\epsilon}^\Omega(fg_j)(x) + \mathcal{H}_{m-1}(\partial B(0;1)) K_\Omega(fg_m)(x).$$

Lemma 5.4.2 and Proposition 5.3.3 give that the limit $L_k^\Omega f(x)$ exists for almost all $x \in \partial\Omega$. Moreover, $L_k^\Omega f \in L^p(\partial\Omega)$ by Lemma 5.2.5.

Let now Ω have compact boundary. Choose $z^1, \dots z^l \in \partial\Omega$ and $\delta > 0$ such that $\partial\Omega \subset B(z^1;\delta) \cup \dots \cup B(z^l;\delta)$, and for each j there exists a coordinate system and $\phi_j \in \mathcal{C}^{0,1}(R^{m-1})$ with $B(z^j;\delta) \cap \Omega = B(z^j;\delta) \cap \{[x', x_m]; x_m > \phi_j(x')\}$. Define $f_j = f$ on $B(z^j;\delta) \cap \partial\Omega$, $f_j = 0$ elsewhere. If $x \in B(z^j;\delta) \cap \partial\Omega$, then $L_k^\Omega(f - f_j)(x)$ exists. We have proved that there exists $L_k^\Omega f_j(x)$ at almost all $x \in B(z^j;\delta) \cap \partial\Omega$. Therefore the limit $L_k^\Omega f(x)$ exists for almost all $x \in \partial\Omega$. Lemma 5.2.6 forces that $L_k^\Omega f \in L^p(\partial\Omega)$.

Proposition 5.4.5 *Let $\Omega \subset R^m$ be open, $m \geq 2$, $1 < p < \infty$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. If $f \in L^p(\partial\Omega)$, then there exists the limit $K'_\Omega f(x)$ given by (5.14) at almost all $x \in \partial\Omega$. If we denote*

$$K'_{\Omega,*} f(x) = \sup_{\epsilon > 0} |K'_{\Omega,\epsilon} f(x)|,$$

then

$$\|K'_\Omega f\|_{L^p(\partial\Omega)} \leq \|K'_{\Omega,*} f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$$

where a constant C depends only on Ω and p.

Proof

$$K'_\Omega f(x) = \sum_{k=1}^m \frac{n_k^\Omega(x)}{\mathcal{H}_{m-1}(\partial B(0;1))} L_k^\Omega f(x)$$

with $L_k^\Omega f$ from Proposition 5.4.4. The rest is a consequence of Proposition 5.4.4 and Lemma 5.2.5.

Lemma 5.4.6 *Let $\Omega \subset R^m$ be open, $m \geq 2$, $1 < p < \infty$, $p' = p/(p-1)$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain. Then K'_Ω on $L^{p'}(\partial\Omega)$ is the adjoint operator of the operator K_Ω on $L^p(\partial\Omega)$.*

Proof Let $f \in L^p(\partial\Omega)$, $g \in L^{p'}(\partial\Omega)$. By virtue of Propositions 5.3.3 and 5.4.5, Lebesgue lemma (Lemma 1.13.2), and Fubini's theorem,

$$\int_{\partial\Omega} gK_\Omega f = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega} gK_{\Omega,\epsilon} f = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega} fK'_{\Omega,\epsilon} g = \int_{\partial\Omega} fK'_\Omega g.$$

Theorem 5.4.7 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary or a graph Lipschitz domain, $w \in R^m \setminus \partial\Omega$, $1 < p < \infty$. If $f \in L^p(R^m)$, then there exist the finite nontangential limits $[\nabla\tilde{S}_w^\Omega f(z)]_+$, $[\nabla\tilde{S}_w^\Omega f(z)]_-$ at almost all $z \in \partial\Omega$ and*

$$[\nabla\tilde{S}_w^\Omega f(z)]_\pm = \pm\frac{1}{2} f(z) n^\Omega(z) + \lim_{\epsilon \downarrow 0} \int_{\partial\Omega \setminus B(z;\epsilon)} \frac{(y-z)f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m} \, d\mathcal{H}_{m-1}.$$

Proof According to Proposition 5.4.4, there exists the limit on the right side, and it is finite at almost all $z \in \partial\Omega$. Suppose first that $\Omega = \{[x', x_m]; x' \in R^{m-1}, x_m > \phi(x')\}$ where ϕ is a Lipschitz function on R^{m-1}. Fix $k \in \{1, \dots, m\}$. Put $\tau = (\delta_{1k}, \dots, \delta_{mk})$. Let $v_1, \dots, v_m$, Z_j^Ω have the sense from Lemmas 5.4.2 and 5.4.3. According to Lemma 5.4.3, there exist $g_1, \dots, g_m \in L^\infty(\partial\Omega)$ such that $\tau = g_1 v_1 + \dots + g_m v_m$. Since $v_m = n^\Omega$ and $v_1, \dots, v_{m-1}$ are orthogonal to v_m, we have $g_m = \tau \cdot v_m = n_k^\Omega$. If $x \in R^m \setminus \partial\Omega$, then

$$\partial_k \tilde{\mathcal{S}}_w^\Omega f(x) = \int\limits_{\partial\Omega} \frac{\tau \cdot (y - x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x - y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$= \sum_{j=1}^{m-1} Z_j(fg_j)(x) + \mathcal{D}^\Omega(n_k^\Omega f)(x).$$

According to Lemma 5.4.2 and Theorem 5.3.5, there exist finite

$$[\partial_k \mathcal{S}_w^\Omega f(z)]_\pm = \sum_{j=1}^{m-1} Z_j(fg_j)(x) \pm \frac{1}{2} n_k^\Omega(z) f(z) + K_\Omega(n_k^\Omega f)(z)$$

$$= \pm \frac{1}{2} n_k^\Omega(z) f(z) + \lim_{\epsilon \downarrow 0} \int\limits_{\partial\Omega \setminus B(z;\epsilon)} \frac{(y_k - z_k) f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z - y|^m}\, \mathrm{d}\mathcal{H}_{m-1}(y)$$

at almost all $z \in \partial\Omega$.

Suppose now that Ω has compact boundary. Then there exist $r > 0$ and $z^1, \dots, z^k \in \partial\Omega$ such that $\partial\Omega \subset B(z^1; r) \cup \dots \cup B(z^k; r)$, $B(z^j; 2r) \cap \Omega = B(z^j, 2r) \cap \omega_j$, where ω_j is a graph Lipschitz domain. Define $f_j = f$ on $B(z^j; 2r) \cap \partial\Omega$, $f_j = 0$ elsewhere, $\tilde{f}_j = f$ on $\partial\Omega \setminus B(z^j; 2r)$, $f_j = 0$ elsewhere. We have proved

$$[\nabla \tilde{\mathcal{S}}_w^\Omega f_j(z)]_\pm = \pm \frac{1}{2} f_j(z) n^\Omega(z) + \lim_{\epsilon \downarrow 0} \int\limits_{\partial\Omega \setminus B(z;\epsilon)} \frac{(y - z) f_j(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z - y|^m}.$$

$\nabla \tilde{\mathcal{S}}_w^\Omega \tilde{f}_j$ is continuous on $B(z^j; r)$ by Theorems 1.13.7 and 1.13.6. Thus

$$[\nabla \tilde{\mathcal{S}}_w^\Omega f(z)]_\pm = \pm \frac{1}{2} f(z) n^\Omega(z) + \lim_{\epsilon \downarrow 0} \int\limits_{\partial\Omega \setminus B(z;\epsilon)} \frac{(y - z) f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z - y|^m}.$$

Theorem 5.4.8 *Let $\Omega \subset R^m$ be open, $m \geq 2$, $1 < p < \infty$. Suppose that Ω has compact Lipschitz boundary or Ω is a graph Lipschitz domain, $w \in R^m \setminus \partial\Omega$. Let $f \in L^p(\partial\Omega)$. If $z \in \partial\Omega$, denote by $[\partial \tilde{\mathcal{S}}_w^\Omega f(z) / \partial n^\Omega(z)]_+$ the nontangential limit of $\partial \tilde{\mathcal{S}}_w^\Omega f(x) / \partial n^\Omega(z)$ at z with respect to Ω and by $[\partial \tilde{\mathcal{S}}_w^\Omega f(z) / \partial n^\Omega(z)]_-$ the nontangential* *limit of* $\partial \tilde{\mathcal{S}}^\Omega_w f(x)/\partial n^\Omega(z)$ *at* z *with respect to* $\omega = R^m \setminus \overline{\Omega}$. *Then*

$$\left[\frac{\partial \tilde{\mathcal{S}}^\Omega_w f(z)}{\partial n(z)}\right]_+ = \frac{1}{2} f(z) - K'_\Omega f(z), \quad \left[\frac{\partial \tilde{\mathcal{S}}^\Omega_w f(z)}{\partial n(z)}\right]_- = -\frac{1}{2} f(z) - K'_\Omega f(z)$$

at almost all $z \in \partial\Omega$.

Proof It is an easy consequence of Theorem 5.4.7 and Proposition 5.4.5.

Lemma 5.4.9 *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary,* $m \geq 2$, $1 < p < \infty$. *Then* $\mathcal{S}^\Omega$ *is a bounded linear operator from* $L^p(\partial\Omega)$ *into* $W^{1,p}(\partial\Omega)$. *Denote* $\nabla_\tau \mathcal{S}^\Omega f = \nabla \mathcal{S}^\Omega - (\nabla \mathcal{S} \cdot n^\Omega) n^\Omega$ *the nontangential gradient of* $\mathcal{S}^\Omega f$. *Then there exists a positive constant* C *such that*

$$C^{-1} \|\mathcal{S}^\Omega f\|_{W^{1,p}(\partial\Omega)} \leq \|\mathcal{S}^\Omega f\|_{L^p(\partial\Omega)} + \|\nabla_\tau \mathcal{S}^\Omega f\|_{L^p(\partial\Omega)} \leq C \|\mathcal{S}^\Omega f\|_{W^{1,p}(\partial\Omega)}$$

for all $f \in L^p(\partial\Omega)$.

Proof Fix $z^1, \ldots, z^k \in \partial\Omega$ and $r > 0$ such that $\partial\Omega \subset B(z^1; r) \cup \cdots \cup B(z^k; r)$. Suppose further that for each j there exist a coordinate system and a Lipschitz function Φ_j such that $\Omega \cap B(z^j; 2r) = B(z^j; 2r) \cap \{[x'; x_m]; x_m > \Phi_j(x')\}$. Choose $\alpha_j \in \mathcal{C}^\infty_c(B(z^j; r))$, $0 \leq \alpha_j \leq 1$ such that $\alpha_1 + \cdots + \alpha_k =$ on a neighborhood of $\partial\Omega$ (see Lemma 1.18.1). For a function g defined on $\partial\Omega$, put $g = 0$ outside $\partial\Omega$ and denote $g_j(x') = \alpha_j(x', \Phi_j(x')) g(x', \Phi_j(x'))$. Since Ω has Lipschitz boundary, there exists a positive constant C_1 such that

$$C_1^{-1} \|g\|_{L^p(\partial\Omega)} \leq \sum_{j=1}^k \|g_j\|_{L^p(R^{m-1})} \leq C_1 \|g\|_{L^p(\partial\Omega)}.$$

If $f \in L^p(\partial\Omega)$, then

$$\|\mathcal{S}^\Omega f\|_{L^p(\partial\Omega)} + \|\nabla \mathcal{S}^\Omega f\|_{L^p(\partial\Omega)} \leq C_2 \|f\|_{L^p(\partial\Omega)}$$

with C_2 independent on f (see Propositions 5.2.2 and 5.2.8). We shall study $\mathcal{S}^\Omega f$ on $B(z^j; r) \cap \partial\Omega$. For simplicity, suppose that $z^j = 0$. For $0 < \epsilon < r$, define $\psi_\epsilon(x') = \mathcal{S}^\Omega f(x', \Phi_j(x') - \epsilon)$. Since $\mathcal{S}^\Omega f \in H(\Omega) \subset \mathcal{C}^\infty(\Omega)$, we deduce that $\psi_\epsilon \in W^{1,p}(U)$ where $U = \{x' \in R^{m-1}; |x'| < r\}$ (see Proposition 1.21.14). Moreover, $\partial_l \psi_\epsilon(x') = \partial_l \mathcal{S}^\Omega f(x', \Phi_j(x') - \epsilon) + \partial_m \mathcal{S}^\Omega f(x', \Phi_j(x') - \epsilon) \partial_l \Phi_j(x')$. It is a derivative in a tangential direction. So, there exists a constant C_3 independent of f and ϵ such that

$$C_3^{-1} \|\nabla \psi_\epsilon\|_{L^p(U)} \leq \|\nabla_\tau \mathcal{S}^\Omega f(x', x_m - \epsilon)\|_{L^p(V)} \leq C_3 \|\nabla \psi_\epsilon\|_{L^p(U)},$$

where $V = \{[x', x_m] \in \partial\Omega \cap B(0; r), |x'| < r\}$. Proposition 5.2.2 gives $\mathcal{S}^\Omega f(x', x_m - \epsilon) \to \mathcal{S}^\Omega f(x' x_m)$ for almost all $x \in V$, $|\mathcal{S}^\Omega f(x', x_m - \epsilon)| \leq M^\Omega_a \mathcal{S}^\Omega f(x', x_m)$, $M^\Omega_a \mathcal{S}^\Omega f \in L^p(\partial\Omega)$. Proposition 5.2.8 and Theorem 5.4.7 give $|\nabla_\tau \mathcal{S}^\Omega f(x', x_m - \epsilon)| \leq$

$M_a^\Omega(\nabla \mathcal{S}^\Omega f)(x', x_m)$, $M_a^\Omega(\nabla \mathcal{S}^\Omega f) \in L^p(\partial\Omega)$, $\nabla_\tau \mathcal{S}^\Omega f(x', x_m - \epsilon) \to \nabla_\tau \mathcal{S}^\Omega f(x' x_m)$ for almost all $x \in V$. Lebesgue lemma (Lemma 1.13.2) gives that ψ_ϵ is a Cauchy sequence in $W^{1,p}(U)$ which converges to $\mathcal{S}^\Omega f$. Thus $\mathcal{S}^\Omega f \in W^{1,p}(\partial\Omega)$.

Corollary 5.4.10 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p, q < \infty$, $0 < s < 1$, $h \in L^\infty(\partial\Omega)$. Define $T_h f = \mathcal{S}^\Omega(hf)$. Then T_h is a compact linear operator on $W^{1,p}(\partial\Omega)$ and on $B_s^{p,q}(\partial\Omega)$.*

Proof The operator $\mathcal{S}^\Omega$ is a bounded linear operator from $L^p(\partial\Omega)$ to $W^{1,p}(\partial\Omega)$ by Lemma 5.4.9. Since the identity operator is a compact linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Lemma 1.21.8, the operator T_h is compact on $W^{1,p}(\partial\Omega)$ (see Theorem 1.7.4).

Fix t, τ such that $0 < \tau < s < t < 1$. The identity is a compact operator from $B_s^{p,q}(\partial\Omega)$ to $B_\tau^{p,p}(\partial\Omega) = W^{\tau,p}(\partial\Omega)$ by Theorems 1.23.14 and 1.23.3. Since $W^{\tau,p}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ and $T_h : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is bounded by Lemma 5.4.9, Theorem 1.7.4 gives that $T_h : B_s^{p,q}(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is compact. Lemma 1.21.16, Theorems 1.23.3 and 1.23.14 give $W^{1,p}(\partial\Omega) \hookrightarrow W^{t,p}(\partial\Omega) = B_t^{p,p}(\partial\Omega) \hookrightarrow B_s^{p,q}(\partial\Omega)$. Hence T_h is compact on $B_s^{p,q}(\partial\Omega)$ by Theorem 1.7.4.

Corollary 5.4.11 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $p' = p/(p-1)$. Then $\mathcal{S}^\Omega$ can be extended as a bounded linear operator from $W^{-1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$, which is the adjoint operator of $\mathcal{S}^\Omega : L^{p'}(\partial\Omega) \to W^{1,p'}(\partial\Omega)$. If $h \in L^\infty(\partial\Omega)$, then $h\mathcal{S}^\Omega$ is a compact linear operator on $W^{-1,p}(\partial\Omega)$. If $-1 < s < 0$ and $1 < p, q < \infty$, then $\mathcal{S}^\Omega$ is a bounded linear operator from $B_s^{p,q}(\partial\Omega)$ to $B_{s+1}^{p,q}(\partial\Omega)$.*

Proof Denote by S the adjoint operator of $\mathcal{S}^\Omega : L^{p'}(\partial\Omega) \to W^{1,p'}(\partial\Omega)$. Then S is a bounded linear operator from $W^{-1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Proposition 1.5.11. If $f \in L^p(\partial\Omega)$, $g \in L^{p'}(\partial\Omega)$, then Fubini's theorem gives

$$\langle Sf, g\rangle = \int\limits_{\partial\Omega} f\mathcal{S}^\Omega g \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} g\mathcal{S}^\Omega f \, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus $Sf = \mathcal{S}^\Omega f$.

$h\mathcal{S}^\Omega$ is a bounded operator from $W^{-1,p}(\partial\Omega)$ to $L^p(\partial\Omega) \subset W^{-1,p}(\partial\Omega)$. Denote $T_h f = \mathcal{S}^\Omega(hf)$. Then T_h is a compact linear operator on $W^{1,p'}(\partial\Omega)$ by Corollary 5.4.10. Since $h\mathcal{S}^\Omega$ is the adjoint operator of T_h, Theorem 1.7.5 gives that $h\mathcal{S}^\Omega$ is a compact linear operator on $W^{-1,p}(\partial\Omega)$.

Let now $-1 < s < 0$ and $1 < p, q < \infty$. An interpolation argument gives that $\mathcal{S}^\Omega$ is a bounded linear operator from $B_s^{p,q}(\partial\Omega)$ to $B_{s+1}^{p,q}(\partial\Omega)$ (see Proposition 1.10.3, Theorems 1.23.16, 1.23.5, and 1.22.1).

5.5 Derivatives of a Double Layer Potential

We show for an open set $\Omega \subset R^m$ with compact Lipschitz boundary and $f \in W^{1,p}(\partial\Omega)$, $1 < p < \infty$, that there exists a nontangential limit of $\nabla\mathcal{D}^\Omega f$ at almost all points of $\partial\Omega$, $\|M_a^\Omega(\nabla\mathcal{D}^\Omega f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\partial\Omega)}$, and K_Ω is a bounded linear operator on $W^{1,p}(\partial\Omega)$.

Lemma 5.5.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$, $f \in W^{1,p}(\partial\Omega)$. If $x \in \Omega$, then*

$$\partial_j\mathcal{D}^\Omega f(x) = \sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}}f)(x),$$

where $\partial_{\tau_{jk}} = n_j^\Omega\partial_k - n_k^\Omega\partial_j$.

Proof Fix $x \in \Omega$. Suppose that $f \in \mathcal{C}^\infty(R^m)$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $\partial\Omega$, $\varphi = 0$ on a neighborhood of x and put $\tilde{f} = f\varphi$.

$$\sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}}f)(x) = \sum_{k=1}^m \int_{\partial\Omega} [-\partial_k h_m(x-\cdot)][n_j\partial_k\tilde{f} - n_k\partial_j\tilde{f}]\, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \sum_{k=1}^m \int_{\partial\Omega} [n_k\partial_j(\tilde{f}\partial_k h_m(x-\cdot)) - n_j\partial_k(\tilde{f}\partial_k h_m(x-\cdot))]\, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int_{\partial\Omega} \tilde{f}\Big[n_j\Delta h_m(x-\cdot) - \sum_{j=1}^k n_k\partial_k\partial_j h(x-\cdot)\Big]\, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int_\Omega \sum_{k=1}^m [\partial_j\partial_k(\tilde{f}h_m(x-\cdot)) - \partial_k\partial_j(\tilde{f}h_m(x-\cdot))]\, \mathrm{d}\mathcal{H}_m + \partial_j\mathcal{D}^\Omega f(x) = \partial_j\mathcal{D}^\Omega f(x).$$

Let now $f \in W^{1,p}(\partial\Omega)$. Choose $z^1, \dots, z^K \in \partial\Omega$ and $r > 0$ such that $\partial\Omega \subset B(z^1; r) \cup \dots \cup B(z^K; r)$, and for each $s \in \{1, \dots, K\}$, there exist a coordinate system and a Lipschitz function Φ_s on R^{m-1} with $B(z^s; r) \cap \Omega = B(z^s; r) \cap \{[x', x_m]; x_m > \Phi_s(x')\}$. According to Lemma 1.18.1, there exist $\alpha_s \in \mathcal{C}_c^\infty(B(z^s; r))$ such that $\alpha_1 + \dots + \alpha_K = 1$ on $\partial\Omega$. Fix s and denote $g(y') = f(y', \Phi_j(y'))\alpha_s(y', \Phi(y'))$. Then $g \in \mathring{W}^{1,p}(R^m)$. So, there exist $g_l \in \mathcal{C}_c^\infty(R^{m-1})$ such that $g_l \to g$ in $W^{1,p}(R^{m-1})$. Putting $\tilde{g}_l(y', y_m) = g_l(y')$, we have

$$\partial_j\mathcal{D}^\Omega(\alpha_s f) = \lim_{l\to\infty} \partial_j\mathcal{D}^\Omega\tilde{g}_l = \lim_{l\to\infty}\sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}}\tilde{g}_l) = \sum_{k=1}^m \partial_k\mathcal{S}^\Omega[\partial_{\tau_{jk}}(\alpha_s f)],$$

$$\partial_j\mathcal{D}^\Omega f = \sum_{s=1}^K \partial_j\mathcal{D}^\Omega(\alpha_s f) = \sum_{s=1}^K\sum_{k=1}^m \partial_k\mathcal{S}^\Omega[\partial_{\tau_{jk}}(\alpha_s f)] = \sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}}f).$$

Theorem 5.5.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$, $0 < a < \infty$. If $f \in W^{1,p}(\partial\Omega)$, then there exists the nontangential limit of $\nabla\mathcal{D}^\Omega f$ at almost all points of $\partial\Omega$, and*

$$\|M_a^\Omega(\nabla\mathcal{D}^\Omega f)\|_{L^p(\partial\Omega)} \leq C\|\nabla_\tau f\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\partial\Omega)} \tag{5.15}$$

with a constant C dependent only on Ω, p, and a. (Here $\nabla_\tau f$ is the tangential gradient of f). Denote by $[\,]_+$ the nontangential limit with respect to Ω and by $[\,]_-$ the nontangential limit with respect to $R^m \setminus \overline{\Omega}$. Then $n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_+ = n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_-$. We define $H^\Omega f = n^\Omega \cdot \nabla\mathcal{D}^\Omega f$. Then the so-called hypersingular operator H^Ω is a bounded linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$.

Proof Suppose first that Ω is bounded. According to Lemma 5.5.1,

$$\partial_j\mathcal{D}^\Omega f(x) = \sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}}f)(x). \tag{5.16}$$

Theorem 5.4.7 and Proposition 5.2.8 give that there exists the nontangential limit of $\nabla\mathcal{D}^\Omega f$ at almost all points of $\partial\Omega$ and (5.15) holds with a constant C dependent only on Ω, p, and a.

Let now Ω be unbounded. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$ and put $\omega = B(0;r) \cap \Omega$. Define $f = 0$ on $\partial\omega \setminus \partial\Omega$. Then $D^\Omega f = D^\omega f$. So, there exists the nontangential limit of $\nabla\mathcal{D}^\Omega f$ at almost all points of $\partial\Omega$ and (5.15) holds.

We now we prove $n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_+ = n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_-$. For the sake of definiteness, suppose that Ω is bounded. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$ and put $\omega = B(0;r) \setminus \overline{\Omega}$. Define $f = 0$ on $\partial\omega \setminus \partial\Omega$. Using (5.16) and Theorem 5.4.7 for Ω and ω, we obtain

$$n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_+ - n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_- = \sum_{j=1}^m n_j^\Omega \sum_{k=1}^m n_k^\Omega \partial_{\tau_{jk}} f = 0.$$

Corollary 5.5.3 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $f \in W^{1,p}(\partial\Omega)$. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$, $g = \partial f/\partial\tau$. Then $\partial_1\mathcal{D}^\Omega f = \partial_2\mathcal{S}^\Omega g$, $\partial_2\mathcal{D}^\Omega f = -\partial_1\mathcal{S}^\Omega g$, and the nontangential limits*

$$\tau \cdot [\nabla\mathcal{D}^\Omega f]_\pm = -n^\Omega \cdot [\nabla\mathcal{S}^\Omega g]_\pm = \mp\frac{g}{2} + K'_\Omega g,$$

$$n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_\pm = \tau \cdot [\nabla\mathcal{S}^\Omega g]_\pm.$$

Proof That is a consequence of Lemma 5.5.1 and Theorem 5.4.8.

Lemma 5.5.4 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. Then K_Ω is a bounded linear operator on $W^{1,p}(\partial\Omega)$.*

Proof If we study $\mathcal{D}^{\Omega} f$ on Ω, then Theorems 5.3.5, 5.5.2 and Lemma 5.5.1 give that $(\frac{1}{2}I + K_{\Omega})$ is a bounded linear operator on $W^{1,p}(\partial\Omega)$. Consequently, K_{Ω} is a bounded linear operator on $W^{1,p}(\partial\Omega)$.

5.6 Integral Representation

In this section, we express a harmonic function as the sum of the double layer potential corresponding to its trace on the boundary and the single layer potential corresponding to the Neumann condition.

Theorem 5.6.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $u \in H(\Omega)$, $M_a^{\Omega}(\nabla u) \in L^p(\partial\Omega)$, $1 < p < \infty$. Then $M_a^{\Omega}(u) \in L^p(\partial\Omega)$, and there exist nontangential limits $u(x)$, $\nabla u(x)$ of u, and ∇u at almost all $x \in \partial\Omega$. Moreover, $u, \partial_j u \in L^p(\partial\Omega)$ and $u = \mathcal{D}^{\Omega} u + \mathcal{S}^{\Omega}(n^{\Omega} \cdot \nabla u)$.*

Proof Suppose first that Ω is connected. Let $\Omega(j)$, Λ_j and ω_j have meaning from Lemma 1.17.13. According to Theorem 2.2.4,

$$u = \mathcal{S}^{\Omega(j)} n^{\Omega(j)} \cdot \nabla u + \mathcal{D}^{\Omega(j)} u \tag{5.17}$$

in $\Omega(j)$. Fix $k \in \{1, \dots, m\}$. By virtue of Lemma 5.5.1,

$$\partial_k u = \partial_k \mathcal{S}^{\Omega(j)}(n^{\Omega(j)} \cdot \nabla u) + \sum_{l=1}^{m} \partial_l \mathcal{S}^{\Omega(j)}[n_k^{\Omega(j)} \partial_l u - n_l^{\Omega(j)} \partial_k u].$$

So, there exist constants a_{rst} such that

$$\partial_k u(x) = \sum_{r,s,t=1}^{m} a_{rst} \int\limits_{\partial\Omega(j)} [\partial_r h_m(x-y)] n_s^{\Omega(j)}(y) \partial_t u(y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Using notations from Lemma 1.17.13,

$$\partial_k u(x) = \sum_{r,s,t=1}^{m} a_{rst} \int\limits_{\partial\Omega} \omega_j(y) [\partial_r h_m(x - \Lambda_j(y))] n_s^{\Omega(j)}(\Lambda_j(y)) \partial_t u(\Lambda_j(y)).$$

Since $M_a^{\Omega}(\nabla u) \in L^p(\partial\Omega)$, the sequence $(\partial_t u) \circ \Lambda_j$ is bounded in $L^p(\partial\Omega)$. According to Theorem 1.5.10, we can suppose that $(\partial_t u) \circ \Lambda_j$ converges weakly to $v_t \in L^p(\partial\Omega)$. Since $\partial_r h_m(x - \Lambda_j(y))$ converges uniformly to $\partial_r h_m(x-y)$, $\omega_j \to 1$ pointwise and in all $L^q(\partial\Omega)$, $n^{\Omega(j)}(\Lambda_j(y)) \to n^{\Omega}(y)$ pointwise and in all $L^q(\partial\Omega)$, we obtain by virtue of Hölder's inequality (Proposition 1.14.1)

$$\partial_k u(x) = \sum_{r,s,t=1}^{m} a_{rst} \left\{ \int_{\partial\Omega} [\partial_r h_m(x-(y))] n_s^{\Omega}(y) \partial_t u(\Lambda_j(y)) \right.$$

$$\left. + \int_{\partial\Omega} [n_s^{\Omega(j)}(\Lambda_j(y))\omega_j(y)\partial_r h_m(x-\Lambda_j(y)) - n_s^{\Omega}(y)\partial_t h_m(x-y))]\partial_t u(\Lambda_j(y)) \right\}$$

$$\to \sum_{r,s,t=1}^{m} a_{rst} \int_{\partial\Omega} [\partial_r h_m(x-(y))] n_s^{\Omega}(y) v_t(y) = \sum_{r,s,t=1}^{m} a_{rst} \partial_r \mathcal{S}^{\Omega}(n_s^{\Omega} v_t)(x).$$

Theorem 5.4.7 gives that there exists the nontangential limit of $\partial_k u$ at almost all points of $\partial\Omega$.

Since Ω has Lipschitz boundary, there exist positive constants a, C_1 such that if $z \in \partial\Omega$, $x \in \Gamma_a^{\Omega}(z)$, then x, z can be connected by a smooth curve in $\Gamma_a^{\Omega}(z)$ with the length at most C_1. So, each $x, y \in \Gamma_a^{\Omega}(z)$ can be connected by a smooth curve in $\Gamma_a^{\Omega}(z)$ with the length at most C_2, where C_2 is a constant. Fix $\epsilon > 0$ small. Since the set $K = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) \geq \epsilon/(2+2a)\}$ is compact, there exists a constant C_3 such that $|u| \leq C_3$ on K. Let now $z \in \partial\Omega$, $x \in \Gamma_a^{\Omega}(z)$. If $|x-z| \geq \epsilon/2$, then $x \in K$ and $|u(x)| \leq C_3$. Let now $x \in B(z; \epsilon/2)$. Fix $y \in \Gamma_a^{\Omega}(z) \setminus B(z; \epsilon/2)$. Then x, y can be connected by a smooth curve γ in $\Gamma_a^{\Omega}(z)$ with the length at most C_2. Since $|\nabla u| \leq M_a^{\Omega}(\nabla u)(z)$ on $\gamma \subset \Gamma_a^{\Omega}(z)$, we have

$$|u(x)| \leq |u(y)| + |u(x) - u(y)| \leq C_3 + \int_{\gamma} |\nabla u| \, d\mathcal{H}_1 \leq C_3 + C_2 M_a^{\Omega}(\nabla u)(z).$$

Hence $M_a^{\Omega}(u) \leq C_3 + C_2 M_a^{\Omega}(\nabla u) \in L^p(\partial\Omega)$.

According to (5.17), we have

$$u(x) = \int_{\partial\Omega} \omega_j(y) h_m(x-\Lambda_j(y)) n^{\Omega(j)}(\Lambda_j(y)) \cdot \nabla u(\Lambda_j(y)) \, d\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega} \omega_j(y) \nabla h_m(x-\Lambda_j(y)) \cdot n^{\Omega(j)}(\Lambda_j(y)) u(\Lambda_j(y)) \, d\mathcal{H}_{m-1}(y).$$

The Lebesgue lemma (Lemma 1.13.2) gives

$$\int_{\partial\Omega} \omega_j(y) h_m(x-\Lambda_j(y)) n^{\Omega(j)}(\Lambda_j(y)) \cdot \nabla u(\Lambda_j(y)) \, d\mathcal{H}_{m-1}(y) \to \mathcal{S}^{\Omega} \frac{\partial u}{\partial n}(x).$$

Since $u \circ \Lambda_j$ is a bounded sequence in $L^p(\partial\Omega)$, we can suppose that there exists $g \in L^p(\partial\Omega)$ such that $u \circ \Lambda_j \to g$ weakly (see Theorem 1.5.10). Thus

$$\int_{\partial\Omega} \nabla h_m(x-y) \cdot n^\Omega(y) u(\Lambda_j(y))\, \mathrm{d}\mathcal{H}_{m-1}(y) \to \mathcal{D}^\Omega u(x).$$

Since $\omega_j(y)\nabla h_m(x-\Lambda_j(y)) \cdot n^{\Omega(j)}(\Lambda_j(y)) \to \nabla h_m(x-y) \cdot n^\Omega(y)$ in $L^{p'}(\partial\Omega)$ with $p' = p/(p-1)$, Hölder's inequality (Proposition 1.14.1) gives

$$\int_{\partial\Omega} [\omega_j(y)\nabla h_m(x-\Lambda_j(y)) \cdot n^{\Omega(j)}(\Lambda_j(y)) - \nabla h_m(x-y) \cdot n^\Omega(y)] u(\Lambda_j(y)) \to 0.$$

Hence

$$u(x) = \mathcal{S}^\Omega(\partial u/\partial n)(x) + \mathcal{D}^\Omega g(x).$$

Proposition 5.2.2 and Theorem 5.3.5 give that there exists the nontangential limit of u at almost all points of $\partial\Omega$. If $\varphi \in L^{p'}(\partial\Omega)$, then the Lebesgue lemma (Lemma 1.13.2) gives

$$\int_{\partial\Omega} g\varphi\, \mathrm{d}\mathcal{H}_{m-1} = \lim_{j\to\infty} \int_{\partial\Omega} \varphi(u \circ \Lambda_j)\, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} u\varphi\, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus $g = u$ and $u = \mathcal{S}^\Omega(\partial u/\partial n) + \mathcal{D}^\Omega u$.

Suppose now that Ω is not connected. Let $\Omega(j)$, Λ_j and ω_j have meaning from Lemma 1.17.13. According to Theorem 2.2.4, we get (5.17). Letting $j \to \infty$, we get $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)$ by the Lebesgue lemma (Lemma 1.13.2).

Corollary 5.6.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $u \in H(\Omega)$, $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$. If Ω is unbounded, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then there are the nontangential limits of u and ∇u at almost all points of $\partial\Omega$ and*

$$\mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in R^m \setminus \overline{\Omega}. \end{cases}$$

Proof Suppose first that Ω is bounded and connected. Then there are the nontangential limits of u and ∇u at almost all points of $\partial\Omega$ and $\mathcal{D}^\Omega u + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u) = u$ in Ω. (See Theorem 5.6.1.) Let now $x \in R^m \setminus \overline{\Omega}$. Choose $\Omega(j)$ from Lemma 1.17.13. We have $\mathcal{D}^{\Omega u(j)}(x) + \mathcal{S}^{\Omega(j)}(n^\Omega \cdot \nabla u)(x) = 0$ by Corollary 2.2.5. Lebesgue lemma (Lemma 1.13.2) forces $\mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)(x) = 0$.

If Ω is bounded, we use the proposition for each component of Ω.

Let now Ω is unbounded. We use the proposition for $\Omega(r) = \Omega \cap B(0;r)$. Since $u(y) \to 0$ as $|y| \to \infty$, we have $\partial^\alpha u(y) = O(|y|^{2-m-|\alpha|})$ as $|y| \to \infty$ for $m > 2$; $\partial^\alpha u(y) = O(|y|^{-1-|\alpha|})$ as $|y| \to \infty$ for $m = 2$. (See Proposition 2.17.3.) If $x \in \Omega$, then

$$u(x) = \lim_{r\to\infty} [\mathcal{D}^{\Omega(r)}u(x) + \mathcal{S}^{\Omega(r)}(n^\Omega \cdot \nabla u)(x)] = \mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)(x).$$

If $x \in R^m \setminus \overline{\Omega}$, then

$$0 = \lim_{r\to\infty} [\mathcal{D}^{\Omega(r)}u(x) + \mathcal{S}^{\Omega(r)}(n^\Omega \cdot \nabla u)(x)] = \mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)(x).$$

Corollary 5.6.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $u \in H(\Omega)$, $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$. If Ω is unbounded, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then there exists the nontangential limit $\nabla u(x)$ of ∇u at almost all $x \in \partial\Omega$. If $a > 0$, then there exists a constant C depending only on Ω, p, and a such that*

$$\|\nabla u\|_{L^p(\partial\Omega)} \le \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \le C\|\nabla u\|_{L^p(\partial\Omega)}.$$

Proof Corollary 5.6.2 gives that there exist the nontangential limits of u and ∇u at almost all points of $\partial\Omega$ and $u = \mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u)(x)$. The rest is a consequence of Proposition 5.2.8 and Theorem 5.5.2.

Definition 5.6.4 Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. If F is a distribution supported on $\partial\Omega$, we say that u is a very weak L^p-solution of the Neumann problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = F \quad \text{on } \partial\Omega,$$

if $u \in H(\Omega)$, $M_a^\Omega(u) \in L^p(\partial\Omega)$, there exists the nontangential limit $u(x)$ of u at almost all $x \in \partial\Omega$, and

$$\int_\Omega u(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi\rangle \quad \forall\varphi \in \mathcal{C}_c^\infty(R^m).$$

Lemma 5.6.5 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, F be a distribution supported on $\partial\Omega$. Let u be a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = F$ on $\partial\Omega$. Let $\omega \subset R^m$ be an open set with compact Lipschitz boundary such that $\partial\omega \subset \Omega$. Define $\tilde{F} = F|\omega + (\partial u/\partial n^\omega)\mathcal{H}_{m-1}|\partial\omega$, $\tilde{\Omega} = \Omega \cap \omega$. Then u is a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in $\tilde{\Omega}$, $\partial u/\partial n = \tilde{F}$ on $\partial\tilde{\Omega}$.*

Proof Let $\varphi \in \mathcal{C}_c^\infty(R^m)$. Since $\partial\omega \subset \Omega$, there exists $\theta \in \mathcal{C}^\infty(R^m)$ such that $\theta = 1$ on a neighborhood of $\overline{\omega}$ and $\theta = 0$ on a neighborhood of $\partial\Omega \setminus \overline{\omega}$. Denote $\tilde{\varphi} = \theta\varphi$.

Choose $\psi \in \mathcal{C}_c^\infty(R^m)$ such that $\psi = 1$ on a neighborhood of $\partial\omega$, $\psi = 0$ on a neighborhood of $R^m \setminus \Omega$. Choose a bounded open set $G \subset \Omega \setminus \tilde{\Omega}$ with smooth boundary such that $(\Omega \setminus \tilde{\Omega}) \cap \operatorname{spt}[\tilde{\varphi}(1-\psi)] \subset G$. By virtue of the Gauss-Green theorem (Theorem 1.19.1),

$$\langle \tilde{F}, \varphi \rangle = \langle F, (1-\psi)\tilde{\varphi} \rangle + \int\limits_{\partial\omega} \psi\tilde{\varphi} \frac{\partial u}{\partial n^\omega} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} u\{-\Delta[(1-\psi)\tilde{\varphi}]\} \, \mathrm{d}\mathcal{H}_m$$

$$+ \int\limits_{\partial\Omega} u \frac{\partial \tilde{\varphi}}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\tilde{\Omega}} u[-\Delta(\psi\tilde{\varphi})]\} \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\omega} u \frac{\partial \tilde{\varphi}}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{\tilde{\Omega}} u(-\Delta\tilde{\varphi}) \, \mathrm{d}\mathcal{H}_m + \int\limits_{G} u\{-\Delta[(1-\psi)\tilde{\varphi}]\} \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\tilde{\Omega}} u \frac{\partial \tilde{\varphi}}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{\tilde{\Omega}} u(-\Delta\tilde{\varphi}) \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial G} \left[\frac{\partial u}{\partial n}(1-\psi)\tilde{\varphi} - u \frac{\partial(1-\psi)\tilde{\varphi}}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int\limits_{\partial\tilde{\Omega}} u \frac{\partial \tilde{\varphi}}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\tilde{\Omega}} u(-\Delta\varphi) \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\tilde{\Omega}} u \frac{\partial \varphi}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}.$$

Proposition 5.6.6 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, f be a distribution supported on $\partial\Omega$. Let u be a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = f$ on $\partial\Omega$. If Ω is unbounded, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then*

$$\mathcal{D}^\Omega u(x) + \mathcal{S}^\Omega f(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in R^m \setminus \overline{\Omega}. \end{cases} \tag{5.18}$$

Proof Suppose first that Ω is bounded. Choose $\psi \in \mathcal{C}_c^\infty(R^m)$ such that $\psi = 1$ on a neighborhood of $\overline{\Omega}$. Fix $x \in R^m \setminus \overline{\Omega}$ and put $\varphi(y) = \psi(y)h_m(x-y)$. Then $\varphi \in H(\Omega)$ by Remark 2.1.3. By the definition of a very weak L^p-solution of the Neumann problem,

$$\mathcal{S}^\Omega f(x) = \langle f, \varphi \rangle = \int\limits_{\Omega} u(-\Delta\varphi) \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = -\mathcal{D}^\Omega u(x).$$

Let now $x \in \Omega$. Fix $r > 0$ such that $\overline{B(x;r)} \subset \Omega$. Define $g = \partial u/\partial n^{B(x;r)}$. Theorem 2.2.4 gives

$$u(x) = \mathcal{D}^{B(x;r)} u(x) + \mathcal{S}^{B(x;r)} g(x).$$

Put $\omega = \Omega \setminus \overline{B(x;r)}$. Lemma 5.6.5 gives that u is a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in ω, $\partial u/\partial n = F := f - g\mathcal{H}_{m-1}|\partial B(x;r)$. We have proved

$$\mathcal{D}^{\omega}u(x) + \mathcal{S}^{\omega}F(x) = 0.$$

Adding

$$u(x) = \mathcal{D}^{B(x;r)}u(x) + \mathcal{S}^{B(x;r)}g(x) + \mathcal{D}^{\omega}u(x) + \mathcal{S}^{\omega}F(x) = \mathcal{D}^{\Omega}u(x) + \mathcal{S}^{\Omega}f(x).$$

Let Ω be unbounded. For $r > 0$, define $\Omega_r = \Omega \cap B(0;r)$, $f_r = f + (\partial u/\partial n)\mathcal{H}_{m-1}|\partial B(0;r)$. Then u is a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in Ω_r, $\partial u/\partial n = f_r$ on $\partial\Omega_r$ (see Lemma 5.6.5). We have proved (5.18) for Ω_r. According to Proposition 2.17.3, there exists a constant C such that

- for $m > 2$, we have $|u| \le Cr^{2-m}$, $|\nabla u| \le Cr^{1-m}$ on $\partial B(0;r)$;
- for $m = 2$, we have $|u| \le Cr^{-1}$, $|\nabla u| \le Cr^{-2}$ on $\partial B(0;r)$.

Letting $r \to \infty$, we obtain (5.18) for Ω.

Corollary 5.6.7 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary,* $1 < p < \infty$, f_+, f_- *be distributions supported on* $\partial\Omega$. *Denote* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Let* $u_\pm$ *be an* L^p*-solution or a very weak* L^p*-solution of the Neumann problem* $\Delta u_\pm = 0$ *in* $\Omega_\pm$, $\partial u_\pm/\partial n^\Omega = f_\pm$ *on* $\partial\Omega_\pm$. *Put* $g = u_+ - u_-$ *on* $\partial\Omega$. *If* $u_-(x) \to 0$ *as* $|x| \to \infty$, *then*

$$u_\pm = \mathcal{D}^{\Omega}g + \mathcal{S}^{\Omega}(f_+ - f_-) \qquad \text{in } \Omega_\pm. \tag{5.19}$$

Proof Corollary 5.6.2 and Proposition 5.6.6 give

$$\pm\mathcal{D}^{\Omega}u_\pm(x) \pm \mathcal{S}^{\Omega}f_\pm(x) = \begin{cases} u_\pm(x), & x \in \Omega_\pm, \\ 0, & x \in \Omega_\mp. \end{cases}$$

Adding, we obtain (5.19).

5.7 Identities for Integral Operators

In this section, we prove two useful relations between integral operators on the boundary.

Proposition 5.7.1 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary and* $1 < p < \infty$. *If* $f \in L^p(\partial\Omega)$, *then*

$$\left(\frac{1}{2}I - K_\Omega\right)\mathcal{S}^{\Omega}f = \mathcal{S}^{\Omega}\left(\frac{1}{2}I - K'_\Omega\right)f.$$

Proof Put $u = \mathcal{S}^\Omega f$. Then $M_a(u) + M_a(\nabla u) \in L^p(\partial\Omega)$, $u = \mathcal{S}^\Omega f$ is the nontangential limit of u and $\partial u/\partial n = \frac{1}{2}f - K'_\Omega f$. (See Propositions 5.2.2, 5.2.8 and Theorem 5.4.8.) Theorem 5.6.1 gives $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(\partial u/\partial n)$. According to the boundary behavior of potentials (Proposition 5.2.2 and Theorem 5.3.5), we have $u = \frac{1}{2}u + K_\Omega u + \mathcal{S}^\Omega(\partial u/\partial n)$ on $\partial\Omega$. Since $u = \mathcal{S}^\Omega f$, $\partial u/\partial n = (\frac{1}{2}I - K'_\Omega)f$, we obtain $(\frac{1}{2}I - K_\Omega)\mathcal{S}^\Omega f = \mathcal{S}^\Omega(\frac{1}{2}I - K'_\Omega)f$.

Proposition 5.7.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p < \infty$. If $f \in W^{1,p}(\partial\Omega)$, then*

$$\mathcal{S}^\Omega\left(\frac{\partial \mathcal{D}^\Omega f}{\partial n}\right) = \left(\frac{1}{2}I - K_\Omega\right)\left(\frac{1}{2}I + K_\Omega\right)f. \tag{5.20}$$

Proof Put $u = \mathcal{D}^\Omega f$ in Ω. Then $u = f/2 + K_\Omega f$ is the nontangential limits of u by Theorem 5.3.5. According to Theorem 5.5.2, there exists the nontangential limit of ∇u at almost all point of boundary and $M_a(\nabla u) \in L^p(\partial\Omega)$. Theorem 5.6.1 gives that $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(\partial u/\partial n)$. Using boundary behavior of potentials (Proposition 5.2.2 and Theorem 5.3.5),

$$u = u/2 + K_\Omega u + \mathcal{S}^\Omega(\partial u/\partial n).$$

Using the fact that $u = [(1/2)I + K_\Omega]f$ on $\partial\Omega$, we obtain (5.20).

5.8 Rellich Inequality

In this section, we prove the inequalities

$$\|n^\Omega \cdot \nabla u\|_{L^2(\partial\Omega)} \le C(\|\nabla_\tau u\|_{L^2(\partial\Omega)} + \|u\|_{L^2(\partial\Omega)}),$$

$$\|\nabla_\tau u\|_{L^2(\partial\Omega)} \le C(\|n^\Omega \cdot \nabla u\|_{L^2(\partial\Omega)} + \|u\|_{L^2(\partial\Omega)})$$

for a harmonic function in Ω with $M_a^\Omega(u) + M_a^\Omega(\nabla u) \in L^2(\partial\Omega)$. These inequalities will play a crucial role in the proof of the existence of an L^2-solution of the Neumann problem.

Lemma 5.8.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $u \in H(\Omega)$, $v \in \mathcal{C}^1(\Omega)$, $1 < p < \infty$, $p' = p/(p-1)$, $M_a^\Omega(u) + M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$, $M_a^\Omega(v) \in L^{p'}(\partial\Omega)$. Assume that $M_a^\Omega(\nabla v) \in L^{p'}(\partial\Omega)$ or $u = v$. Suppose that at almost all $x \in \partial\Omega$, there exist the nontangential limits of ∇u and v. If Ω is unbounded and $m > 2$, suppose moreover that $v(x) \to 0$ as $|x| \to \infty$. Then*

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = \int_\Omega \nabla u \cdot \nabla v\, \mathrm{d}\mathcal{H}_m. \tag{5.21}$$

Proof Suppose first that Ω is bounded. Lemma 1.28.1 gives that $|\nabla u| \in L^p(\Omega)$. If $u \neq v$, then $|\nabla v| \in L^{p'}(\Omega)$ and Hölder's inequality (Proposition 1.14.1) forces that $\nabla u \cdot \nabla v \in L^1(\Omega)$. Let $\Omega(j)$ be an increasing sequence of open sets from Lemma 1.17.13. According to the Gauss-Green theorem (Theorem 1.19.1) and the Lebesgue lemma (Lemma 1.13.2), we have

$$\int\limits_{\partial\Omega} v\frac{\partial u}{\partial n} = \lim_{j\to\infty} \int\limits_{\partial\Omega(j)} v\frac{\partial u}{\partial n} = \lim_{j\to\infty} \int\limits_{\Omega(j)} \nabla u \cdot \nabla v = \int\limits_{\Omega} \nabla u \cdot \nabla v.$$

Let now Ω be unbounded. Since u is bounded in a neighborhood of the infinity, there exists a constant c such that $u(x) \to c$ as $|x| \to \infty$ (see Proposition 2.17.1). Proposition 2.17.3 gives that $\nabla u(x) = \nabla[u(x) - c] = O(|x|^{1-m})$ for $m > 2$ and $\nabla u(x) = O(|x|^{-2})$ for $m = 2$. Similarly for v. If Ω is unbounded and $m > 2$, then $v(x) = O(|x|^{2-m})$ by Proposition 2.17.3. If $r > 0$, then

$$\int\limits_{\partial[\Omega\cap B(0;r)]} v\frac{\partial u}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega\cap B(0;r)} \nabla u \cdot \nabla v\, \mathrm{d}\mathcal{H}_m.$$

If $r \to \infty$ we get (5.21).

Lemma 5.8.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Let $h \in \mathcal{C}^1(R^m, R^m)$ be compactly supported, $u \in H(\Omega)$. Suppose that there exists the nontangential limit of ∇u at almost all $x \in \partial\Omega$, $M_a^{\Omega}(|\nabla u|) \in L^2(\partial\Omega)$. Then*

$$\int\limits_{\partial\Omega} |\nabla u|^2 (n\cdot h)\, \mathrm{d}\mathcal{H}_{m-1} = 2\int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(h\cdot\nabla u)\, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int\limits_{\Omega} |\nabla u|^2 \sum_{j=1}^m \partial_j h_j\, \mathrm{d}\mathcal{H}_m - 2\int\limits_{\Omega} \sum_{j,k=1}^m (\partial_j h_k)(\partial_j u)(\partial_k u)\, \mathrm{d}\mathcal{H}_m.$$

Proof Without loss of generality, we can suppose that Ω is connected. Suppose that Ω is bounded. Suppose first that Ω has smooth boundary and u is harmonic on a neighborhood of $\overline{\Omega}$. By virtue of the Gauss-Green theorem (Theorem 1.19.1),

$$\int\limits_{\partial\Omega} |\nabla u|^2 (n\cdot h)\, \mathrm{d}\mathcal{H}_{m-1} - 2\int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(h\cdot\nabla u)\, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} \Bigg[|\nabla u|^2 \sum_{j=1}^m \partial_j h_j$$

$$+ \sum_{j=1}^m h_j \sum_{k=1}^m 2(\partial_k u)(\partial_j\partial_k u) - 2(\Delta u)(h\cdot\nabla u) - 2\sum_{j=1}^m (\partial_j u)\sum_{k=1}^m (\partial_j h_k)(\partial_k u)$$

$$-2\sum_{j=1}^{m}(\partial_j u)\sum_{k=1}^{m}h_k(\partial_j\partial_k u)\Bigg]\,\mathrm{d}\mathcal{H}_m=\int_{\Omega}|\nabla u|^2\sum_{j=1}^{m}\partial_j h_j\,\mathrm{d}\mathcal{H}_m$$

$$-2\int_{\Omega}\sum_{j,k=1}^{m}(\partial_j h_k)(\partial_j u)(\partial_k u)\,\mathrm{d}\mathcal{H}_m.$$

Let now $\Omega(l)$ be domains from Lemma 1.17.13. Since $|\nabla u|\in L^2(\Omega)$ (Lemma 1.28.1), we obtain by virtue of Lebesgue lemma (Lemma 1.13.2)

$$\int_{\partial\Omega}|\nabla u|^2(n\cdot h)\,\mathrm{d}\mathcal{H}_{m-1}=\lim_{l\to\infty}\int_{\partial\Omega(l)}|\nabla u|^2(n\cdot h)\,\mathrm{d}\mathcal{H}_{m-1}$$

$$=\lim_{l\to\infty}\Bigg[2\int_{\partial\Omega(l)}\frac{\partial u}{\partial n}(h\cdot\nabla u)\,\mathrm{d}\mathcal{H}_{m-1}+\int_{\Omega(l)}|\nabla u|^2\sum_{j=1}^{m}\partial_j h_j\,\mathrm{d}\mathcal{H}_m$$

$$-2\int_{\Omega(l)}\sum_{j,k=1}^{m}(\partial_j h_k)(\partial_j u)(\partial_k u)\,\mathrm{d}\mathcal{H}_m\Bigg]=2\int_{\partial\Omega}\frac{\partial u}{\partial n}(h\cdot\nabla u)\,\mathrm{d}\mathcal{H}_{m-1}$$

$$+\int_{\Omega}|\nabla u|^2\sum_{j=1}^{m}\partial_j h_j\,\mathrm{d}\mathcal{H}_m-2\int_{\Omega}\sum_{j,k=1}^{m}(\partial_j h_k)(\partial_j u)(\partial_k u)\,\mathrm{d}\mathcal{H}_m.$$

If Ω is unbounded, choose $r>0$ such that $\partial\Omega\cup\operatorname{spt}h\subset B(0;r)$ and use the proposition for $B(0;r)\cap\Omega$.

Lemma 5.8.3 *Let* $0<a,b,c<\infty$. *If*

$$a^2\le b^2+cab,$$

then

$$a\le(1+c)b.$$

Proof $(a-bc/2)^2=a^2-cab+b^2c^2/4\le b^2+b^2c^2/4\le b^2(1+c/2)^2$. Thus $a-bc/2\le b(1+c/2)$. Hence $a\le b(1+c)$.

Proposition 5.8.4 *Let* $\Omega\subset R^m$ *be an open set with compact Lipschitz boundary,* $m\ge 2$. *Let* $u\in H(\Omega)$, $M_a^{\Omega}(u)+M_a^{\Omega}(\nabla u)\in L^2(\partial\Omega)$. *If* Ω *is unbounded and* $m>2$, *suppose moreover that* $u(x)\to 0$ *as* $|x|\to\infty$. *Then*

$$\|n^{\Omega}\cdot\nabla u\|_{L^2(\partial\Omega)}\le C(\|\nabla_\tau u\|_{L^2(\partial\Omega)}+\|u\|_{L^2(\partial\Omega)}),$$

$$\|\nabla_\tau u\|_{L^2(\partial\Omega)}\le C(\|n^{\Omega}\cdot\nabla u\|_{L^2(\partial\Omega)}+\|u\|_{L^2(\partial\Omega)})$$

where C *depends only on* Ω.

Proof At almost all points of the boundary, there exist the nontangential limits of u and ∇u by Theorem 5.6.1. According to Lemma 1.17.7, there exists $h \in \mathcal{C}_c^\infty(R^m)$ and $C_1 \in (0,1)$ such that $h \cdot n^\Omega > C_1$ on $\partial\Omega$. By virtue of Lemma 5.8.2,

$$\int\limits_{\partial\Omega} \left[|\nabla_\tau u|^2 + \left(\frac{\partial u}{\partial n}\right)^2 \right] (n \cdot h) \, \mathrm{d}\mathcal{H}_{m-1} = 2 \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} \Bigg[(h \cdot \nabla_\tau u)$$

$$+ h \cdot n \frac{\partial u}{\partial n} \Bigg] \, \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\Omega} |\nabla u|^2 \sum_{j=1}^m \partial_j h_j \, \mathrm{d}\mathcal{H}_m - 2 \int\limits_{\Omega} \sum_{j,k=1}^m (\partial_j h_k)(\partial_j u)(\partial_k u) \, \mathrm{d}\mathcal{H}_m.$$

Subtracting $\int_{\partial\Omega} (h \cdot n)(\partial u / \partial n)^2 \, \mathrm{d}\mathcal{H}_{m-1}$ from both sides,

$$\int\limits_{\partial\Omega} |\nabla_\tau u|^2 (n \cdot h) \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 (n \cdot h) \, \mathrm{d}\mathcal{H}_{m-1} + 2 \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} (h \cdot \nabla_\tau u) \, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int\limits_{\Omega} |\nabla u|^2 \sum_{j=1}^m \partial_j h_j \, \mathrm{d}\mathcal{H}_m - 2 \int\limits_{\Omega} \sum_{j,k=1}^m (\partial_j h_k)(\partial_j u)(\partial_k u) \, \mathrm{d}\mathcal{H}_m.$$

Fix a constant $C_2 \geq 1$ such that $|h| + |\nabla h| \leq C_2$. Since $C_1 \leq n \cdot h \leq C_2$, we obtain by Hölder's inequality (Proposition 1.14.1) for $\| \ \| = \| \ \|_{L^2(\partial\Omega)}$

$$C_1 \|\nabla_\tau u\|^2 \leq C_2 \|n \cdot \nabla u\|^2 + 2C_2 \|\nabla_\tau u\| \cdot \|n \cdot \nabla u\| + C_2 3m^2 \|\nabla u\|^2_{L^2(\Omega)},$$

$$C_1 \|n \cdot \nabla u\|^2 \leq C_2 \|\nabla_\tau u\|^2 + 2C_2 \|\nabla_\tau u\| \cdot \|n \cdot \nabla u\| + C_2 3m^2 \|\nabla u\|^2_{L^2(\Omega)}.$$

According to Lemma 5.8.1 and Hölder's inequality, (Proposition 1.14.1)

$$\int\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m = \int\limits_{\partial\Omega} u \frac{\partial u}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} \leq \|u\| \cdot \|n \cdot \nabla u\|.$$

Therefore

$$C_1^2 \|\nabla_\tau u\|^2 \leq 4C_2^2 m^2 (\|n \cdot \nabla u\| + \|u\|)^2 + 2C_2 m (\|n \cdot \nabla u\| + \|u\|) \|\nabla_\tau u\|,$$

$$C_1^2 \|n \cdot \nabla u\|^2 \leq C_2^2 (\|\nabla_\tau u\| + \|u\|)^2 + 3C_2 m^2 (\|\nabla_\tau u\| + \|u\|) \|n \cdot \nabla u\|.$$

According to Lemma 5.8.3,

$$C_1 \|\nabla_\tau u\| \leq (1 + C_1^{-1}) 2C_2 m (\|n \cdot \nabla u\| + \|u\|),$$

$$C_1 \|n \cdot \nabla u\| \leq (1 + 3m^2 C_1^{-1}) C_2 (\|\nabla_\tau u\| + \|u\|).$$

5.9 An L^2-Solution of the Neumann Problem

In this section, we study L^2-solutions of the Neumann problem on a domain $\Omega \subset R^m$ with compact Lipschitz boundary. We look for a solution in the form of a single layer potential with density $f \in L^2(\partial\Omega)$. This leads to the integral equation $\frac{1}{2}f - K'_\Omega f = g$.. We find a necessary and sufficient condition for the solvability of this equation. As a consequence, we find a necessary and sufficient condition for the solvability of the Neumann problem.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $f, g \in L^p(\partial\Omega)$, $1 < p < \infty$. If $m = 2$ and Ω is unbounded, suppose moreover that $\int_{\partial\Omega} f\, \mathcal{H}_{m-1} = 0$. Then the single layer potential $\mathcal{S}^\Omega f$ is an L^p-solution of the Neumann problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega$$

if and only if

$$\frac{1}{2}f - K'_\Omega f = g.$$

(See Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.8 and 5.4.7.)

Lemma 5.9.1 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \geq 2$, $g \in L^p(\partial\Omega)$, $1 < p < \infty$. If Ω is unbounded, suppose moreover that $m = 2$. If there exists an L^p-solution u of the Neumann problem for the Laplace equation with the boundary condition g, then*

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Proof Put $v \equiv 1$. Lemma 5.8.1 gives

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = 0.$$

Lemma 5.9.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \geq 2$. If u is an L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$, then u is constant.*

Proof If Ω is unbounded, then u is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1, there exists a constant c such that $u(x) \to c$ as $x \to \infty$. Put $v = u - c$. If Ω is bounded, put $v = u$. Then v is an L^2-solution of the Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = 0$ on $\partial\Omega$. According to Lemma 5.8.1,

$$0 = \int\limits_{\partial\Omega} v \frac{\partial v}{\partial n} \, d\mathcal{H}_{m-1} = \int\limits_{\Omega} |\nabla v|^2 \, d\mathcal{H}_m.$$

Since $\nabla v = 0$ in Ω, the function v is constant (see Lemma 1.20.1).

Lemma 5.9.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $f \in L^2(\partial\Omega)$. If $m = 2$, suppose moreover that $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. If*

$$\int\limits_{\partial\Omega} f \mathcal{S}^{\Omega} f \, d\mathcal{H}_{m-1} = 0, \tag{5.22}$$

then $f \equiv 0$.

Proof Put $\mu = f\mathcal{H}_{m-1}|\partial\Omega$. Since $\mathcal{S}^{\Omega} f \in L^2(\partial\Omega)$ by Proposition 5.2.2, the real measure μ has finite energy. Theorem 3.39.3 and Lemma 3.41.4 give that $\mu = 0$.

Proposition 5.9.4 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $f \in L^2(\partial\Omega)$. If $m = 2$ or Ω is bounded, suppose moreover that $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. If $f/2 - K'_{\Omega} f = 0$, then $f = 0$.*

Proof $\mathcal{S}^{\Omega} f$ is an L^2-solution of the Neumann problem for the Laplace equation with the zero boundary condition. Lemma 5.9.2 gives that there exists a constant c such that $\mathcal{S}^{\Omega} f = c$ in Ω. Propositions 5.2.2 and 5.2.3 force that $\mathcal{S}^{\Omega} f = c$ on $\partial\Omega$. If Ω is bounded, then (5.22) holds true because $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. If Ω is unbounded, then $c = 0$ because $\mathcal{S}^{\Omega} f(x) \to 0$ as $|x| \to \infty$ (see Proposition 3.1.6). Thus (5.22) holds too. According to Lemma 5.9.3, we have $f = 0$.

Theorem 5.9.5 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. If $m > 2$, put $X = L^2(\partial\Omega)$, for $m = 2$ put $X = \{f \in L^2(\partial\Omega), \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$. Then there exists a positive constant C such that*

$$\|f/2 - K'_{\Omega} f\|_{L^2(\partial\Omega)} \leq C(\|\nabla_{\tau} \mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)}),$$

$$\|\nabla_{\tau} \mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)} \leq C(\|f/2 - K'_{\Omega} f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)}),$$

$$C^{-1} \|f\|_{L^2(\partial\Omega)} \leq \|f/2 - K'_{\Omega} f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)} \leq C \|f\|_{L^2(\partial\Omega)},$$

$$C^{-1} \|f\|_{L^2(\partial\Omega)} \leq \|f/2 + K'_{\Omega} f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)} \leq C \|f\|_{L^2(\partial\Omega)}$$

for all $f \in X$.

Proof Since K'_{Ω} and $\mathcal{S}^{\Omega}$ are bounded linear operators on $L^2(\partial\Omega)$ (see Propositions 5.4.5 and 5.2.2), there exists a constant C_1 such that

$$\|f/2 - K'_{\Omega} f\|_{L^2(\partial\Omega)} + \|f/2 + K'_{\Omega} f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^{\Omega} f\|_{L^2(\partial\Omega)} \leq C_1 \|f\|_{L^2(\partial\Omega)}$$

for all $f \in L^2(\partial\Omega)$. Set $\omega = R^m \setminus \overline{\Omega}$. Using Proposition 5.8.4 for $u = \mathcal{S}^\Omega f$ and Theorem 5.4.8, we obtain that there exists a constant C_2 such that

$$\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} \le C_2[\|\nabla_\tau \mathcal{S}^\Omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}],$$
$$\|\nabla_\tau \mathcal{S}^\Omega f\|_{L^2(\partial\Omega)} \le C_2[\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}],$$
$$\|f/2 - K'_\omega f\|_{L^2(\partial\Omega)} \le C_2\left[\|\nabla_\tau \mathcal{S}^\omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\omega f\|_{L^2(\partial\Omega)}\right],$$
$$\|\nabla_\tau \mathcal{S}^\omega f\|_{L^2(\partial\Omega)} \le C_2\left[\|f/2 - K'_\omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\omega f\|_{L^2(\partial\Omega)}\right]$$

for $f \in X$. Remember that the nontangential limits of the tangential part of $\nabla\mathcal{S}^\Omega f$ with respect to Ω and ω are the same (see Theorem 5.4.7). Easy calculation yields that $K'_\omega f = -K'_\Omega f$. Therefore

$$\|f\|_{L^2(\partial\Omega)} \le \|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + \|f/2 + K'_\Omega f\|_{L^2(\partial\Omega)} = \|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)}$$
$$+\|f/2 - K'_\omega f\|_{L^2(\partial\Omega)} \le \|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + C_2\big[\|\nabla_\tau \mathcal{S}^\omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2}\big]$$
$$\le \|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + C_2^2\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + (C_2 + C_2^2)\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}.$$

We have proved that there exists a constant C_3 such that

$$\|f\|_{L^2(\partial\Omega)} \le C_3\left[\|f/2 - K'_\omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}\right], \qquad \forall f \in X.$$

Since $K'_\Omega f = -K'_\omega f$, we have

$$\|f\|_{L^2(\partial\Omega)} \le C_3\|f/2 + K'_\Omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}, \qquad \forall f \in X.$$

Lemma 5.9.6 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $X = \{f \in L^p(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0\}$. If Ω is bounded, then $[(1/2)I - K'_\Omega](L^p(\partial\Omega)) \subset X$. If Ω is unbounded and $f \in L^p(\partial\Omega)$, then $[(1/2)I - K'_\Omega]f \in X$ if and only if $f \in X$.*

Proof Suppose first that Ω is bounded. If $f \in X$, then $\mathcal{S}^\Omega f$ is an L^p-solution of the Neumann problem with the boundary condition $f/2 - K'_\Omega f$ (see Theorem 5.4.8, Proposition 5.2.2, Theorem 5.4.7 and Proposition 5.2.8). Lemma 5.9.1 gives that $(f/2 - K'_\Omega f) \in X$.

Let now Ω is unbounded. Then $\omega = R^m \setminus \overline{\Omega}$ is bounded. Since $f/2 + K'_\Omega f = f/2 - K'_\omega f \in X$, we have

$$\int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} [(f/2 - K'_\Omega f) + (f/2 + K'_\Omega f)] = \int_{\partial\Omega} [(f/2 - K'_\Omega f) \, \mathrm{d}\mathcal{H}_{m-1}.$$

Theorem 5.9.7 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \ge 2$. If Ω is unbounded and $m > 2$, put $X = L^2(\Omega)$. If Ω is bounded or $m = 2$, put*

$X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0\}$. *Then there exists a positive constant C such that*

$$C^{-1}\|f\|_{L^2(\partial\Omega)} \le \|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\partial\Omega)}, \quad f \in X.$$

Proof $\mathcal{S}^\Omega$ is a compact linear operator on $L^2(\partial\Omega)$ by Proposition 5.2.2. Theorem 5.9.5 gives that $\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}$ is an equivalent norm on $L^2(\partial\Omega)$. If Ω is unbounded and $m > 2$, set $Mf = 0$. If Ω is bounded or $m = 2$, define

$$Mf = \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1}.$$

If $f \in L^2(\partial\Omega)$, $f/2 - K'_\Omega f = 0$ and $Mf = 0$, then $f = 0$ by Proposition 5.9.4. Lemma 1.7.9 forces that $\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + |Mf|$ is an equivalent norm on $L^2(\partial\Omega)$. Hence $\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)}$ is an equivalent norm on X.

Lemma 5.9.8 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $h \in \mathcal{C}_c^\infty(R^m, R^m)$, $h \cdot n^\Omega \ge \alpha > 0$ on $\partial\Omega$, $1/2 < \lambda < \infty$. For $f \in L^2(\partial\Omega)$, define*

$$Hf(x) = \int_{\partial\Omega} \frac{1}{\mathcal{H}_{m-1}(\partial B(0;1))} \frac{(x-y)\cdot(h(x)-h(y))}{|x-y|^m} f(y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Then there exists a positive constant C such that

$$\|f\|_{L^2(\partial\Omega)} \le C\big[\|(\lambda I - K'_\Omega)f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)} + \|Hf\|_{L^2(\partial\Omega)}\big]$$

for all $f \in L^2(\partial\Omega)$.

Proof Fix $f \in L^2(\partial\Omega)$. According to Lemma 5.8.2 and Theorem 5.4.8,

$$0 \le \int_{\partial\Omega} |\nabla\mathcal{S}^\Omega f|^2 (n\cdot h) \, \mathrm{d}\mathcal{H}_{m-1} = 2\int_{\partial\Omega} \Big(\frac{1}{2}f - K'_\Omega f\Big)(h\cdot\nabla\mathcal{S}^\Omega f) \, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int_\Omega |\nabla\mathcal{S}^\Omega f|^2 \sum_{j=1}^m \partial_j h_j \, \mathrm{d}\mathcal{H}_m - 2\int_\Omega \sum_{j,k=1}^m (\partial_j h_k)(\partial_j\mathcal{S}^\Omega f)(\partial_k\mathcal{S}^\Omega f) \, \mathrm{d}\mathcal{H}_m.$$

Fix a constant C_1 such that $|\nabla h| \le C_1$. Denote

$$K_h f(z) = \lim_{\epsilon\downarrow 0} \int_{\partial\Omega\setminus B(z;\epsilon)} \frac{h(z)\cdot(y-z)f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m} \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

According to Theorem 5.4.7 and Lemma 5.2.7,

$$0 \le 2\int\limits_{\partial\Omega} \Big[\Big(\frac{1}{2}-\lambda\Big)f + \lambda f - K'_\Omega f\Big]\Big[\frac{1}{2}(h\cdot n^\Omega)f + K_h f\Big] + \int\limits_{\Omega} C_1 3m^2|\nabla\mathcal{S}^\Omega f|^2,$$

$$\|(1/2)(h\cdot n^\Omega)f + K_h f\|_{L^2(\partial\Omega)} \le C_2\|f\|_{L^2(\partial\Omega)},$$

where C_2 does not depend on f. Since

$$\int\limits_{\Omega} |\nabla\mathcal{S}^\Omega f|^2\,\mathrm{d}\mathcal{H}_m = \int\limits_{\partial\Omega}\Big(\frac{1}{2}f - K'_\Omega f\Big)\mathcal{S}^\Omega f\,\mathrm{d}\mathcal{H}_{m-1}$$

by Lemma 5.8.1 and Theorem 5.4.8, we have

$$\int\limits_{\partial\Omega}\Big(\lambda-\frac{1}{2}\Big)(h\cdot n^\Omega)f^2\,\mathrm{d}\mathcal{H}_{m-1} \le \int\limits_{\partial\Omega}(1-2\lambda)fK_h f\,\mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\partial\Omega}(\lambda f$$

$$-K'_\Omega f)\Big[\frac{1}{2}h\cdot n^\Omega f + K_h f\Big]\,\mathrm{d}\mathcal{H}_{m-1} + 3C_1m^2\int\limits_{\partial\Omega}\Big(\frac{1}{2}f - K'_\Omega f\Big)\mathcal{S}^\Omega f\,\mathrm{d}\mathcal{H}_{m-1}.$$

By virtue of Hölder's inequality (Proposition 1.14.1),

$$\alpha\Big(\lambda-\frac{1}{2}\Big)\|f\|^2_{L^2(\partial\Omega)} \le (1-2\lambda)\int\limits_{\partial\Omega} fK_h f\,\mathrm{d}\mathcal{H}_{m-1}$$

$$+C_2\|\lambda f - K'_\Omega f\|_{L^2(\partial\Omega)}\|f\|_{L^2(\partial\Omega)} + 3C_1m^2\|\frac{1}{2}f - K'_\Omega f\|_{L^2(\partial\Omega)}\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}.$$

The adjoint operator K'_h of K_h is given by

$$K'_h f(x) = \lim_{\epsilon\downarrow 0}\int\limits_{\partial\Omega\setminus B(z;\epsilon)} \frac{h(y)\cdot(x-y)f(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m}\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

Thus

$$\int\limits_{\partial\Omega} fK_h f = \int\limits_{\partial\Omega} fK'_h f = \int\limits_{\partial\Omega} f\frac{K_h f + K'_h f}{2} = \int\limits_{\partial\Omega} f\frac{Hf}{2}.$$

Hence

$$\alpha\Big(\lambda-\frac{1}{2}\Big)\|f\|^2_{L^2(\partial\Omega)} \le (1-2\lambda)\int\limits_{\partial\Omega} f\frac{Hf}{2}\,\mathrm{d}\mathcal{H}_{m-1}$$

$$+C_2\|\lambda f - K'_\Omega f\|_{L^2(\partial\Omega)}\|f\|_{L^2(\partial\Omega)} + 3C_1m^2\|\frac{1}{2}f - K'_\Omega f\|_{L^2(\partial\Omega)}\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}.$$

According to Proposition 5.4.5, there exists a constant C_3 independent of f such that $\|\frac{1}{2}f - K'_\Omega f\|_{L^2} \leq C_3\|f\|_{L^2}$. Using Hölder's inequality (Proposition 1.14.1),

$$\alpha\Big(\lambda - \frac{1}{2}\Big)\|f\|^2_{L^2(\partial\Omega)} \leq \lambda\|f\|_{L^2(\partial\Omega)}\|Hf\|_{L^2(\partial\Omega)}$$

$$+C_2\|\lambda f - K'_\Omega f\|_{L^2(\partial\Omega)}\|f\|_{L^2(\partial\Omega)} + 3C_1 m^2 C_3\|f\|_{L^2(\partial\Omega)}\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}.$$

Now we divide this inequality by $\|f\|_{L^2(\partial\Omega)}$.

Proposition 5.9.9 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $\lambda \in R$, $|\lambda| \geq 1/2$. Then $\lambda I - K'_\Omega$ is a Fredholm operator with index* 0 *in* $L^2(\partial\Omega)$.

Proof Suppose first that $\lambda \geq 1/2$ and Ω is a bounded domain. Put $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$. Then $\lambda I - K'_\Omega$ is a bounded linear operator on X by Proposition 5.4.5 and Lemma 5.9.6. The operator $(1/2)I - K'_\Omega$ is bounded below on X by Theorem 5.9.7. The range $[(1/2)I - K'_\Omega](X)$ is closed by Proposition 1.4.13. Denote $g \equiv 1$. Since $(1/2)I - K'_\Omega$ is one-to-one on X and $L^2(\partial\Omega) = X \oplus \{ag; a \in R\}$, we have $\dim \operatorname{Ker}[(1/2)I - K'_\Omega] \leq 1$. Since $[(1/2)I - K'_\Omega](L^2(\partial\Omega)) = [(1/2)I - K'_\Omega](X) + \{a[(1/2)I - K'_\Omega]g; a \in R\}$, Proposition 1.3.7 gives that $[(1/2)I - K'_\Omega](L^2(\Omega))$ is closed. Thus $[(1/2)I - K'_\Omega]$ is an upper semi-Fredholm operator in $L^2(\partial\Omega)$. Let now $1/2 < \lambda$. According to Lemma 1.17.7, there exist $h \in \mathcal{C}^\infty_c(R^m, R^m)$ and a positive constant α such that $h \cdot n^\Omega \geq \alpha$ on $\partial\Omega$. Let H be the integral operator from Lemma 5.9.8. Then $\|[(1/2)I - K'_\Omega]f\|_{L^2} + \|\mathcal{S}^\Omega f\|_{L^2} + \|Hf\|_{L^2}$ is an equivalent norm on $L^2(\partial\Omega)$ by Lemma 5.9.8. The operator $\mathcal{S}^\Omega$ is compact on $L^2(\partial\Omega)$ by Proposition 5.2.2. Since $h \in \mathcal{C}^\infty(R^m)$, there exists a constant C_1 such that $|(x-y)\cdot(h(x)-h(y))|/|x-y|^m \leq C_1|x-y|^{2-m}$ for $x, y \in \partial\Omega$. Thus the operator H is compact on $L^2(\partial\Omega)$ by Proposition 1.26.6. Lemma 1.7.9 gives that $\lambda I - K'_\Omega$ is an upper semi-Fredholm operator on $L^2(\partial\Omega)$. If $\lambda > \|K'_\Omega\|$, then $\lambda I - K'_\Omega$ is an isomorphism by Proposition 1.6.7, and therefore, $i(\lambda I - K'_\Omega) = 0$. By the continuity of the index (Proposition 1.8.5), we infer that $i(\lambda I - K'_\Omega) = 0$ for all $1/2 \leq \lambda < \infty$. So, $\lambda I - K'_\Omega$ is a Fredholm operator on $L^2(\partial\Omega)$ for all $1/2 \leq \lambda < \infty$.

Let now Ω be an unbounded domain and $1/2 \leq \lambda < \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$ and denote $\omega = B(0;r) \cap \Omega$. We have proved that $\lambda I - K'_\omega$ is a Fredholm operator with index 0 on $L^2(\partial\omega)$. For $f \in L^2(\partial\Omega)$, $g \in L^2(\partial B(0;r))$, denote $T(f,g) = [(\lambda I - K'_\Omega)f, (\lambda I - K'_{B(0;r)})g]$. Then $T - (\lambda I - K'_\omega)$ is an integral operator with a bounded continuous kernel and therefore a compact operator on $L^2(\partial\Omega)$ (see Proposition 1.26.2). Theorem 1.8.3 gives that T is a Fredholm operator with index 0 on $L^2(\partial\omega)$. Since $\lambda I - K'_{B(0;r)}$ is a Fredholm operator with index 0 on $L^2(\partial B(0;r))$, we deduce that $\lambda I - K'_\Omega$ is a Fredholm operator with index 0 on $L^2(\partial\Omega)$.

Let now Ω be open and $1/2 \leq \lambda < \infty$. Denote by $\Omega(1), \ldots, \Omega(k)$ components of Ω. For $f_j \in L^2(\partial\Omega_j)$, denote

$$T(f_1, \ldots, f_k) = [(\lambda I - K'_{\Omega(1)})f_1, \ldots, (\lambda I - K'_{\Omega(k)})f_k].$$

Then $T - (\lambda I - K'_\Omega)$ is an integral operator with a bounded continuous kernel and therefore a compact operator on $L^2(\partial\Omega)$ (see Proposition 1.26.2). Since T is a Fredholm operator with index 0 on $L^2(\partial\Omega)$, Theorem 1.8.3 gives that $\lambda I - K'_\Omega$ is a Fredholm operator with index 0 on $L^2(\partial\Omega)$.

Let now $-\infty < \lambda \le -1/2$. Denote $\omega = R^m \setminus \overline{\Omega}$. Since $-\lambda I - K'_\omega$ is a Fredholm operator with index 0 on $L^2(\partial\omega) = L^2(\partial\Omega)$, the operator $\lambda I - K'_\Omega = \lambda I + K'_\omega$ is a Fredholm operator with index 0 on $L^2(\partial\Omega)$.

Theorem 5.9.10 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Denote by T the restriction of $(1/2)I - K'_\Omega$ onto $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \,\mathrm{d}\mathcal{H}_{m-1} = 0\}$. Then T is an isomorphism. If $g \in L^2(\partial\Omega)$, then there exists an L^2-solution of the Neumann problem*

$$\Delta u = 0 \text{ in } \Omega, \qquad \partial u/\partial n = g \text{ on } \partial\Omega$$

if and only if

$$\int_{\partial\Omega} g \,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

The general form of a solution of this problem is $u = \mathcal{S}^\Omega(T^{-1}g) + c$ where c is an arbitrary constant. If u is an L^2-solution of the problem, then

$$\|M_a(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}; \tag{5.23}$$

if moreover

$$\int_{\Omega} u \,\mathrm{d}\mathcal{H}_m = 0,$$

then

$$\|M_a(u)\|_{L^2(\partial\Omega)} + \|M_a(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}. \tag{5.24}$$

Here a constant C does not depend on g.

Proof $[(1/2)I - K'_\Omega](X) \subset [(1/2)I - K'_\Omega](L^2(\partial\Omega)) \subset X$ by Lemma 5.9.6. Hence $\beta((1/2)I - K'_\Omega) \ge 1$. If $f \in L^2(\partial\Omega)$, then $\mathcal{S}^\Omega f$ is an L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = f/2 - K'_\Omega f$ on $\partial\Omega$ by Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.8 and 5.4.7. If u is an L^2-solution of the Neumann problem with the boundary condition $g \equiv 0$, then u is constant by Lemma 5.9.2. So, if $f \in \operatorname{Ker}((1/2)I - K'_\Omega)$, then $\mathcal{S}^\Omega f$ is constant in Ω. Proposition 5.2.2 gives that $\mathcal{S}^\Omega f$ is constant on $\overline{\Omega}$. If moreover $f \in X$, then

$$\int_{\partial\Omega} f\mathcal{S}^\Omega f \,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

Lemma 5.9.3 gives that $f \equiv 0$. Thus $\alpha((1/2)I - K'_\Omega) \leq 1$. Since $(1/2)I - K'_\Omega$ is a Fredholm operator with index 0 (see Proposition 5.9.9), we deduce that $\alpha((1/2)I - K'_\Omega) = \beta((1/2)I - K'_\Omega) = 1$. Therefore there exists $\varphi \in L^2(\partial\Omega) \setminus X$ such that $\mathrm{Ker}((1/2)I - K'_\Omega) = \{a\varphi; a \in R\}$. Hence $((1/2)I - K'_\Omega)(X) = ((1/2)I - K'_\Omega)(L^2(\partial\Omega)) = X$. Thus T is an isomorphism by Theorem 1.6.3. If $g \in X$ and c is a constant, then $\mathcal{S}^\Omega(T^{-1}g) + c$ is an L^2-solution of the Neumann problem with the boundary condition g. On the other hand, if there exists an L^2-solution of the Neumann problem with the boundary condition g, then $g \in X$ by Lemma 5.9.1.

Let $g \in X$. Then $u = \mathcal{S}^\Omega(T^{-1}g) - \oint_\Omega \mathcal{S}^\Omega(T^{-1}g)\, \mathrm{d}\mathcal{H}_m$ is a unique L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that $\oint_\Omega u\, \mathrm{d}\mathcal{H}_m = 0$. Since $f \mapsto \oint_\Omega \mathcal{S}^\Omega f\, \mathrm{d}\mathcal{H}_m$ is a bounded linear functional on $L^2(\partial\Omega)$ (see Corollary 5.2.9 and Proposition 1.14.1), Propositions 5.2.2 and 5.2.8 give that there exists a constant C such that (5.24) holds. If v is another solution of the problem, then $v - u$ is constant. Therefore $\|M_a(\nabla v)\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}$.

Theorem 5.9.11 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary. The operator $(1/2)I - K'_\Omega$ is an isomorphism on $L^2(\partial\Omega)$. Fix $g \in L^2(\partial\Omega)$. If $m > 2$, then there exists an L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$. If $m = 2$, then there exists an L^2-solution of this problem if and only if $\int_{\partial\Omega} g\, \mathrm{d}\mathcal{H}_{m-1} = 0$. If u is a solution of this problem, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + u_\infty$ is a unique L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. This solution satisfies*

$$\|M_a(u)\|_{L^2(\partial\Omega)} + \|M_a(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)} + |u_\infty| \tag{5.25}$$

where a constant C depends only on Ω and a.

Proof Let $f \in L^2(\partial\Omega)$, $[(1/2)I - K'_\Omega]f = 0$. Then $\int_{\partial\Omega} f\, \mathrm{d}\mathcal{H}_{m-1} = 0$ by Lemma 5.9.6. Theorem 5.4.8 gives that $\mathcal{S}^\Omega f$ is an L^2-solution of the Neumann problem with zero boundary condition. Thus $\mathcal{S}^\Omega f$ is constant by Lemma 5.9.2. Since $\mathcal{S}^\Omega f(x) \to 0$ as $|x| \to \infty$, we infer that $\mathcal{S}^\Omega f = 0$ on Ω. Proposition 5.2.3 gives that $\mathcal{S}^\Omega f = 0$ on $\overline{\Omega}$. Lemma 5.9.3 gives that $f \equiv 0$. Since $(1/2)I - K'_\Omega$ is a Fredholm operator with index 0 (see Proposition 5.9.9), we deduce that $[(1/2)I - K'_\Omega](L^2(\partial\Omega)) = L^2(\partial\Omega)$. Thus $(1/2)I - K'_\Omega$ is an isomorphism by Theorem 1.6.3. Lemma 5.9.6 gives that $(1/2)I - K'_\Omega$ is an isomorphism on $\{f \in L^2(\partial\Omega); \int_{\partial\Omega} f\, \mathrm{d}\mathcal{H}_{m-1} = 0\}$.

Let $g \in L^2(\partial\Omega)$. If $m = 2$, suppose moreover $\int_{\partial\Omega} g\, \mathrm{d}\mathcal{H}_{m-1} = 0$. If $u_\infty \in R$, then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + u_\infty$ is an L^2-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. (See Theorem 5.4.8.) If v is an L^2-solution of the Neumann problem with the boundary condition g, then there exists a constant c such that $v = u + c$ by Lemma 5.9.2. Hence there exists the limit v_∞ of v at infinity. If $v_\infty = u_\infty$, then $v = u$.

Let now $m = 2$ and u be an L^2-solution of the Neumann problem with the boundary condition $g \in L^2(\partial\Omega)$. Then $\int_{\partial\Omega} g \, d\mathcal{H}_{m-1} = 0$ by Lemma 5.9.1.

Propositions 5.2.3 and 5.2.8 give that there exists a constant C such that (5.25) holds.

5.10 An L^2-Solution of the Robin Problem

In this section, we study L^2-solutions of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$ for a domain $\Omega \subset R^m$ with compact Lipschitz boundary. We assume that $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} \neq 0$. We prove the existence of a solution for arbitrary $g \in L^2(\partial\Omega)$. We discuss also the uniqueness of a solution.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $f, g \in L^p(\partial\Omega)$, $1 < p < \infty$. If $m = 2$ and Ω is unbounded, suppose moreover that $\int_{\partial\Omega} f \, \mathcal{H}_{m-1} = 0$. Then the single layer potential $\mathcal{S}^\Omega f$ is an L^p-solution of the Robin problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \quad \text{on } \partial\Omega$$

if and only if

$$\frac{1}{2} f - K'_\Omega f + h\mathcal{S}^\Omega f = g.$$

(See Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.8 and 5.4.7.)

If $m = 2, f \in L^p(\partial\Omega)$, then the problem might be a behavior of a single layer potential at infinity. So, we look for a solution of the problem in a bit modified form

$$\mathcal{S}^\Omega(f - \oint f) + \oint f$$

with an unknown density $f \in L^p(\partial\Omega)$. This modified single layer potential is a solution of the problem if and only if

$$\frac{1}{2}(f - \oint f) - K'_\Omega(f - \oint f) + h[\mathcal{S}^\Omega(f - \oint f) + \oint f] = g.$$

Proposition 5.10.1 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \geq 2$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\mathcal{H}_{m-1}(\{x \in \partial\Omega; h(x) > 0\}) > 0$. Let u be an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof According to Lemma 5.8.1,

$$0 = \int\limits_{\partial\Omega} u\left[\frac{\partial u}{\partial n} + hu\right] \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} |\nabla u|^2 \,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} hu^2 \,\mathrm{d}\mathcal{H}_{m-1}.$$

Thus $\nabla u = 0$ in Ω, $hu^2 = 0$ on $\partial\Omega$. Since $\nabla u = 0$ in Ω, there exists a constant c such that $u \equiv c$ (see Lemma 1.20.1). Hence $hc^2 = 0$ what forces $c = 0$.

Proposition 5.10.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m > 2$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$. If $f \in L^2(\partial\Omega)$ and $f/2 - K'_\Omega f + h\mathcal{S}^\Omega f = 0$, then $f = 0$.*

Proof $u = \mathcal{S}^\Omega f$ is an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. Moreover, $\mathcal{S}^\Omega f(x) \to 0$ as $|x| \to \infty$. Proposition 5.10.1 gives that $\mathcal{S}^\Omega f = 0$ in Ω. Propositions 5.2.2 and 5.2.3 force that $\mathcal{S}^\Omega f = 0$ on $\partial\Omega$. According to Lemma 5.9.3, we have $f = 0$.

Proposition 5.10.3 *Let $\Omega \subset R^2$ be a domain with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$, $f \in L^2(\partial\Omega)$. If $(1/2)(f - \oint f) - K'_\Omega(f - \oint f) + h[\mathcal{S}^\Omega(f - \oint f) + \oint f] = 0$, then $f = 0$.*

Proof $u = \mathcal{S}^\Omega(f - \oint f) + \oint f$ is an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. Proposition 5.10.1 gives that $\mathcal{S}^\Omega(f - \oint f) + \oint f = 0$ in Ω. Propositions 5.2.2 and 5.2.3 force that $\mathcal{S}^\Omega(f - \oint f) + \oint f = 0$ on $\partial\Omega$. Hence $(1/2)(f - \oint f) - K'_\Omega(f - \oint f) = 0$. Proposition 5.9.4 gives that $f - \oint f = 0$. Therefore we have in Ω that $0 = \mathcal{S}^\Omega(f - \oint f) + \oint f = \oint f$ and thus $f = (f - \oint f) + \oint f = 0$.

Theorem 5.10.4 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$. If $m > 2$, put $Tf = f/2 - K'_\Omega f + h\mathcal{S}^\Omega f$. For $m = 2$, define $Tf = (1/2)(f - \oint f) - K'_\Omega(f - \oint f) + h[\mathcal{S}^\Omega(f - \oint f) + \oint f]$. Then T is an isomorphism on $L^2(\partial\Omega)$. If $g \in L^2(\partial\Omega)$, then there exists an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$.*

- *If Ω is bounded and $m > 2$, then $u = \mathcal{S}^\Omega(T^{-1}g)$ is a unique solution of the problem. Moreover, there exists a constant C dependent only on Ω, h, and a such that the estimate (5.24) holds.*
- *If $m = 2$, $f = T^{-1}g$, then $\mathcal{S}^\Omega(f - \oint f) + \oint f$ is a unique solution of the problem. Moreover, there exists a constant C dependent only on Ω, h, and a such that the estimate (5.24) holds.*
- *If Ω is unbounded, $m > 2$, and u is a solution of the problem, then there exists $u_\infty \in R$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^\Omega T^{-1}(g - hu_\infty) + u_\infty$ is a unique solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover, there exists a constant C dependent only on Ω, h, and a such that the estimate (5.25) holds.*

Proof T is a Fredholm operator with index 0 by Propositions 5.9.9, 5.2.2 and Theorem 1.8.3. Since T is one-to-one by Propositions 5.10.2 and 5.10.3, it is onto. Hence T is an isomorphism by Theorem 1.6.3.

Let now $g \in L^2(\partial\Omega), f = T^{-1}g$. If Ω is bounded and $m > 2$, then $\mathcal{S}^\Omega f$ is a unique solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$ by Theorem 5.4.7 and Proposition 5.10.1. If $m = 2$, then $\mathcal{S}^\Omega(f - \oint f) + \oint f$ is a unique solution of the problem by Theorem 5.4.7 and Proposition 5.10.1. Let now Ω be unbounded, $m > 2$. If $u_\infty \in R$ is given, then $u = \mathcal{S}^\Omega T^{-1}(g - hu_\infty) + u_\infty$ is an L^2-solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$. Let now v be an L^2-solution of the Robin problem with the boundary condition g. Since v is bounded in a neighborhood of infinity, there exists a constant v_∞ such that $v(x) \to v_\infty$ as $|x| \to \infty$ (see Proposition 2.17.1). If $v_\infty = u_\infty$, then $u - v \equiv 0$ by Proposition 5.10.1.

The estimates (5.24), (5.25) are consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

5.11 L^p-Solution of the Neumann and Robin Problems

In this section, we study an L^p-solution of the Robin problem under assumption that $\Omega \subset R^m$ is a domain with compact Ljapunov boundary (i.e., of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.) Example 5.11.6 shows that this problem is not solvable for domains with Lipschitz boundary.

Proposition 5.11.1 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < \infty$. Then K_Ω, K'_Ω are compact linear operators in $L^p(\partial\Omega)$.*

Proof According to Lemma 1.17.9, there exists a constant C such that $|K^\Omega(x,y)| \le C|x-y|^{\alpha-m+1}$ for $x, y \in \partial\Omega$. Proposition 1.26.6 gives that K_Ω, K'_Ω are compact linear operators in $L^p(\partial\Omega)$.

Lemma 5.11.2 *Let $\Omega \subset R^m$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < q < \infty$, $h \in L^\infty(\partial\Omega)$. Let u be an L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$. Then $u \in L^q(\partial\Omega)$ if and only if $\mathcal{S}^\Omega g \in L^q(\partial\Omega)$.*

Proof Suppose first that Ω is bounded. Then $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(\partial u/\partial n) = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(g - hu)$ by Theorem 5.6.1. Using boundary properties of a single layer potential and a double layer potential (Proposition 5.2.2 and Theorem 5.3.5), we obtain $u = \frac{1}{2}u + K_\Omega u + \mathcal{S}^\Omega(g - hu)$ on $\partial\Omega$. The operator $u \mapsto -K_\Omega u + \mathcal{S}^\Omega hu$ is compact in $L^p(\partial\Omega)$ and in $L^q(\partial\Omega)$ by Propositions 5.11.1, 5.2.2 and Theorem 1.7.4. Since $\frac{1}{2}u - K_\Omega u + \mathcal{S}^\Omega hu = \mathcal{S}^\Omega g$ and the operator $u \mapsto \frac{1}{2}u - K_\Omega u + \mathcal{S}^\Omega hu$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ and in $L^q(\partial\Omega)$ by Theorem 1.8.3, Lemma 1.8.4 gives that $u \in L^q(\partial\Omega)$ if and only if $\mathcal{S}^\Omega g \in L^q(\partial\Omega)$.

Let now Ω be unbounded. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$. Put $\omega = \Omega \cap B(0;r)$, $g = \partial u/\partial n^\omega + hu$ on $\partial\omega \setminus \partial\Omega$. Then $u, g \in L^\infty(\partial\omega \setminus \partial\Omega)$. Moreover, u

is an L^p-solution of the Robin problem $\Delta u = 0$ in ω, $\partial u/\partial n + hu = g$ on $\partial\omega$. Thus $u \in L^q(\partial\Omega)$ if and only if $u \in L^q(\partial\omega)$; $u \in L^q(\partial\omega)$ if and only if $\mathcal{S}^\omega g \in L^q(\partial\omega)$; $\mathcal{S}^\omega g \in L^q(\partial\omega)$ if and only if $\mathcal{S}^\Omega g \in L^q(\partial\Omega)$.

Theorem 5.11.3 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < \infty$. Denote by T_p the restriction of $(1/2)I - K'_\Omega$ onto $X_p = \{f \in L^p(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$. Then T is an isomorphism on X_p. If $g \in L^p(\partial\Omega)$, then there exists an L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ if and only if $\int_{\partial\Omega} g \, d\mathcal{H}_{m-1} = 0$. The general form of a solution of this problem is $u = \mathcal{S}^\Omega(T_p^{-1} g) + c$ where c is an arbitrary constant. If u is an L^p-solution of the problem, then*

$$\|M_a(\nabla u)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}; \tag{5.26}$$

if moreover

$$\int_\Omega u \, d\mathcal{H}_m = 0.$$

then

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)} \tag{5.27}$$

Here a constant C does not depend on g.

Proof Lemma 5.9.6 gives that $T_p(X_p) \subset X_p$. The operator K'_Ω is a compact operator on $L^p(\partial\Omega)$ by Proposition 5.11.1. Therefore T_p is a Fredholm operator with index 0 by Theorem 1.8.3. The operator T_2 is an isomorphism by Theorem 5.9.10. Lemma 1.8.4 gives that T_p is an isomorphism.

If there exists an L^p-solution of the Neumann problem with the boundary condition g, then $g \in X_p$ by Lemma 5.9.1. If $g \in X_p$ and c is a constant, then $\mathcal{S}^\Omega(T_p^{-1} g) + c$ is an L^p-solution of the Neumann problem with the boundary condition g.

Let now u be an L^p-solution of the Neumann problem with the boundary condition $g \equiv 0$. Theorem 5.6.1 gives that $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega g = \mathcal{D}^\Omega u$. Put $p' = p/(p-1)$. Since $\mathcal{S}^\Omega(\partial u/\partial n) = 0 \in L^{p'}(\partial\Omega)$, we have $u \in L^{p'}(\partial\Omega)$ by Lemma 5.11.2. Since $u = \mathcal{D}^\Omega u$, we have $M_a^\Omega(u) \in L^{p'}(\partial\Omega)$ (see Theorem 5.3.5). Lemma 5.8.1 gives

$$0 = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \, d\mathcal{H}_{m-1} = \int_\Omega |\nabla u|^2 \, d\mathcal{H}_m.$$

Since $\nabla u = 0$ in Ω, the function u is constant.

Let $g \in X_p$. Then $u = \mathcal{S}^\Omega(T_p^{-1} g) - \oint_\Omega \mathcal{S}^\Omega(T_p^{-1} g) \, d\mathcal{H}_m$ is a unique L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that

$\oint_\Omega u \, \mathrm{d}\mathcal{H}_m = 0$. Since $f \mapsto \oint_\Omega \mathcal{S}^\Omega f \, \mathrm{d}\mathcal{H}_m$ is a bounded linear functional on $L^p(\partial\Omega)$ (see Corollary 5.2.9 and Proposition 1.14.1), Propositions 5.2.2 and 5.2.8 give that there exists a constant C such that (5.27) holds. Let now v be another L^p-solution of the problem. Then $v - u$ is constant. Thus $\|M_a(\nabla v)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}$.

Theorem 5.11.4 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1 < p < \infty$. Then $(1/2)I - K'_\Omega$ is an isomorphism on $L^p(\partial\Omega)$. Fix $g \in L^p(\partial\Omega)$. If $m > 2$, then there exists an L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$. If $m = 2$, then there exists an L^p-solution of this problem if and only if $\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0$. Suppose that the Neumann problem with the boundary condition g is solvable. If u is a solution of this problem, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + u_\infty$ is a unique L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. This solution satisfies*

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \leq C(\|g\|_{L^p(\partial\Omega)} + |u_\infty|) \tag{5.28}$$

where a constant C depends only on Ω, p, and a.

Proof Proposition 5.11.1 gives that K'_Ω is a compact operator on $L^p(\partial\Omega)$. Therefore $(1/2)I - K'_\Omega$ is a Fredholm operator with index 0 by Theorem 1.8.3. The operator $(1/2)I - K'_\Omega$ is an isomorphism on $L^2(\partial\Omega)$ by Theorem 5.9.11. Lemma 1.8.4 gives that $(1/2)I - K'_\Omega$ is an isomorphism on $L^p(\partial\Omega)$.

If $m = 2$ and there exists an L^p-solution of the Neumann problem with the boundary condition g, then $\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0$ by Lemma 5.9.1.

Let now $g \in L^p(\partial\Omega)$. If $m = 2$, suppose moreover that $\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0$. If $u_\infty \in R$ then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + u_\infty$ is an L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. (See Theorem 5.4.8.)

Let now u be an L^p-solution of the Neumann problem with the boundary condition g. Lemma 1.28.2 and Proposition 2.17.1 give that there exists u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Suppose now that $g = 0$, $u_\infty = 0$. Corollary 5.6.2 gives that $u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega g = \mathcal{D}^\Omega u$. Put $p' = p/(p-1)$. Since $\mathcal{S}^\Omega(\partial u/\partial n) = 0 \in L^{p'}(\partial\Omega)$, we have $u \in L^{p'}(\partial\Omega)$ by Lemma 5.11.2. Since $u = \mathcal{D}^\Omega u$, we have $M_a^\Omega(u) \in L^{p'}(\partial\Omega)$ (see Theorem 5.3.5). Lemma 5.8.1 gives

$$0 = \int\limits_{\partial\Omega} u \frac{\partial u}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m.$$

Since $\nabla u = 0$ in Ω, the function u is constant. Since $u(x) \to 0$ as $|x| \to \infty$, we deduce that $u \equiv 0$.

Propositions 5.2.3 and 5.2.8 give that there exists a constant C such that (5.28) holds.

Theorem 5.11.5 *Let $\Omega \subset R^m$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$. If $m > 2$, put $Tf = f/2 - K'_\Omega f + h\mathcal{S}^\Omega f$. For $m = 2$, define $Tf = (1/2)(f - \oint f) - K'_\Omega(f - \oint f) + h[\mathcal{S}^\Omega(f - \oint f) + \oint f]$. Then T is an isomorphism on $L^p(\partial\Omega)$. If $g \in L^p(\partial\Omega)$, then there exists an L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$.*

- *If Ω is bounded and $m > 2$, then $u = \mathcal{S}^\Omega(T^{-1}g)$ is a unique solution of the problem. This solution satisfies (5.27) with a constant C depending only on Ω, h, p, and a.*
- *If $m = 2$, $f = T^{-1}g$, then $\mathcal{S}^\Omega(f - \oint f) + \oint f$ is a unique solution of the problem. This solution satisfies (5.27) with a constant C depending only on Ω, h, p, and a.*
- *If Ω is unbounded, $m > 2$, and u is a solution of the problem, then there exists $u_\infty \in R$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^\Omega T^{-1}(g - hu_\infty) + u_\infty$ is a unique solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$. This solution satisfies (5.28) with a constant C depending only on Ω, h, p, and a.*

Proof $T - \frac{1}{2}I$ is a compact operator on $L^p(\partial\Omega)$ by Propositions 5.11.1, 5.2.2 and Theorem 1.7.4. Thus T is a Fredholm operator with index 0 by Theorem 1.8.3. Since T is an isomorphism in $L^2(\partial\Omega)$ by Theorem 5.10.4, it is an isomorphism on $L^p(\partial\Omega)$ by Lemma 1.8.4.

Let now $g \in L^p(\partial\Omega)$, $f = T^{-1}g$. If Ω is bounded and $m > 2$, then $\mathcal{S}^\Omega f$ is an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$ by Theorem 5.4.7. If $m = 2$, then $\mathcal{S}^\Omega(f - \oint f) + \oint f$ is an L^p-solution of the problem by Theorem 5.4.7. Let now Ω be unbounded, $m > 2$. If $u_\infty \in R$ is given, then $u = \mathcal{S}^\Omega T^{-1}(g - hu_\infty) + u_\infty$ is an L^p-solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$.

Let now u be an L^p-solution of the Robin problem with the boundary condition g. If Ω is unbounded, then there exists a constant u_∞ such that $u(x) \to u_\infty$ by Lemma 1.28.2 and Proposition 2.17.1. Let now $g \equiv 0$. If Ω is unbounded and $m > 2$, suppose moreover that $u_\infty = 0$. Lemma 5.11.2 gives that $u \in L^{p'}(\partial\Omega)$ with $p' = p/(p-1)$. Since $\partial u/\partial n + hu = 0$, we have $\partial u/\partial n = -hu \in L^{p'}(\partial\Omega)$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$ and put $\omega = \Omega \cap B(0; r)$. Then $u = \mathcal{D}^\omega u + \mathcal{S}^\omega(\partial u/\partial n^\omega)$ in ω by Theorem 5.6.1. Since $u, \partial u/\partial n \in L^{p'}(\partial\omega)$, we have $M^\omega_a u \in L^{p'}(\partial\omega)$ by Proposition 5.2.2 and Theorem 5.3.5. If Ω is unbounded, then u is bounded at infinity and thus $M^\Omega_a u \in L^{p'}(\partial\Omega)$. Lemma 5.8.1 gives

$$0 = \int_{\partial\Omega} u\left[\frac{\partial u}{\partial n} + hu\right] d\mathcal{H}_{m-1} = \int_{\Omega} |\nabla u|^2 \, d\mathcal{H}_m + \int_{\partial\Omega} hu^2 \, d\mathcal{H}_{m-1}.$$

Thus $\nabla u = 0$ in Ω, $hu = 0$ on $\partial\Omega$. Hence there exists a constant c such that $u(x) = c$. Since

$$0 = \int\limits_{\partial\Omega} hu^2 \, \mathrm{d}\mathcal{H}_{m-1} = c^2 \int\limits_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1}$$

and

$$\int\limits_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0,$$

we infer that $c = 0$.

The estimates (5.27), (5.28) follow from Propositions 5.2.3 and 5.2.8.

Example 5.11.6 Fix $2 < p < \infty$. Choose $\alpha \in (\pi p/(p-1), 2\pi)$. Denote $G_\alpha = \{x = [x_1, x_2]; x_1 = r\cos\varphi, x_2 = r\sin\varphi, r > 0, 0 < \varphi < \alpha\}$. Define $u(x) = [u_1(x), u_2(x)] = [r^{\pi/\alpha}\cos(\varphi\pi/\alpha), r^{\pi/\alpha}\sin(\varphi\pi/\alpha)]$ for $x = [r\cos\varphi, r\sin\varphi]$. Easy calculation yields that $u_1 + iu_2$ is a holomorphic function in G_α. If $z \in \partial G_\alpha$, then there exists a finite limit $u(z)$ of u and $u_2(z) = 0$. So, the tangential derivative $\partial u_2(z)/\partial\tau = 0$. Choose a bounded open set Ω such that $\Omega \cap B(0;\rho) = G_\alpha \cap B(0;\rho)$ for some $\rho > 0$ and $\partial\Omega$ is of class $\mathcal{C}^\infty$ away from 0. Put $v = u_1$ on Ω. Then $v \in H(\Omega)$ by Proposition 2.20.2. If $z \in \partial\Omega \setminus \{0\}$, then there exist finite limits of v and ∇v in z. Denote $f = \partial v/\partial n$ on $\partial\Omega \setminus \{0\}$. Clearly, $f \in L^\infty(\partial\Omega \setminus B(0;\rho)) \subset L^p(\partial\Omega \setminus B(0;\rho))$. Proposition 2.20.1 gives that $f = \partial v/\partial n = \partial u_2/\partial\tau = 0$ on $\partial\Omega \cap B(0;\rho) \setminus \{0\}$. Therefore $f \in L^p(\partial\Omega)$. Fix $a > 0$; then there exist positive constants C_1, C_2 such that $C_1|z|^{\pi/\alpha} \le M_a^\Omega(v)(z) \le C_2|z|^{\pi/\alpha}$, $C_1|z|^{\pi/\alpha-1} \le M_a^\Omega(\nabla v)(z) \le C_2|z|^{\pi/\alpha-1}$. Therefore $M_a^\Omega(u) \in L^p(\partial\Omega) \cap L^2(\partial\Omega)$, $M_a^\Omega(\nabla u) \in L^2(\partial\Omega) \setminus L^p(\partial\Omega)$. Thus v is an L^2-solution of the Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = f$. But it is not an L^p-solution of the problem in spite $f \in L^p(\partial\Omega)$.

Lemma 5.11.7 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Then there exists $f \in L^p(\partial\Omega)$ with $\int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 1$ such that $\mathcal{S}^\Omega f$ is constant on $\overline{\Omega}$ and* $\mathrm{Ker}(\frac{1}{2}I - K'_\Omega) = \{cf; c \in R^1\}$.

Proof $\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$ by Propositions 5.9.9, 5.11.1 and Theorem 1.8.3. Since $(\frac{1}{2}I - K'_\Omega)(L^p(\partial\Omega)) \neq L^p(\partial\Omega)$ by Lemma 5.9.6, $\mathrm{Ker}(\frac{1}{2}I - K'_\Omega)$ is nontrivial. Since $\mathrm{Ker}(\frac{1}{2}I - K'_\Omega) \cap \{g \in L^p(\partial\Omega); \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0\} = \{0\}$ by Theorems 5.9.10 and 5.11.3, there exists a unique $f \in \mathrm{Ker}(\frac{1}{2}I - K'_\Omega) \cap L^p(\partial\Omega)$ with $\int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 1$ and $\mathrm{Ker}(\frac{1}{2}I - K'_\Omega) = \{cf; c \in R^1\}$. Since $\mathcal{S}^\Omega f$ is an L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = 0$, there exists a constant c such that $\mathcal{S}^\Omega f = c$ on Ω (see Lemma 5.9.2 and Theorem 5.11.3). So, $\mathcal{S}^\Omega f = c$ on $\partial\Omega$ by Proposition 5.2.2.

5.12 L^p-Solution of the Transmission Problem

This section is devoted to L^p-solutions of the transmission problem. The Euclidean space is divided into two open sets Ω_+, Ω_- with Lipschitz boundary, where Ω_+ is bounded. If $p \neq 2$, we suppose that $\partial\Omega_\pm$ is Ljapunov. We find a necessary and sufficient condition for the solvability of the problem $\Delta u_\pm = 0$ in $\Omega_\pm$, $a_+u_+ - a_-u_- = g$, $b_+\frac{\partial u_+}{\partial n} - b_-\frac{\partial u_-}{\partial n} + hu_+ = f$ on $\partial\Omega$ and discuss the uniqueness.

Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Denote by $n_\pm$ the exterior unit normal of $\Omega_\pm$, $n = n_+$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega_+)$, $g \in W^{1,p}(\partial\Omega_+)$, $f \in L^p(\partial\Omega_+)$, $1 < p < \infty$. We shall study the transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.29}$$

$$a_+u_+ - a_-u_- = g, \quad b_+\frac{\partial u_+}{\partial n} - b_-\frac{\partial u_-}{\partial n} + hu_+ = f \quad \text{on } \partial\Omega. \tag{5.30}$$

If $\tilde{u}_\pm = a_\pm u_\pm$, $\tilde{b}_\pm = b_\pm/a_\pm$, $\tilde{h} = h/a_+$, then $u_\pm$ is an L^p-solution of the transmission problem (5.29), (5.30) if and only if $\tilde{u}_\pm$ is an L^p-solution of the transmission problem

$$\Delta\tilde{u}_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.31}$$

$$\tilde{u}_+ - \tilde{u}_- = g, \quad \tilde{b}_+\frac{\partial\tilde{u}_+}{\partial n} - \tilde{b}_-\frac{\partial\tilde{u}_-}{\partial n} + \tilde{h}\tilde{u}_+ = f \quad \text{on } \partial\Omega. \tag{5.32}$$

If $\tilde{u}_\pm = v_\pm + \mathcal{D}^\Omega g$, then $\tilde{u}_\pm$ is an L^p-solution of the transmission problem (5.31), (5.32) if and only if $v_\pm$ is an L^p-solution of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.33}$$

$$v_+ - v_- = 0, \quad \tilde{b}_+\frac{\partial v_+}{\partial n} - \tilde{b}_-\frac{\partial v_-}{\partial n} + \tilde{h}v_+ = F, \quad \text{on } \partial\Omega \tag{5.34}$$

where

$$F = f - (\tilde{b}_+ - \tilde{b}_-)H^\Omega g - \tilde{h}\left(\frac{1}{2}I + K_\Omega\right)g \in L^p(\partial\Omega) \tag{5.35}$$

(see Theorems 5.3.5 and 5.5.2).

For $m > 2$, we shall look for a solution of the problem (5.33), (5.34) in the form of a single layer potential $\mathcal{S}^\Omega\varphi$ with an unknown density $\varphi \in L^p(\partial\Omega)$. The function $v_\pm = \mathcal{S}^\Omega\varphi$ is an L^p-solution of the problem (5.33), (5.34) if and only if

$$\frac{\tilde{b}_+ + \tilde{b}_-}{2}\varphi - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega\varphi + \tilde{h}\mathcal{S}^\Omega\varphi = F$$

(see Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8).

If $m = 2$, we shall look for a solution of the problem (5.33), (5.34) in the modified form

$$v_\pm = \mathcal{S}^\Omega \left(\varphi - \oint\limits_{\partial\Omega} \varphi \right) + \oint\limits_{\partial\Omega} \varphi$$

with $\varphi \in L^p(\partial\Omega)$. Then $v_\pm$ is an L^p-solution of the problem (5.33), (5.34) if and only if

$$\frac{\tilde{b}_+ + \tilde{b}_-}{2}(\varphi - \oint \varphi) - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega(\varphi - \oint \varphi) + \tilde{h}[\mathcal{S}^\Omega(\varphi - \oint \varphi) + \oint \varphi] = F$$

(see Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8).

Lemma 5.12.1 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$. Let $\tilde{b}_+$, $\tilde{b}_-$ be positive constants, $\tilde{h} \equiv 0$, $F \equiv 0$. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$\tilde{b}_+ = \tilde{b}_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

If $v_\pm$ is an L^p-solution of the transmission problem (5.33), (5.34) such that $v_-(x) \to 0$ as $|x| \to \infty$, then $v_\pm \equiv 0$.

Proof Define $v = v_\pm$ on $\Omega_\pm$. Since $v_+ = v_-$ on $\partial\Omega$, the function v can be continuously extended onto R^m. Denote $f_\pm = \partial u_\pm/\partial n_\pm$. Then $\tilde{b}_+ f_+ + \tilde{b}_- f_- = 0$. Lemma 5.9.1 gives $\int_{\partial\Omega} f_+ \, \mathrm{d}\mathcal{H}_{m-1} = 0$. So, $\int_{\partial\Omega} f_- \, \mathrm{d}\mathcal{H}_{m-1} = 0$.

Suppose first that $p = 2$. Lemma 5.8.1 gives

$$0 = \int\limits_{\partial\Omega} v(\tilde{b}_+ f_+ + \tilde{b}_- f_-) \, \mathrm{d}\mathcal{H}_{m-1} = \tilde{b}_+ \int\limits_{\Omega_+} |\nabla v_+|^2 + \tilde{b}_- \int\limits_{\Omega_-} |\nabla v_-|^2.$$

Since $\nabla v = 0$ on $R^m \setminus \partial\Omega$, the function v is constant on each component of $R^m \setminus \partial\Omega$. Since $v \in \mathcal{C}^0(R^m)$ and v is constant on each component of $R^m \setminus \partial\Omega$, we deduce that v is constant. Since $v(x) \to 0$ as $|x| \to \infty$, we infer that $v \equiv 0$.

Let now p be general. Corollary 5.6.2 gives

$$\mathcal{D}^{\Omega\pm} v + \mathcal{S}^\Omega f_\pm = \begin{cases} v_\pm, & \text{in } \Omega_\pm, \\ 0, & \text{in } \Omega_\mp. \end{cases}$$

Since $\mathcal{D}^{\Omega-} v = -\mathcal{D}^{\Omega+} v$, we get adding

$$v = \mathcal{S}^\Omega (f_+ + f_-) \qquad \text{in } R^m \setminus \partial\Omega.$$

If $\tilde{b}_+ = \tilde{b}_-$, then $f_+ + f_- = 0$ and $v = 0$. Let now Ω has boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$. Then

$$\frac{\tilde{b}_+ + \tilde{b}_-}{2}(f_+ + f_-) - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega(f_+ + f_-) = 0.$$

The operator $(\tilde{b}_+/2 + \tilde{b}_-/2)I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ and in $L^2(\partial\Omega)$ by Proposition 5.11.1 and Theorem 1.8.3. Thus $f_+ + f_- \in L^2(\partial\Omega)$ by Lemma 1.8.4. So, $v = \mathcal{S}^\Omega(f_+ + f_-)$ is an L^2-solution of the problem (5.33), (5.34) by Propositions 5.2.2, 5.2.3, and 5.2.8. We have proved that $v \equiv 0$.

Proposition 5.12.2 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $m > 2$, $1 < p < \infty$. Denote $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let $\tilde{b}_+$, $\tilde{b}_-$ be positive constants. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$\tilde{b}_+ = \tilde{b}_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

Then the operator $T = [(\tilde{b}_+ + \tilde{b}_-)/2]I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega$ is an isomorphism on $L^p(\partial\Omega)$. Let $h \equiv 0$, $F \in L^p(\partial\Omega)$. Then there exists an L^p-solution of the transmission problem (5.33), (5.34). If $u_\pm$ is an L^p-solution of this problem, then there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then $u_\pm = \mathcal{S}^\Omega T^{-1}F + u_\infty$ is a unique L^p-solution of the transmission problem (5.33), (5.34) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover,

$$\|M_a(u_\pm)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u_\pm)\|_{L^p(\partial\Omega)} \le C(\|f\|_{L^p(\partial\Omega)} + |u_\infty|) \tag{5.36}$$

where C depends only on Ω, $\tilde{b}_\pm$, p, and a.

Proof The operator T is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ (see Propositions 5.9.9, 5.11.1 and Theorem 1.8.3). Let $\varphi \in L^p(\partial\Omega)$, $T\varphi = 0$. Then $u_\pm = \mathcal{S}^\Omega\varphi$ is an L^p-solution of the transmission problem (5.33), (5.34) with $F \equiv 0$. Lemma 5.12.1 gives that $u_\pm \equiv 0$. So, $\varphi = \partial u_+/\partial n - \partial u_-/\partial n = 0$ by Theorem 5.4.8. Since the index $i(T) = 0$, the operator T is onto. Hence T is an isomorphism on $L^p(\partial\Omega)$ by Theorem 1.6.3.

If $F \in L^p(\partial\Omega)$ and $u_\infty \in R^1$, then $u_\pm = \mathcal{S}^\Omega T^{-1}F + u_\infty$ is an L^p-solution of the transmission problem (5.33), (5.34) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Let now $u_\pm$ be an L^p-solution of the transmission problem (5.33), (5.34). According to Lemma 1.28.2 and Proposition 2.17.1, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. If $F = 0$ and $u_\infty = 0$, then $u_\pm \equiv 0$ by Lemma 5.12.1.

Propositions 5.2.2, 5.2.3, and 5.2.8 force (5.36).

Proposition 5.12.3 *Let $\Omega = \Omega_+ \subset R^2$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. Denote $\Omega_- = R^2 \setminus \overline{\Omega_+}$. Let $\tilde{b}_+$, $\tilde{b}_-$ be positive constants.*

Suppose that one of the following conditions is fulfilled:

1. $p = 2$;
2. $\tilde{b}_+ = \tilde{b}_-$;
3. Ω *has boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$.

Let $\tilde{h} \equiv 0$, $F \in L^p(\partial\Omega)$. *Then there exists an* L^p*-solution of the transmission problem (5.33), (5.34) if and only if*

$$\int\limits_{\partial\Omega} F \, \mathrm{d}\mathcal{H}_1 = 0. \tag{5.37}$$

Denote by X_p *the set of all* $F \in L^p(\partial\Omega)$ *satisfying (5.37). Then the operator* $T = [(\tilde{b}_+ + \tilde{b}_-)/2]I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega$ *is an isomorphism on* X_p. *If* $u_\pm$ *is an* L^p*-solution of the problem (5.33), (5.34), then there exists a constant* u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *On the other hand, if* $F \in X_p$, $u_\infty \in R^1$ *are given, then* $u_\pm = \mathcal{S}^\Omega T^{-1}F + u_\infty$ *is a unique* L^p*-solution of the transmission problem (5.33), (5.34) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover, (5.36) holds with a constant C depending only on* Ω, $\tilde{b}_\pm$, p, *and* a.

Proof Let $u_\pm$ be an L^p-solution of the problem (5.33), (5.34). Then $u_\pm$ is an L^p-solution of the Neumann problem for the Laplace equation on $\Omega_\pm$ with the boundary condition $\partial u_\pm/\partial n_\pm$. Lemma 5.9.1 gives that $\partial u_\pm/\partial n_\pm \in X_p$. Thus $F = \tilde{b}_+\partial u_+/\partial n_+ + \tilde{b}_-\partial u_-/\partial n_- \in X_p$.

The operator $[(\tilde{b}_+ + \tilde{b}_-)/2]I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ (see Propositions 5.9.9, 5.11.1 and Theorem 1.8.3). Lemmas 5.9.6 and 1.8.8 give that T is a Fredholm operator with index 0 in X_p. Let $\varphi \in X_p$, $T\varphi = 0$. Then $u_\pm = \mathcal{S}^\Omega\varphi$ is an L^p-solution of the transmission problem (5.33), (5.34) with $F \equiv 0$. Lemma 5.12.1 gives that $u_\pm \equiv 0$. Theorem 5.4.8 gives $\varphi = \partial u_+/\partial n - \partial u_-/\partial n = 0$. Since the index $i(T) = 0$, the operator T is onto. So, T is an isomorphism on X_p by Theorem 1.6.3.

If $F \in X_p$ and $u_\infty \in R^1$, then $u_\pm = \mathcal{S}^\Omega T^{-1}F + u_\infty$ is an L^p-solution of the transmission problem (5.33), (5.34) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Let now $u_\pm$ be an L^p-solution of the transmission problem (5.33), (5.34). According to Lemma 1.28.2 and Proposition 2.17.1, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. If $F = 0$ and $u_\infty = 0$, then $u_\pm \equiv 0$ by Lemma 5.12.1.

Equation (5.36) is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Proposition 5.12.4 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary,* $m > 2$, $1 < p < \infty$. *Denote* $\Omega_- = R^m \setminus \overline{\Omega}_+$. *Let* $\tilde{b}_+$, $\tilde{b}_-$ *be positive constants. Suppose that one of the following conditions is fulfilled:*

1. $p = 2$;
2. $\tilde{b}_+ = \tilde{b}_-$;
3. Ω *has boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$.

Let $\tilde{h} \in L^\infty(\partial\Omega)$, $\tilde{h} \geq 0$, $\int_{\partial\Omega} \tilde{h}\, d\mathcal{H}_{m-1} > 0$. *Then the operator* $T = [(\tilde{b}_+ + \tilde{b}_-)/2]I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega + \tilde{h}\mathcal{S}^\Omega$ *is an isomorphism on* $L^p(\partial\Omega)$. *Let* $F \in L^p(\partial\Omega)$. *Then there exists an* L^p*-solution of the transmission problem (5.33), (5.34). If* $u_\pm$ *is an* L^p*-solution of this problem, then there exists a constant* u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *On the other hand, if* $u_\infty \in R^1$ *is given, then* $u_\pm = \mathcal{S}^\Omega T^{-1}(F - \tilde{h}u_\infty) + u_\infty$ *is a unique* L^p*-solution of the transmission problem (5.33), (5.34) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover, (5.36) holds with a constant C depending only on* Ω, $\tilde{b}_\pm$, $\tilde{h}$, p, *and* a.

Proof Suppose first that $v_\pm$ is an L^2-solution of the problem (5.33), (5.34) with $F = 0$ such that $v_-(x) \to 0$ as $|x| \to \infty$. Since $v_+ = v_-$ on $\partial\Omega$, Lemma 5.8.1 gives

$$0 = \int_{\partial\Omega} v_+ \left(\tilde{b}_+ \frac{\partial v_+}{\partial n_+} + \tilde{b}_- \frac{\partial v_-}{\partial n_-} + \tilde{h} v_+ \right) d\mathcal{H}_{m-1}$$

$$= \tilde{b}_+ \int_{\Omega_+} |\nabla v_+|^2 \, d\mathcal{H}_m + \tilde{b}_- \int_{\Omega_-} |\nabla v_-|^2 \, d\mathcal{H}_m + \int_{\partial\Omega} \tilde{h} v_+^2 \, d\mathcal{H}_{m-1}.$$

Denote $v = v_\pm$ on $\Omega_\pm$. Since $\nabla v = 0$ on $R^m \setminus \partial\Omega$, the function v is constant on each component of $R^m \setminus \partial\Omega$. Since $v_+ = v_-$ on $\partial\Omega$ and v is constant on each component of $R^m \setminus \partial\Omega$, we deduce that v is constant. Since $v(x) \to 0$ as $|x| \to \infty$, we infer that $v \equiv 0$.

The operator $T - \tilde{h}\mathcal{S}^\Omega$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ by Proposition 5.12.2. Since $\tilde{h}\mathcal{S}^\Omega$ is a compact operator on $L^p(\partial\Omega)$ (see Proposition 5.2.2 and Theorem 1.7.4), Theorem 1.8.3 gives that T is a Fredholm operator with index 0 in $L^p(\partial\Omega)$. Let $\varphi \in L^p(\partial\Omega)$, $T\varphi = 0$. Then $\varphi \in L^2(\partial\Omega)$ by Lemma 1.8.4 and $u_\pm = \mathcal{S}^\Omega\varphi$ is an L^2-solution of the transmission problem (5.33), (5.34) with $F \equiv 0$. We have proved that $u_\pm \equiv 0$. Theorem 5.4.8 gives $\varphi = \partial u_+/\partial n - \partial u_-/\partial n = 0$. Since the index $i(T) = 0$, the operator T is onto. So, T is an isomorphism on $L^p(\partial\Omega)$ by Theorem 1.6.3.

If $F \in L^p(\partial\Omega)$ and $u_\infty \in R^1$, then $u_\pm = \mathcal{S}^\Omega T^{-1}(F - \tilde{h}u_\infty) + u_\infty$ is an L^p-solution of the transmission problem (5.33), (5.34) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Let now $u_\pm$ be an L^p-solution of the transmission problem (5.33), (5.34). According to Lemma 1.28.2 and Proposition 2.17.1, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Let $F = 0$ and $u_\infty = 0$. Since $u_\pm$ is an L^p-solution of the transmission problem $\Delta u_\pm = 0$ in $\Omega_\pm$, $u_+ - u_- = 0$, $\tilde{b}_+ \partial u_+/\partial n - \tilde{b}_- \partial u_-/\partial n = -\tilde{h}u_+$ on $\partial\Omega$, Proposition 5.12.2 gives that there exists $\varphi \in L^p(\partial\Omega)$ such that $u_\pm = \mathcal{S}^\Omega\varphi$. Since $T\varphi = 0$, and T is an isomorphism, we deduce that $\varphi = 0$. Hence $u_\pm = \mathcal{S}^\Omega\varphi \equiv 0$.

Propositions 5.2.2, 5.2.3, and 5.2.8 force (5.36).

Proposition 5.12.5 *Let* $\Omega = \Omega_+ \subset R^2$ *be a bounded open set with Lipschitz boundary,* $1 < p < \infty$. *Denote* $\Omega_- = R^2 \setminus \overline{\Omega_+}$. *Let* $\tilde{b}_+$, $\tilde{b}_-$ *be positive constants.*

Suppose that one of the following conditions is fulfilled:

1. $p = 2$;
2. $\tilde{b}_+ = \tilde{b}_-$;
3. Ω *has boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$.

Let $\tilde{h} \in L^\infty(\partial\Omega)$, $\tilde{h} \geq 0$, $\int_{\partial\Omega} \tilde{h}\, \mathrm{d}\mathcal{H}_1 > 0$. *Denote* $T\varphi = (\tilde{b}_+ + \tilde{b}_-)(\varphi - \oint\varphi)/2 - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega(\varphi - \oint\varphi) + \tilde{h}[\mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi]$. *Then* T *is an isomorphism on* $L^p(\partial\Omega)$. *Let* $F \in L^p(\partial\Omega)$. *Put* $\varphi = T^{-1}F$. *Then* $u_\pm = \mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi$ *is a unique* L^p*-solution of the transmission problem (5.33), (5.34). Moreover,*

$$\|M_a(u_\pm)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u_\pm)\|_{L^p(\partial\Omega)} \leq C\|F\|_{L^p(\partial\Omega)} \tag{5.38}$$

where C *depends only on* Ω, p, $\tilde{b}_\pm$, $\tilde{h}$, *and* a.

Proof Suppose first that $v_\pm$ is an L^2-solution of the problem (5.33), (5.34) with $F = 0$. Since $v_+ = v_-$ on $\partial\Omega$, Lemma 5.8.1 gives

$$0 = \int\limits_{\partial\Omega} v_+ \left(\tilde{b}_+ \frac{\partial v_+}{\partial n_+} + \tilde{b}_- \frac{\partial v_-}{\partial n_-} + \tilde{h} v_+\right) \mathrm{d}\mathcal{H}_1$$

$$= \tilde{b}_+ \int\limits_{\Omega_+} |\nabla v_+|^2\, \mathrm{d}\mathcal{H}_2 + \tilde{b}_- \int\limits_{\Omega_-} |\nabla v_-|^2\, \mathrm{d}\mathcal{H}_2 + \int\limits_{\partial\Omega} \tilde{h} v_+^2\, \mathrm{d}\mathcal{H}_1.$$

This gives $\nabla v_\pm = 0$ in $\Omega_\pm$, $\tilde{h}v_+ = 0$ on $\partial\Omega$. Denote $v = v_\pm$ on $\Omega_\pm$. Since $\nabla v = 0$ on $R^m \setminus \partial\Omega$, the function v is constant on each component of $R^m \setminus \partial\Omega$. Since $v_+ = v_-$ on $\partial\Omega$ and v is constant on each component of $R^m \setminus \partial\Omega$, we deduce that v is constant. Since $\tilde{h}v = 0$ on $\partial\Omega$ and $\tilde{h} \not\equiv 0$, we deduce that $v \equiv 0$.

The operator $\tau = [(\tilde{b}_+ + \tilde{b}_-)/2]I - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ (see Propositions 5.9.9, 5.11.1 and Theorem 1.8.3). Since $T - \tau$ is a compact operator on $L^p(\partial\Omega)$ (see Proposition 5.2.2 and Theorem 1.7.4), Theorem 1.8.3 gives that T is a Fredholm operator with index 0 in $L^p(\partial\Omega)$. Let $\varphi \in L^p(\partial\Omega)$, $T\varphi = 0$. Then $\varphi \in L^2(\partial\Omega)$ by Lemma 1.8.4 and $u_\pm = \mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi$ is an L^2-solution of the transmission problem (5.33), (5.34) with $F \equiv 0$. We have proved that $u_\pm \equiv 0$. Theorem 5.4.8 gives $\varphi = \partial u_+/\partial n - \partial u_-/\partial n = 0$. Since the index $i(T) = 0$, the operator T is onto. So, T is an isomorphism on $L^p(\partial\Omega)$ by Theorem 1.6.3.

Let $F \in L^p(\partial\Omega)$. Put $\varphi = T^{-1}F$. Then $u_\pm = \mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi$ is an L^p-solution of the transmission problem (5.33), (5.34).

Let now $v_\pm$ be an L^p-solution of the transmission problem (5.33), (5.34) with $F = 0$. Since $v_\pm$ is an L^p-solution of the transmission problem $\Delta v_\pm = 0$ in $\Omega_\pm$, $v_+ - v_- = 0$, $\tilde{b}_+\partial v_+/\partial n - \tilde{b}_-\partial v_-/\partial n = -\tilde{h}v_+$ on $\partial\Omega$, Proposition 5.12.3 gives that there exist a constant c and $\psi \in L^p(\partial\Omega)$ with $\int_{\partial\Omega} \psi\, \mathcal{H}_{m-1} = 0$ such that $v_\pm = \mathcal{S}^\Omega\psi + c$. Put $\varphi = \psi + c$. Then $v_\pm = \mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi$ and therefore $T\varphi = 0$. Since T is an isomorphisms, we obtain $\varphi = 0$. Thus $v_\pm = \mathcal{S}^\Omega(\varphi - \oint\varphi) + \oint\varphi = 0$.

Propositions 5.2.2, 5.2.3, and 5.2.8 force (5.38).

Corollary 5.12.6 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$. Let $\tilde{b}_+$, $\tilde{b}_-$ be positive constants. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$\tilde{b}_+ = \tilde{b}_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

Let $\tilde{h} \in L^\infty(\partial\Omega)$, $\tilde{h} \geq 0$, $\tau \in L^\infty(\partial\Omega, R^m)$ be a tangential vector function on $\partial\Omega$, $F \in L^p(\Omega)$, $u_\infty \in R^1$.

- *If $m > 2$, then there exists a unique L^p-solution of the transmission problem (5.33),*

$$u_+ - u_- = 0, (\tilde{b}_+ n + \tau) \cdot \nabla u_+ - (\tilde{b}_- n + \tau) \cdot \nabla u_- + \tilde{h} u_+ = F \tag{5.39}$$

  on $\partial\Omega$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover, (5.36) holds with a constant C dependent only on Ω, p, $\tilde{b}_+$, $\tilde{b}_-$, $\tilde{h}$, and a.
- *If $m = 2$ and $\int_{\partial\Omega} \tilde{h}\, \mathrm{d}\mathcal{H}_1 > 0$, then there exists a unique L^p-solution of the transmission problem (5.33), (5.39). Moreover, (5.38) holds with a constant C dependent only on Ω, p, $\tilde{b}_+$, $\tilde{b}_-$, $\tilde{h}$, and a.*
- *If $m = 2$ and $\tilde{h} \equiv 0$, then there exists an L^p-solution of the transmission problem (5.33), (5.39) if and only if $\int_{\partial\Omega} F\, \mathrm{d}\mathcal{H}_1 = 0$. If this condition is fulfilled, then there exists a unique L^p-solution of the transmission problem (5.33), (5.39) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover, the estimate (5.36) holds with a constant C dependent only on Ω, p, $\tilde{b}_+$, $\tilde{b}_-$, $\tilde{h}$, and a.*

Proof If $u_\pm$ is an L^p-solution of the problem (5.33), (5.39) or (5.33), (5.34), then $\tilde{b}_+ n \cdot \nabla u_+ - \tilde{b}_- n \cdot \nabla u_- = G$ on $\partial\Omega$ for some $G \in L^p(\partial\Omega)$. According to Propositions 5.12.2 and 5.12.3, there exists $\varphi \in L^p(\partial\Omega)$ and $c \in R^1$ such that $u_\pm = \mathcal{S}^\Omega \varphi + c$. Since the tangential derivative of a single layer has no skip on the boundary (see Theorem 5.4.7), $u_\pm$ is an L^p-solution of the problem (5.33), (5.34) if and only if it is an L^p-solution of the problem (5.33), (5.39). The rest is a consequence of Propositions 5.12.2, 5.12.3, 5.12.4, and 5.12.5.

Theorem 5.12.7 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$b_+/a_+ = b_-/a_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

Let $f \in L^p(\partial\Omega)$, $g \in W^{1,p}(\partial\Omega)$.

- *If $m > 2$, then there exists an L^p solution of the transmission problem (5.29), (5.30). If $u_\pm$ is an L^p-solution of this problem, then there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then there exists a unique L^p-solution $u_\pm$ of the transmission problem (5.29), (5.30) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u_\pm)\|_{L^p} + \|M_a(\nabla u_\pm)\|_{L^p} \le C(\|f\|_{L^p} + \|g\|_{W^{1,p}} + |u_\infty|) \tag{5.40}$$

  with a constant C dependent only on Ω, a_+, a_-, b_+, b_-, h, p, and a.
- *If $m = 2$ and $h \not\equiv 0$, then there exists a unique L^p-solution of the transmission problem (5.29), (5.30). Moreover*

$$\|M_a(u_\pm)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u_\pm)\|_{L^p(\partial\Omega)} \le C(\|f\|_{L^p(\partial\Omega)} + \|g\|_{W^{1,p}(\partial\Omega)})$$

  with a constant C dependent only on Ω, a_+, a_-, b_+, b_-, h, p, and a.
- *Let $m = 2$ and $h \equiv 0$. Then there exists an L^p solution of the transmission problem (5.29), (5.30) if and only if*

$$\int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{5.41}$$

  If $u_\pm$ is an L^p-solution of the transmission problem (5.29), (5.30), then there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if (5.41) is satisfied and $u_\infty \in R^1$ is given, then there exists a unique L^p-solution $u_\pm$ of the transmission problem (5.29), (5.30) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover, the estimate (5.40) holds with a constant C dependent only on Ω, a_+, a_-, b_+, b_-, p, and a.

Proof The theorem is a consequence of Corollary 5.12.6 and reasoning at the beginning of the section.

5.13 L^p-Solution of the Jump Problem

In this section, we study L^p-solutions of the jump problem on cracks $\Delta u = 0$ in $R^m \setminus \Gamma$, $[u]_+ - [u]_- = g$, $[\partial u/\partial n]_+ - [\partial u/\partial n]_- + h[u]_+ = f$ on Γ. Here Γ is a closed subset of a bounded Lipschitz surface. We find a necessary and sufficient condition for the solvability of the problem and discuss the uniqueness.

Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. If v is a function defined on $R^m \setminus \partial\Omega$, denote by $[v]_\pm$ the nontangential limit of v with respect to $\Omega_\pm$ and $M_a^\pm v$ the nontangential maximal function of v with respect to $\Omega_\pm$. Denote by $n_\pm$ the exterior unit normal of $\Omega_\pm$, $n = n_+$. Let Γ

be a closed subset of $\partial\Omega$, $g \in W^{1,p}(\partial\Omega), f \in L^p(\partial\Omega)$, $1 < p < \infty$. The problem

$$\Delta u = 0 \quad \text{in } R^m \setminus \Gamma,$$

$$[u]_+ - [u]_- = g, \quad \left[\frac{\partial u}{\partial n}\right]_+ - \left[\frac{\partial u}{\partial n}\right]_- = f \quad \text{on } \Gamma$$

is so-called *jump problem.* We shall study a generalized jump problem. Let $h \in L^\infty(\partial\Omega)$. *We say that u is an L^p-solution of the generalized jump problem*

$$\Delta u = 0 \quad \text{in } R^m \setminus \Gamma, \tag{5.42}$$

$$[u]_+ - [u]_- = g, \quad \left[\frac{\partial u}{\partial n}\right]_+ - \left[\frac{\partial u}{\partial n}\right]_- + h[u]_+ = f \quad \text{on } \Gamma \tag{5.43}$$

if $u \in H(R^m \setminus \Gamma)$, $M_a^+ u + M_a^- u + M_a^+|\nabla u| + M_a^-|\nabla u| \in L^p(\partial\Omega)$, there exist nontangential limits of u and ∇u with respect to Ω_+ and with respect to Ω_- at almost all points of $\partial\Omega$, and the jump conditions (5.43) on the crack Γ are satisfied in the sense of nontangential limits at $\mathcal{H}_{m-1}$-almost all points of Γ.

Lemma 5.13.1 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let Γ be a closed subset of $\partial\Omega$, $h \in L^\infty(\partial\Omega)$, $g \in W^{1,p}(\partial\Omega)$, $f \in L^p(\partial\Omega)$, $1 < p < \infty$. Define $\tilde h = h\chi_\Gamma$, $\tilde g = g\chi_\Gamma$, $\tilde f = f\chi_\Gamma$.*

- *Suppose that there exists an L^p-solution u of the generalized jump problem (5.42), (5.43). Define $u_\pm = u$ on $\Omega_\pm$. Then $u_\pm$ is an L^p-solution of the transmission problem*

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.44}$$

$$u_+ - u_- = \tilde g, \quad \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} + \tilde h u_+ = \tilde f \quad \text{on } \partial\Omega. \tag{5.45}$$

- *Let $u_\pm$ be an L^p-solution of the transmission problem (5.44), (5.45). Then there exist the nontangential limits $u_+(x) = u_-(x)$ at all $x \in \partial\Omega \setminus \Gamma$. Define $u = u_\pm$ in $\Omega_\pm$, $u = u_+$ on $\partial\Omega \setminus \Gamma$. Then u is an L^p-solution of the generalized jump problem (5.42), (5.43).*

Proof Suppose first that u is an L^p-solution of the generalized jump problem (5.42), (5.43). Clearly, $u_\pm$ is an L^p-solution of the transmission problem (5.44), (5.45).

Let now $u_\pm$ be an L^p-solution of the transmission problem (5.44), (5.45). It is enough to prove that $u \in H(R^m \setminus \Gamma)$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Denote $\omega = \Omega_- \cap B(0; r)$. Corollary 5.6.2 gives

$$\mathcal{D}^\Omega u_+ + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u_+) = \begin{cases} u \text{ in } \Omega, \\ 0 \text{ in } R^m \setminus \overline{\Omega}, \end{cases}$$

$$\mathcal{D}^\omega u_- + \mathcal{S}^\omega(n^\omega \cdot \nabla u_-) = \begin{cases} u \text{ in } \omega, \\ 0 \text{ in } R^m \setminus \overline{\omega}. \end{cases}$$

Adding

$$u = \mathcal{D}^{\Omega}\tilde{g} + \mathcal{S}^{\Omega}(\tilde{f} - \tilde{h}u_+) + \mathcal{D}^{B(0;r)}u_- + \mathcal{S}^{B(0;r)}(n^{B(0;r)} \cdot \nabla u_-)$$

in $B(0;r) \setminus \partial\Omega$. Since the densities of these potentials are supported on $\Gamma \cup \partial B(0;r)$, these potentials are harmonic in $B(0;r) \setminus \Gamma$. So, $u \in H(B(0;r) \setminus \Gamma)$.

Theorem 5.13.2 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let Γ be a closed subset of $\partial\Omega$. Let $1 < p < \infty$, $g \in W^{1,p}(\partial\Omega)$, $f \in L^p(\partial\Omega)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that $f = 0$, $g = 0$, $h = 0$ on $\partial\Omega \setminus \Gamma$.*

- *If $m > 2$, then there exists an L^p solution of the generalized jump problem (5.42), (5.43). If u is an L^p-solution of this problem, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then there exists a unique L^p-solution u of the generalized jump problem (5.42), (5.43) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u)\|_{L^p} + \|M_a(\nabla u)\|_{L^p} \leq C(\|f\|_{L^p} + \|g\|_{W^{1,p}} + |u_\infty|) \tag{5.46}$$

  with a constant C dependent only on Ω, p, h, and a. If $h = 0$, then $u = \mathcal{D}^{\Omega}g + \mathcal{S}^{\Omega}f$ is a unique L^p-solution of the generalized jump problem (5.42), (5.43) such that $u(x) \to 0$ as $|x| \to 0$.
- *If $m = 2$ and $h \not\equiv 0$, then there exists a unique L^p-solution of the generalized jump problem (5.42), (5.43). Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \leq C(\|f\|_{L^p(\partial\Omega)} + \|g\|_{W^{1,p}(\partial\Omega)})$$

  with a constant C dependent only on Ω, h, p, and a.
- *Let $m = 2$ and $h \equiv 0$. Then there exists an L^p-solution of the generalized jump problem (5.42), (5.43) if and only if*

$$\int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{5.47}$$

  If u is an L^p-solution of the generalized jump problem (5.42), (5.43), then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if (5.47) is satisfied and $u_\infty \in R^1$ is given, then there exists a unique L^p-solution u of the generalized jump problem (5.42), (5.43) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover, the estimate (5.46) holds with a constant C dependent only on Ω, p, and a. The function $u = \mathcal{D}^{\Omega}g + \mathcal{S}^{\Omega}f$ is a unique L^p-solution of the generalized jump problem (5.42), (5.43) such that $u(x) \to 0$ as $|x| \to 0$.

Proof The theorem is an easy consequence of Lemma 5.13.1, Theorem 5.12.7, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.3.5, 5.4.8, and 5.5.2.

5.14 Regular Solutions of the Dirichlet Problem

In this section, we study the Dirichlet problem for the Laplace equation with a boundary condition $g \in W^{1,p}(\partial\Omega)$, where $\Omega \subset R^m$ is an open set with compact Lipschitz boundary. If $p \neq 2$, we suppose that $\partial\Omega$ is Ljapunov. We prove the existence of such L^p-solution u of the problem that $M_a(\nabla u) \in L^p(\partial\Omega)$.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$, $g \in W^{1,p}(\partial\Omega)$. In this section, we shall look for an L^p-solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega$$

such that $M_a(\nabla u) \in L^p(\partial\Omega)$.

Proposition 5.14.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. Let u be an L^2-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ such that $M_a(\nabla u) \in L^2(\partial\Omega)$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof Without loss of generality, we can suppose that Ω is connected. According to Theorem 5.6.1, there exists a nontangential limit of ∇u at almost all points of $\partial\Omega$. Lemma 5.8.1 gives

$$0 = \int_{\partial\Omega} u \frac{\partial u}{\partial n}\, d\mathcal{H}_{m-1} = \int_{\Omega} |\nabla u|^2\, d\mathcal{H}_m.$$

Since $\nabla u = 0$ in Ω, the function u is constant. Since $u = 0$ on $\partial\Omega$, we infer that $u \equiv 0$.

Proposition 5.14.2 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < \infty$. Let u be an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ such that $M_a(\nabla u) \in L^p(\partial\Omega)$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$ and put $\omega = \Omega \cap B(0; r)$, $G = R^m \setminus \overline{\omega}$. According to Corollary 5.6.2, there exists a nontangential limit of ∇u at almost all points of $\partial\Omega$ and $0 = \mathcal{D}^\omega u + \mathcal{S}^\omega f$ in G, where $f = \partial u/\partial n^\omega$. The fact $M_a(\nabla u) \in L^p(\partial\Omega)$ forces $f \in L^p(\partial\omega)$. Since $\mathcal{S}^G f = \mathcal{D}^G u$ in G and $u \in W^{1,2}(\partial G) \cap W^{1,p}(\partial G)$, Theorems 5.4.8 and 5.5.2 give that $f/2 - K'_G f = \partial\mathcal{S}^G f/\partial n^G = \partial\mathcal{D}^G u/\partial n^G \in L^2(\partial G) \cap L^p(\partial G)$. The operator K'_G is a compact operator in $L^p(\partial G)$ and in $L^2(\partial G)$ by Proposition 5.11.1. Thus $f \in L^2(\partial G)$ by Lemma 1.8.4 and Theorem 1.8.3. Since $u = \mathcal{D}^\omega u + \mathcal{S}^\omega f$ in ω by Corollary 5.6.2, we have $M_a(u), M_a(\nabla u) \in L^2(\partial\omega)$ (see Propositions 5.2.2, 5.2.8, Theorems 5.5.2 and 5.3.5). If Ω is unbounded, then u and ∇u are bounded at infinity. Thus $M_a(u), M_a(\nabla u) \in L^2(\partial\Omega)$. Proposition 5.14.1 gives $u \equiv 0$.

Proposition 5.14.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $\lambda \in R$, $|\lambda| > 1/2$. Then $\lambda I + K_\Omega$ is an isomorphism on $W^{1,2}(\partial\Omega)$.*

Proof Suppose first that $\lambda > 1/2$. The operator K_Ω is bounded on $W^{1,2}(\partial\Omega)$ by Lemma 5.5.4. Put $\Omega_+ = \Omega$, $\Omega_- = R^m \setminus \overline{\Omega}$, $a_+ = (2\lambda + 1)/2$, $a_- = (2\lambda - 1)/2$. Fix $g \in W^{1,2}(\partial\Omega)$. According to Theorem 5.12.7, there exists a unique L^2-solution of the transmission problem

$$\Delta u_\pm = 0 \text{ in } \Omega_\pm, \quad a_+u_+ - a_-u_- = g \text{ on } \partial\Omega, \quad \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} = 0 \text{ on } \partial\Omega$$

such that $u_\pm(x) \to 0$ as $|x| \to \infty$. According to Corollary 5.6.7,

$$u_\pm = \mathcal{D}^\Omega(u_+ - u_-) + \mathcal{S}^\Omega\left[\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}\right] = \mathcal{D}^\Omega(u_+ - u_-) \quad \text{in } \Omega_\pm.$$

Theorem 5.3.5 gives $u_\pm = \pm\frac{1}{2}(u_+ - u_-) + K_\Omega(u_+ - u_-)$ on $\partial\Omega$. Clearly $f = u_+ - u_- \in W^{1,2}(\partial\Omega)$. We have

$$g = a_+u_+ - a_-u_- = a_+\frac{1}{2}f + K_\Omega f - a_-(-f/2 + K_\Omega f) = \lambda f + K_\Omega f.$$

Thus the operator $\lambda I + K_\Omega$ is lower semi-Fredholm in $W^{1,2}(\partial\Omega)$. If $\lambda > \|K_\Omega\|$, then $\lambda I + K_\Omega$ is an isomorphism by Proposition 1.6.7 and therefore Fredholm with index 0. The continuity of the index (Proposition 1.8.5) gives that $i(\lambda I + K_\Omega) = 0$ for all $\lambda > 1/2$ and the operator $\lambda I + K_\Omega$ is Fredholm. Since it is onto, it is one-to-one. Theorem 1.6.3 gives that $\lambda I + K_\Omega$ is an isomorphism on $W^{1,2}(\partial\Omega)$.

Let now $\lambda < -1/2$. Put $\omega = R^m \setminus \overline{\Omega}$. We have proved that $-\lambda I + K_\omega = -(\lambda I + K_\Omega)$ is an isomorphism on $W^{1,2}(\partial\Omega)$.

Proposition 5.14.4 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Then $\frac{1}{2}I + K_\Omega$, $\frac{1}{2}I - K_\Omega$ are Fredholm operators with index* 0 *on* $W^{1,2}(\partial\Omega)$.

Proof The operator K_Ω is bounded on $W^{1,2}(\partial\Omega)$ by Lemma 5.5.4. Denote $\omega = R^m \setminus \overline{\Omega}$. Fix $f \in W^{1,2}(\partial\Omega)$ and put $u = \mathcal{D}^\Omega f$. Then $u \in H(\Omega) \cap H(\omega)$, $M_a^\Omega(u) + M_a^\Omega(\nabla u) + M_a^\omega(u) + M_a^\omega(\nabla u) \in L^2(\partial\Omega)$, $u(x) \to 0$ as $|x| \to \infty$. (See Remark 2.1.5, Theorems 5.3.5 and 5.5.2.) The nontangential limits of u are given by $[u]_\Omega = f/2 + K_\Omega f$, $[u]_\omega = -f/2 + K_\Omega f$ (see Theorem 5.3.5). According to Proposition 5.8.4, there exists a constant $C_1 \in (1, \infty)$ depending only on Ω such that

$$\|n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_\Omega\|_{L^2(\partial\Omega)} \leq C_1\{\|\nabla_\tau(f/2 + K_\Omega f)\|_{L^2(\partial\Omega)} + \|(f/2 + K_\Omega f)\|_{L^2(\partial\Omega)}\},$$

$$\|\nabla_\tau(f/2 + K_\Omega f)\|_{L^2(\partial\Omega)} \leq C_1\{\|n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_\Omega\|_{L^2(\partial\Omega)} + \|(f/2 + K_\Omega f)\|_{L^2(\partial\Omega)}\},$$

$$\|n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_\omega\|_{L^2(\partial\Omega)} \leq C_1\{\|\nabla_\tau(f/2 - K_\Omega f)\|_{L^2(\partial\Omega)} + \|(f/2 - K_\Omega f)\|_{L^2(\partial\Omega)}\},$$

$$\|\nabla_\tau(f/2 - K_\Omega f)\|_{L^2(\partial\Omega)} \leq C_1\{\|n^\Omega \cdot [\nabla\mathcal{D}^\Omega f]_\omega\|_{L^2(\partial\Omega)} + \|(f/2 - K_\Omega f)\|_{L^2(\partial\Omega)}\}.$$

According to Proposition 5.3.3, there exists a positive constant C_2 depending only on Ω such that $\|K_\Omega f\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)}$. Since $n^\Omega \cdot [\nabla \mathcal{D}^\Omega f]_\Omega = n^\Omega \cdot [\nabla \mathcal{D}^\Omega f]_\omega$ by Theorem 5.5.2, we have

$$\|f\|_{W^{1,2}(\partial\Omega)} = \|f\|_{L^2(\partial\Omega)} + \|\nabla_\tau f\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)} + \|\nabla_\tau (f/2 + K_\Omega f)\|_{L^2}$$

$$+ \|\nabla_\tau (f/2 - K_\Omega f)\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)} + \|\nabla_\tau (f/2 + K_\Omega f)\|_{L^2(\partial\Omega)}$$

$$+ C_1 \{ \|n^\Omega \cdot [\nabla \mathcal{D}^\Omega f]_\omega \|_{L^2(\partial\Omega)} + \|(f/2 - K_\Omega f)\|_{L^2(\partial\Omega)} \} \leq (1 + C_1) \|f\|_{L^2(\partial\Omega)}$$

$$+ C_1 \|(f/2 + K_\Omega f)\|_{L^2(\partial\Omega)} \} + \|\nabla_\tau (f/2 + K_\Omega f)\|_{L^2(\partial\Omega)} + C_1^2 \{ \|f/2 + K_\Omega f\|_{L^2}$$

$$+ \|\nabla_\tau (f/2 + K_\Omega f)\|_{L^2} \} \leq (1 + C_1) \|f\|_{L^2(\partial\Omega)} + 2C_1^2 \|(f/2 + K_\Omega f)\|_{W^{1,2}(\partial\Omega)}.$$

The identity operator is a compact operator from $W^{1,2}(\partial\Omega)$ to $L^2(\partial\Omega)$ by Lemma 1.21.8. Since $\|(\frac{1}{2} I + K_\Omega) f\|_{W^{1,2}(\partial\Omega)} + \|f\|_{L^2(\partial\Omega)}$ is an equivalent norm on $W^{1,2}(\partial\Omega)$, Lemma 1.7.9 gives that $\frac{1}{2} I + K_\Omega$ is an upper semi-Fredholm operator on $W^{1,2}(\partial\Omega)$. If $\lambda > 1/2$, then $\lambda I + K_\Omega$ is an isomorphism on $W^{1,2}(\partial\Omega)$ (see Proposition 5.14.3). Theorem on the continuity of the index (Proposition 1.8.5) gives that $i(\frac{1}{2} I + K_\Omega) = 0$. Hence $\frac{1}{2} I + K_\Omega$ is Fredholm.

We have proved that $\frac{1}{2} I + K_\omega = \frac{1}{2} I - K_\Omega$ is a Fredholm operator with index 0 in $W^{1,2}(\partial\Omega)$.

Lemma 5.14.5 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, $0 < \alpha < 1$, $1 < p < \infty$. For $j, k \in \{1, \dots, m\}$ denote $\partial_{\tau_{jk}} = n_j^\Omega \partial_k - n_k^\Omega \partial_j$,*

$$T_{jk} f(z) = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega \setminus B(z;\epsilon)} \frac{(y_j - z_j)(n_k^\Omega(y) - n_k^\Omega(z))}{\mathcal{H}_{m-1}(\partial B(0;1)) |z - y|^m} f(y) \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T_{jk} is a compact linear operator on $L^p(\partial\Omega)$. If $f \in W^{1,p}(\partial\Omega)$, then

$$\partial_{\tau_{jk}} K f(z) = \sum_{i=1}^m [T_{ik} \partial_{\tau_{ji}} f(z) - T_{ij} \partial_{\tau_{ki}} f(z)] + K_\Omega \partial_{\tau_{jk}} f(z).$$

Proof Since there exists a constant C_1 such that $|(y_j - z_j)(n_k^\Omega(y) - n_k^\Omega(z))| / |z - y|^m \leq C_1 |y - z|^{1-m+\alpha}$ for $y, z \in \partial\Omega$, the operator T_{jk} is a compact operator on $L^p(\partial\Omega)$ (see Proposition 1.26.3).

If $f \in \mathcal{C}^1(R^m)$, then we have for almost all $z \in \partial\Omega$ by Theorem 5.3.5, Lemma 5.5.1, Theorems 5.5.2 and 5.4.7

$$\partial_{\tau_{jk}} [f(z)/2 + K f(z)] = \partial_{\tau_{jk}} [\nabla \mathcal{D}^\Omega f(z)]_\Omega = n_j(z) \sum_{i=1}^m [\partial_i \mathcal{S}^\Omega (\partial_{\tau_{ki}} f)(z)]_\Omega$$

$$- n_k(z) \sum_{i=1}^m [\partial_i \mathcal{S}^\Omega (\partial_{\tau_{ji}} f)(z)]_\Omega = \sum_{i=1}^m \frac{1}{2} n_i(z) [n_j(z) \partial_{\tau_{ki}} f(z) - n_k(z) \partial_{\tau_{ji}} f(z)]$$

$$+\lim_{\epsilon\downarrow 0}\int\limits_{\partial\Omega\setminus B(z;\epsilon)}\sum_{i=1}^{m}\frac{y_i-z_i}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m}[n_j(z)\partial_{\tau_{ki}}f(y)-n_k(z)\partial_{\tau_{ji}}f(y)]$$

$$=\frac{1}{2}\partial_{\tau_{jk}}f(z)+\sum_{i=1}^{m}[T_{ik}\partial_{\tau_{ji}}f(z)-T_{ij}\partial_{\tau_{ki}}f(z)]$$

$$+\lim_{\epsilon\downarrow 0}\int\limits_{\partial\Omega\setminus B(z;\epsilon)}\sum_{i=1}^{m}\frac{(y_i-z_i)n_i^{\Omega}(y)}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m}\partial_{\tau_{jk}}f(y)\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

Since $\mathcal{C}^1(R^m)$ is a dense subset of $W^{1,p}(\partial\Omega)$, we have

$$\partial_{\tau_{jk}}Kf(z)=\sum_{i=1}^{m}[T_{ik}\partial_{\tau_{ji}}f(z)-T_{ij}\partial_{\tau_{ki}}f(z)]+K_{\Omega}\partial_{\tau_{jk}}f(z)$$

for all $f\in W^{1,p}(\partial\Omega)$.

Proposition 5.14.6 *Let $\Omega\subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m\geq 2$, $0<\alpha<1$, $1<p<\infty$. Then K_{Ω} is a compact linear operator on $W^{1,p}(\partial\Omega)$.*

Proof K_{Ω} is a compact operator on $L^p(\partial\Omega)$ by Proposition 5.11.1. K_{Ω} is a bounded linear operator on $W^{1,p}(\partial\Omega)$ by Lemma 5.5.4.

For $j,k\in\{1,\dots,m\}$ and $f\in W^{1,p}(\partial\Omega)$ denote $\partial_{\tau_{jk}}=n_j^{\Omega}\partial_k-n_k^{\Omega}\partial_j$,

$$T_{jk}f(z)=\lim_{\epsilon\downarrow 0}\int\limits_{\partial\Omega\setminus B(z;\epsilon)}\frac{(y_j-z_j)(n_k^{\Omega}(y)-n_k^{\Omega}(z))}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m}f(y)\,\mathrm{d}\mathcal{H}_{m-1}.$$

Then

$$\partial_{\tau_{jk}}Kf(z)=\sum_{i=1}^{m}[T_{ik}\partial_{\tau_{ji}}f(z)-T_{ij}\partial_{\tau_{ki}}f(z)]+K_{\Omega}\partial_{\tau_{jk}}f(z)$$

by Lemma 5.14.5. Since T_{ik} and K_{Ω} are compact operators on $L^p(\partial\Omega)$ by Lemma 5.14.5, the operator K_{Ω} is a compact operator on $W^{1,p}(\partial\Omega)$.

Proposition 5.14.7 *Let $\Omega\subset R^m$ be an open set with compact Lipschitz boundary, $m>2$, $1<p<\infty$. Suppose that $p=2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0<\alpha<1$. Then $\frac{1}{2}I+K_{\Omega}+\mathcal{S}^{\Omega}$ is an isomorphism on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$.*

Proof Denote $\omega=R^m\setminus\overline{\Omega}$. Let $\omega(1),\dots,\omega(k)$ be all components of ω. Then $\frac{1}{2}I-K'_{\omega(j)}+\mathcal{S}^{\omega(j)}$ is a Fredholm operator with index 0 on $L^2(\partial\omega(j))$. (See Theorem 5.10.4.) Define $Tf=[\frac{1}{2}I-K'_{\omega(j)}+\mathcal{S}^{\omega(j)}](f\chi_{\omega(j)})$ on $\partial\omega(j)$. Clearly, T is a Fredholm operator with index 0 on $L^2(\partial\Omega)$. The operator $\frac{1}{2}I-K'_{\omega}+\mathcal{S}^{\omega}-T$

is an integral operator with continuous kernel and therefore a compact operator on $L^2(\partial\Omega)$. (See Proposition 1.26.2.) So, $\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega$ is a Fredholm operator with index 0 on $L^2(\partial\Omega)$ by Theorem 1.8.3. If $f \in L^2(\partial\Omega)$ and $[\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega]f = 0$, then $\mathcal{S}^\omega f$ is an L^2-solution of the Robin problem $\Delta u = 0$ in ω, $\partial u/\partial n + u = 0$ on $\partial\omega$, $u(x) \to 0$ as $|x| \to \infty$. (See Theorems 5.4.8, 5.4.7, Propositions 5.2.2, 5.2.3, and 5.2.8.) The uniqueness of a solution of the Robin problem forces $\mathcal{S}^\omega f = 0$ on ω. (See Proposition 5.10.1.) According to the continuity of a single layer potential (Propositions 5.2.2 and 5.2.3), we have $\mathcal{S}^\omega f = 0$ on $\partial\omega$. Lemma 5.9.3 gives that $f \equiv 0$. Since the operator $\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega = \frac{1}{2}I + K'_\Omega + \mathcal{S}^\Omega$ is a Fredholm operator with index 0, it is an isomorphism on $L^2(\partial\Omega)$ (see Theorem 1.6.3). Since K_Ω and K'_Ω are adjoint operators (see Lemma 5.4.6), Proposition 1.6.4 gives that $\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$ is an isomorphism on $L^2(\partial\Omega)$.

The operator $\frac{1}{2}I + K_\Omega$ is a Fredholm operator with index 0 in $W^{1,p}(\partial\Omega)$ and in $L^p(\partial\Omega)$ (see Propositions 5.14.4, 5.14.6, 5.11.1 and Theorem 1.8.3). The operator $\mathcal{S}^\Omega$ is compact on $W^{1,p}(\partial\Omega)$ and on $L^p(\partial\Omega)$ by Corollary 5.4.10 and Proposition 5.2.2. Theorem 1.8.3 gives that the operator $\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$ is a Fredholm operator with index 0 in $W^{1,p}(\partial\Omega)$ and in $L^p(\partial\Omega)$. We have proved that this operator is an isomorphism in $L^2(\partial\Omega)$. Lemma 1.8.4 gives that $\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$ is an isomorphism on $W^{1,p}(\partial\Omega)$ and on $L^p(\partial\Omega)$.

Proposition 5.14.8 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Denote $Pf = \oint_{\partial\Omega} f \,\mathrm{d}\mathcal{H}_1$. Then $(\frac{1}{2}I + K_\Omega)(I - P) + \mathcal{S}^\Omega(I - P) + P$ is an isomorphism on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$.*

Proof Denote $\omega = R^m \setminus \overline{\Omega}$. Let $\omega(1), \dots, \omega(k)$ be all components of ω. Then $\frac{1}{2}I - K'_{\omega(j)} + \mathcal{S}^{\omega(j)}$ is a Fredholm operator with index 0 on $L^2(\partial\omega(j))$ by Theorems 5.10.4 and 1.8.3. Define $Tf = [\frac{1}{2}I - K'_{\omega(j)} + \mathcal{S}^{\omega(j)}](f\chi_{\omega(j)})$ on $\partial\omega(j)$. Clearly, T is a Fredholm operator with index 0 on $L^2(\partial\Omega)$. The operator $\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega - T$ is a compact operator on $L^2(\partial\Omega)$ by Proposition 1.26.2 and Theorem 1.7.4. So, $(I - P)[\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega] + P$ is a Fredholm operator with index 0 on $L^2(\partial\Omega)$ by Theorem 1.8.3.

Let now $f \in L^2(\partial\Omega)$, $(I-P)[\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega]f + Pf = 0$. Then $0 = P\{(I-P)[\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega]f + Pf\} = Pf$. Lemma 5.9.6 gives $P(\frac{1}{2}I - K_\omega)f = 0$. Denote $c = P\mathcal{S}^\omega f$. According to Lemma 5.8.1, Theorem 5.4.8, Propositions 5.2.2, 5.2.3, and 5.2.8,

$$0 = \int_{\partial\omega} [\mathcal{S}^\omega f]\Big[(I-P)\Big(\frac{1}{2}I - K'_\omega\Big)f + (I-P)\mathcal{S}^\omega f\Big] \,\mathrm{d}\mathcal{H}_{m-1}$$

$$= \int_{\partial\omega} [\mathcal{S}^\omega f]\Big(\frac{1}{2}I - K'_\omega\Big)f \,\mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\omega} [(I-P)\mathcal{S}^\omega f]^2 \,\mathrm{d}\mathcal{H}_{m-1}$$

$$+c\int_{\partial\omega} (I-P)\mathcal{S}^\omega f \,\mathrm{d}\mathcal{H}_{m-1} = \int_{\omega} |\nabla\mathcal{S}^\omega f|^2 \,\mathrm{d}\mathcal{H}_m + \int_{\partial\omega} [(I-P)\mathcal{S}^\omega f]^2 \,\mathrm{d}\mathcal{H}_{m-1}.$$

Thus $0 = (I - P)\mathcal{S}^\omega f = \mathcal{S}^\omega f - c$ on $\partial\omega$. Since $\oint_{\partial\omega} f \, \mathrm{d}\mathcal{H}_{m-1} = Pf = 0$ we have

$$\int_{\partial\omega} f\mathcal{S}^\omega f \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\omega} cf \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Lemma 5.9.3 gives that $f \equiv 0$. Since the operator $(I - P)[\frac{1}{2}I - K'_\omega + \mathcal{S}^\omega] + P = (I-P)[\frac{1}{2}I + K'_\Omega + \mathcal{S}^\Omega] + P$ is a Fredholm operator with index 0, it is an isomorphism on $L^2(\partial\Omega)$ (see Theorem 1.6.3). The operator $(\frac{1}{2}I + K_\Omega\mathcal{S}^\Omega)(I-P) + P$ is the adjoint operator of the operator $(I-P)[\frac{1}{2}I + K'_\Omega + \mathcal{S}^\Omega] + P$ (see Lemmas 5.4.6 and 1.5.12). Proposition 1.6.4 gives that it is an isomorphism on $L^2(\partial\Omega)$.

The operator $\frac{1}{2}I + K_\Omega$ is a Fredholm operator with index 0 in $W^{1,p}(\partial\Omega)$ and in $L^p(\partial\Omega)$ (Propositions 5.14.4, 5.14.6, 5.11.1 and Theorem 1.8.3). The operator $\mathcal{S}^\Omega$ is compact on $W^{1,p}(\partial\Omega)$ and on $L^p(\partial\Omega)$ by Corollary 5.4.10 and Proposition 5.2.2. Theorem 1.8.3 gives that the operator $[\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega](I - P) + P$ is a Fredholm operator with index 0 in $W^{1,p}(\partial\Omega)$ and in $L^p(\partial\Omega)$. We have proved that this operator is an isomorphism in $L^2(\partial\Omega)$. Lemma 1.8.4 gives that $[\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega](I - P) + P$ is an isomorphism on $W^{1,p}(\partial\Omega)$ and on $L^p(\partial\Omega)$.

Theorem 5.14.9 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m > 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For a fixed $g \in W^{1,p}(\partial\Omega)$, put $f = (\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)^{-1}g$. Then $u = \mathcal{D}^\Omega f + \mathcal{S}^\Omega f$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$. Moreover,*

$$\|M_a^\Omega(u)\|_{L^p(\partial\Omega)} + \|M_a^\Omega(\nabla u)\|_{L^p(\partial\Omega)} \le C\|g\|_{W^{1,p}(\partial\Omega)} \tag{5.48}$$

with a constant C dependent only on Ω, p, and a.

Proof According to Proposition 5.14.7, there exists continuous $(\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)^{-1}$ on $W^{1,p}(\partial\Omega)$. The function u is an L^p-solution of the problem by Proposition 5.2.2 and Theorem 5.3.5. Moreover, $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ by Proposition 5.2.8 and Theorem 5.5.2. This solution is unique by Propositions 5.14.1 and 5.14.2. The relation (5.48) is a consequence of Proposition 5.2.2, Theorem 5.3.5, Proposition 5.2.8, and Theorem 5.5.2.

Theorem 5.14.10 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Define $P\varphi = \oint_{\partial\Omega} \varphi \, \mathrm{d}\mathcal{H}_{m-1}$, $T = [\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega](I - P) + P$. Fix $g \in W^{1,p}(\partial\Omega)$. Then $u = \mathcal{D}^\Omega(I - P)T^{-1}g + \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$. Moreover, the relation (5.48) holds with a constant C dependent only on Ω, p, and a.*

Proof According to Proposition 5.14.8, there exists a continuous T^{-1} on $W^{1,p}(\partial\Omega)$. The function u is an L^p-solution of the problem by Proposition 5.2.2 and Theorem 5.3.5. Moreover, $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ by Proposition 5.2.8 and Theorem 5.5.2. This solution is unique by Propositions 5.14.1 and 5.14.2. The relation (5.48) is a consequence of Propositions 5.2.2, 5.2.3, Theorem 5.3.5, Proposition 5.2.8, and Theorem 5.5.2.

Theorem 5.14.11 *Let $\Omega \subset R^m$ be an unbounded open set with compact Lipschitz boundary, $m > 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Put $T = \frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$. Fix $g \in W^{1,p}(\partial\Omega)$. If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given then $u = \mathcal{D}^\Omega T^{-1}(g - u_\infty) + \mathcal{S}^\Omega T^{-1}(g - u_\infty) + u_\infty$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ and $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a^\Omega(u)\|_{L^p(\partial\Omega)} + \|M_a^\Omega(\nabla u)\|_{L^p(\partial\Omega)} \le C[\|g\|_{W^{1,p}(\partial\Omega)} + |u_\infty|] \tag{5.49}$$

with a constant C dependent only on Ω, p, and a.

Proof If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then u is bounded at infinity by Lemma 1.28.2, and therefore there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$ (see Proposition 2.17.1).

Let now u_∞ be given. According to Proposition 5.14.7, there exists continuous T^{-1} on $W^{1,p}(\partial\Omega)$. The function $u = \mathcal{D}^\Omega T^{-1}(g - u_\infty) + \mathcal{S}^\Omega T^{-1}(g - u_\infty) + u_\infty$ is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $M_a^\Omega(\nabla u) \in L^p(\partial\Omega)$ and $u(x) \to u_\infty$ as $|x| \to \infty$ (see Proposition 5.2.3, Theorem 5.3.5, Proposition 5.2.8, Theorem 5.5.2). This solution is unique by Propositions 5.14.1 and 5.14.2. The relation (5.49) is a consequence of Proposition 5.2.3, Theorem 5.3.5, Proposition 5.2.8, and Theorem 5.5.2.

Proposition 5.14.12 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$. Fix $g \in W^{1,p}(\partial\Omega)$. Denote by u_g an L^p-solution of the Dirichlet problem $\Delta u_g = 0$ in Ω, $u_g = g$ on $\partial\Omega$ such that $M_a^\Omega(\nabla u_g) \in L^p(\partial\Omega)$. If Ω is unbounded and $m > 2$, suppose moreover that $u_g(x) \to 0$ as $|x| \to \infty$. Then u_g is determined uniquely, and there exists the nontangential limit of ∇u_g at almost all points of the boundary. Define the Dirichlet-Robin operator DR_h by $DR_h g = n^\Omega \cdot \nabla u_g + hg$. Then DR_h is a bounded linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$.*

- *If $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$, then $DR_h : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism.*
- *If Ω is unbounded and $m > 2$, then $DR_0 : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism.*
- *If Ω is bounded or $m = 2$, then $DR_0 : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator with index 0, $DR_0(W^{1,p}(\partial\Omega)) = \{f \in L^p(\partial\Omega); \int_{\partial\Omega} f = 0\}$, and the kernel of DR_0 is formed by constant functions.*

Proof There exists unique u_g by Theorems 5.14.9, 5.14.10, and 5.14.11. According to Theorem 5.6.1, there exists the nontangential limit of ∇u_g at almost all points of the boundary. Using the integral representation of u_g (Theorems 5.14.9, 5.14.10, and 5.14.11) and properties of a single layer potential and a double layer potential

(Proposition 5.2.8 and Theorem 5.5.2), we infer that DR_h is a bounded linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$.

If $DR_h g = 0$, then u_g is an L^p-solution of the Robin problem $\Delta u_g = 0$ in Ω, $\partial u_g/\partial n + hu_g = 0$ on $\partial\Omega$. If $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$, then $u_g \equiv 0$ by Proposition 5.10.1 and Theorem 5.11.5, and therefore $g = 0$. Let now $h \equiv 0$. Then u_g is constant (see Theorems 5.9.10, 5.9.11, 5.11.3, and 5.11.4). Hence g is constant. If Ω is unbounded and $m > 2$, then the condition $u_g(x) \to 0$ as $|x| \to \infty$ forces $u_g \equiv 0$. Hence $g \equiv 0$. On the other hand, if Ω is bounded or $m = 2$, and g is constant, then u_g is constant and $DR_0 g = 0$.

Let now Ω be unbounded and $m > 2$. If $f \in L^p(\partial\Omega)$, then there exists $\varphi \in L^p(\partial\Omega)$ such that $v = \mathcal{S}^\Omega\varphi$ is an L^p-solution of the Robin problem $\Delta v = 0$ in Ω, $\partial v/\partial n + hv = f$ on $\partial\Omega$ such that $v(x) \to 0$ as $|x| \to \infty$ (see Theorems 5.9.11, 5.11.4, 5.10.4, and 5.11.5). Denote by g the nontangential limit of v. Then $g \in W^{1,p}(\partial\Omega)$ by Lemma 5.4.9. Clearly, $u_g = v$. Thus $DR_h g = f$. Theorem 1.6.3 gives that $DR_h : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism.

Suppose now that $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$. Let Ω be bounded or $m = 2$. If $f \in L^p(\partial\Omega)$, then there exist $\varphi \in L^p(\partial\Omega)$ and a constant c such that $v = \mathcal{S}^\Omega\varphi + c$ is an L^p-solution of the Robin problem $\Delta v = 0$ in Ω, $\partial v/\partial n + hv = f$ on $\partial\Omega$ (see Theorems 5.10.4 and 5.11.5). Denote by g the nontangential limit of v. Then $g \in W^{1,p}(\partial\Omega)$ by Lemma 5.4.9. Clearly, $u_g = v$. Thus $DR_h g = f$. Theorem 1.6.3 gives that $DR_h : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism.

Let now Ω be bounded or $m = 2$. If $g \in W^{1,p}(\partial\Omega)$, $f = DR_0 g$, then u_g is an L^p-solution of the Neumann problem $\Delta u_g = 0$ in Ω, $\partial u_g/\partial n = f$ on $\partial\Omega$. According to Lemma 5.9.1, we have $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. Suppose now that $f \in L^p(\partial\Omega)$, $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. According to Theorems 5.9.10, 5.9.11, 5.11.3, and 5.11.4, there exist $\varphi \in L^p(\partial\Omega)$ with $\int_{\partial\Omega} \varphi \, d\mathcal{H}_{m-1} = 0$ such that $v = \mathcal{S}^\Omega\varphi$ is an L^p-solution of the Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = f$ on $\partial\Omega$. Denote by g the nontangential limit of v. Then $g \in W^{1,p}(\partial\Omega)$ by Lemma 5.4.9. Clearly, $u_g = v$. Thus $DR_0 g = f$.

5.15 L^p-Solution of the Dirichlet Problem

In this section, we study L^p-solutions of the Dirichlet problem on an open set Ω with compact Lipschitz boundary. If $p \neq 2$, we suppose that $\partial\Omega$ is Ljapunov. We prove the existence of a solution and discuss the uniqueness.

Proposition 5.15.1 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$, then $u \equiv 0$.*

Proof Put $p' = p/(p-1)$. Fix $x \in \Omega$. Let $\Omega(j)$ be a sequence of domains from Lemma 1.17.13. We can suppose that $B(x; r) \subset \Omega(j)$ for some $r > 0$. Fix $b > a > 0$. According to Theorems 5.14.9 and 5.14.10, there exists an $L^{p'}$-solution v_x of the problem $\Delta v_x = 0$ in Ω, $v_x = h_x$ on $\partial\Omega$, such that $M_b(|\nabla v_x|) \in L^{p'}(\partial\Omega)$. Put

$w_x(y) = h_m(y-x) - v_x(y)$. Theorem on three potentials (Theorem 2.2.4) gives

$$u = \mathcal{S}^{\Omega(j)} \frac{\partial u}{\partial n} + \mathcal{D}^{\Omega(j)} u.$$

According to Corollary 2.2.2, we have

$$\int_{\partial\Omega(j)} \left(u \frac{\partial v_x}{\partial n} - v_x \frac{\partial u}{\partial n} \right) \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Adding

$$u(x) = \int_{\partial\Omega(j)} \left[w_x \frac{\partial u}{\partial n} - \frac{\partial w_x}{\partial n} u \right] \mathrm{d}\mathcal{H}_{m-1}. \tag{5.50}$$

Put $\tilde{\Omega} = \Omega \setminus \overline{B(x;r)}$. Since $M_b^{\tilde{\Omega}}(u) \in L^p(\partial\Omega)$ and $M_b^{\tilde{\Omega}}(|\nabla v_x|) \in L^{p'}(\partial\Omega)$, Hölder's inequality (Proposition 1.14.1) gives that $M_b^{\tilde{\Omega}}(u) M_b^{\tilde{\Omega}}(|\nabla w_x|) \in L^1(\partial\Omega)$. If $y \in \partial\Omega$, then $\Lambda_j(y) \in \Gamma_b(y)$, $\Lambda_j(y) \to y$ as $j \to \infty$. Thus $|[u(\partial w_x/\partial n)](\Lambda_j(y))| \le M_b^{\tilde{\Omega}}(u)(y) M_b^{\tilde{\Omega}}(|\nabla w_x|)(y)$, $[u(\partial w_x/\partial n)](\Lambda_j(y)) \to 0$ as $j \to \infty$ for almost all $y \in \partial\Omega$. Lebesgue lemma (Lemma 1.13.2) gives

$$\lim_{j\to\infty} \int_{\partial\Omega(j)} u \frac{\partial w_x}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{j\to\infty} \int_{\partial\Omega} u(\Lambda_j(y)) \frac{\partial w_x(\Lambda_j(y))}{\partial n} \omega_j(y) \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Fix $y \in \partial\Omega$ such that $M_b^{\tilde{\Omega}}(u)(y) < \infty$, $M_b^{\tilde{\Omega}}(|\nabla w_x|)(y) < \infty$, 0 is the nontangential limit of u at y, and 0 is the nontangential limits of w_x at y. (Almost all $y \in \partial\Omega$ satisfy this condition.) Fix $\epsilon > 0$. Let $z \in \Gamma_b^{\tilde{\Omega}}(y)$. Then $B(z; |y-z|/(1+b)) \subset \Gamma_b^{\tilde{\Omega}}(y)$. Since $|u| \le M_b^{\tilde{\Omega}}(u)(y)$, Theorem 2.9.2 gives $|\nabla u(z)| \le [M_b^{\tilde{\Omega}}(u)(y)] m(1+b)/|y-z|$. There exists $\delta > 0$ such that $|u| \le \epsilon$ on $B(y;\delta) \cap \Gamma_b^{\tilde{\Omega}}(y)$. If $|y-z| < \delta/2$, then $|u| \le \epsilon$ on $B(z; |y-z|/(a+1))$ and Theorem 2.9.2 gives $|\nabla u(z)| \le \epsilon m(1+b)/|y-z|$. Now there exists a constant c independent on y and a curve C in $\Gamma_b^{\tilde{\Omega}}(y)$ connecting y and z such that the length of C is at most $c|y-z|$. Since $u(y) = 0$ and $|\nabla w_x| \le M_b^{\tilde{\Omega}}(|\nabla w_x|)(y)$ on C, we deduce that $|w_x(z)| \le [M_b^{\tilde{\Omega}}(|\nabla w_x|)(y)] c|y-z|$. Thus $|w_x(z)\nabla u(z)| \le [M_b^{\tilde{\Omega}}(|\nabla w_x|)(y)] c|y-z| [M_b^{\tilde{\Omega}}(u)(y)] m(1+b)/|y-z| = cm(b+1)[M_b^{\tilde{\Omega}}(|\nabla w_x|)(y)][M_b^{\tilde{\Omega}}(u)(y)]$ and $|w_x(\Lambda_j(y))\nabla u(\Lambda_j(y)) \to 0|$ as $j \to \infty$. Lebesgue lemma (Lemma 1.13.2) gives

$$\lim_{j\to\infty} \int_{\partial\Omega(j)} w_x \frac{\partial u}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{j\to\infty} \int_{\partial\Omega} w_x(\Lambda_j(y)) \frac{\partial u(\Lambda_j(y))}{\partial n} \omega_j(y) \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Using (5.50), we get $u(x) = 0$.

Theorem 5.15.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m > 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For a fixed $g \in L^p(\partial\Omega)$, put $f = (\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)^{-1}g$. Then $u = \mathcal{D}^\Omega f + \mathcal{S}^\Omega f$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)} \tag{5.51}$$

with a constant C dependent only on Ω, p, and a.

Proof According to Proposition 5.14.7, there exists continuous $(\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)^{-1}$ on $L^p(\partial\Omega)$. The function u is an L^p-solution of the problem by Proposition 5.2.2 and Theorem 5.3.5. This solution is unique by Proposition 5.15.1. The relation (5.51) is a consequence of Proposition 5.2.2 and Theorem 5.3.5.

Theorem 5.15.3 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Define $P\varphi = \oint_{\partial\Omega} \varphi \, \mathrm{d}\mathcal{H}_1$, $T = [\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega](I - P) + P$. Fix $g \in L^p(\partial\Omega)$. Then $u = \mathcal{D}^\Omega(I - P)T^{-1}g + \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Moreover, the estimate (5.51) holds with a constant C dependent only on Ω, p, and a.*

Proof According to Proposition 5.14.8, there exists continuous T^{-1} on $L^p(\partial\Omega)$. The function u is an L^p-solution of the problem by Propositions 5.2.2, 5.2.3 and Theorem 5.3.5.

Let now u be an L^p-solution of the problem and $g \equiv 0$. If Ω is bounded, then $u \equiv 0$ by Proposition 5.15.1. Suppose now that Ω is unbounded. Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$ and put $\omega = \Omega \cap B(0; r)$. Define $\tilde{g} = 0$ on $\partial\Omega$ and $\tilde{g} = u$ on $\partial B(0; r)$. Then $\tilde{g} \in W^{1,p}(\partial\omega)$. According to Theorem 5.14.10, there exists an L^p-solution $\tilde{u}$ of the Dirichlet problem $\Delta\tilde{u} = 0$ in ω, $\tilde{u} = \tilde{g}$ on $\partial\omega$ such that $M_a(\nabla\tilde{u}) \in L^p(\partial\omega)$. According to Proposition 5.15.1, we have $u - \tilde{u} = 0$ in ω. Since u is bounded at infinity, Proposition 2.12.1 forces that $\nabla u(x) \to 0$ as $|x| \to \infty$. Therefore $M_a(\nabla u) \in L^p(\partial\Omega)$. Theorem 5.14.10 gives that $u \equiv 0$.

Equation (5.51) is a consequence of Propositions 5.2.2, 5.2.3 and Theorem 5.3.5.

Theorem 5.15.4 *Let $\Omega \subset R^m$ be an unbounded open set with compact Lipschitz boundary, $m > 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Put $T = \frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$. Fix $g \in L^p(\partial\Omega)$. If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then $u = \mathcal{D}^\Omega T^{-1}(g - u_\infty) + \mathcal{S}^\Omega T^{-1}(g - u_\infty) + u_\infty$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \le C\left[\|g\|_{L^p(\partial\Omega)} + |u_\infty|\right] \tag{5.52}$$

with a constant C dependent only on Ω, p, and a.

Proof If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then u is bounded at infinity by Lemma 1.28.2, and therefore there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$ (see Proposition 2.17.1).

Let now u_∞ be given. According to Proposition 5.14.7, there exists continuous T^{-1} on $L^p(\partial\Omega)$. The function $u = \mathcal{D}^\Omega T^{-1}(g - u_\infty) + \mathcal{S}^\Omega T^{-1}(g - u_\infty) + u_\infty$ is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$ (see Proposition 5.2.3 and Theorem 5.3.5).

Let now $g \equiv 0$ and u be an L^p-solution of the problem such that $u(x) \to 0$ as $|x| \to \infty$. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$ and put $\omega = \Omega \cap B(0;r)$. Define $\tilde{g} = 0$ on $\partial\Omega$ and $\tilde{g} = u$ on $\partial B(0;r)$. Then $\tilde{g} \in W^{1,p}(\partial\omega)$. According to Theorem 5.14.9, there exists an L^p-solution $\tilde{u}$ of the Dirichlet problem $\Delta\tilde{u} = 0$ in ω, $\tilde{u} = \tilde{g}$ on $\partial\omega$ such that $M_a(\nabla\tilde{u}) \in L^p(\partial\omega)$. According to Proposition 5.15.1, we have $u - \tilde{u} = 0$ in ω. Since u is bounded at infinity, Proposition 2.12.1 forces that $\nabla u(x) \to 0$ as $|x| \to \infty$. Therefore $M_a(\nabla u) \in L^p(\partial\Omega)$. Theorem 5.14.11 gives that $u \equiv 0$.

The relation (5.52) is a consequence of Proposition 5.2.3 and Theorem 5.3.5.

Proposition 5.15.5 *Let $\Omega \subset R^m$ be a bounded domain with connected Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Then the operator $\frac{1}{2}I + K_\Omega$ is an isomorphism on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$. If $g \in L^p(\Omega)$, then*

$$u = \mathcal{D}^\Omega[(1/2)I + K_\Omega]^{-1} g$$

is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If $g \in W^{1,p}(\partial\Omega)$, then $M_a(\nabla u) \in L^p(\partial\Omega)$.

Proof Denote $\omega = R^m \setminus \overline{\Omega}$. The operator $(1/2)I - K'_\omega = (1/2)I + K'_\Omega$ is an isomorphism on $L^2(\partial\Omega)$ by Theorem 5.9.11. So, the operator $(1/2)I + K_\Omega$ is an isomorphism on $L^2(\partial\Omega)$ by Proposition 1.6.4. The operator $(1/2)I + K_\Omega$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$ by Propositions 5.14.4, 5.11.1, 5.14.6, 5.9.9, Theorems 1.8.3 and 1.8.2. Since $(1/2)I + K_\Omega$ is an isomorphism on $L^2(\partial\Omega)$, Lemma 1.8.4 gives that $\frac{1}{2}I + K_\Omega$ is an isomorphism on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$. If $g \in L^p(\Omega)$, then u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ by Theorem 5.3.5. This solution is unique by Theorems 5.15.2 and 5.15.3. If $g \in W^{1,p}(\partial\Omega)$, then $[(1/2)I + K_\Omega]^{-1} g \in W^{1,p}(\partial\Omega)$. Theorem 5.5.2 gives $M_a(\nabla u) \in L^p(\partial\Omega)$.

5.16 Harmonic Measure and the Normal Derivative of Green's Function

In this section, we compare L^p-solutions of the Dirichlet problem and PWB-solutions of the Dirichlet problem. We suppose that $1 < p < \infty$ and $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary. If $p \neq 2$, we suppose that

$\partial\Omega$ is Ljapunov. We prove that every L^p-solution of the Dirichlet problem is also a PWB-solution of the problem. We show that the harmonic measure μ_x^Ω and the surface measure on $\partial\Omega$ are mutually absolutely continuous, and $\mu_x^\Omega = (-\partial G_\Omega(x,\cdot)/\partial n)\mathcal{H}_{m-1}|\partial\Omega$ where G_Ω is Green's function of Ω. As a consequence, we obtain that for every bounded open set Ω with Lipschitz boundary and $f \in L^\infty(\partial\Omega)$, there exists a unique L^∞-solution of the Dirichlet problem with the boundary condition f and this solution is bounded.

Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$. If $g \in \mathcal{C}^0(\partial\Omega)$, then there exists a unique $H_g^\Omega \in H(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ such that $H_g^\Omega = g$ on $\partial\Omega$ (see Theorems 4.2.7, 4.5.7, and 4.5.8). If $x \in \Omega$, then there exists a unique nonnegative Radon measure μ_x^Ω on $\partial\Omega$ such that

$$H_g^\Omega(x) = \int_{\partial\Omega} g \,\mathrm{d}\mu_x^\Omega. \tag{5.53}$$

The measure μ_x^Ω is called harmonic measure relative to Ω and x. If $z \in \Omega$, then $L^1(\mu_x^\Omega) = L^1(\mu_z^\Omega)$ (see Corollary 4.4.7). The functions g from $L^1(\mu_x^\Omega)$ are so-called resolutive functions (Corollary 4.4.7), and H_g^Ω given by (5.53) is the so-called PWB-solution of the Dirichlet problem with the boundary condition g (see Theorem 4.4.5).

Proposition 5.16.1 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary,* $m \geq 2$, $1 < p < \infty$, $p' = p/(p-1)$. *Suppose that* $p = 2$ *or* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x-y)$ *be Green's function of* Ω. *Fix* $x \in \Omega$. *If* $z \in \partial\Omega$, *then*

$$\lim_{\Omega \ni y \to z} G_\Omega(y,x) = \lim_{\Omega \ni y \to z} G_\Omega(x,y) = 0. \tag{5.54}$$

There exists a nontangential limit of $\nabla G_\Omega(x,\cdot) = \nabla G_\Omega(\cdot,x)$ *at almost all* $z \in \partial\Omega$ *and*

$$\|M_a(g_\Omega(\cdot,x))\|_{L^p(\partial\Omega)} + \|M_a(\nabla g_\Omega(\cdot,x))\|_{L^p(\partial\Omega)} \leq C\,\mathrm{dist}(x,\partial\Omega)^{1-m},$$

where a constant C depends only on Ω, *p, and a.*

Proof $G_\Omega(x,y) = G_\Omega(y,x)$ by Proposition 3.9.12.

Equation (5.54) is a consequence of Corollary 4.6.5.

Define $g(y) = h_m(x-y)$. The function g is resolutive by Theorem 4.2.7. Since Ω has Lipschitz boundary, it is regular by Theorems 4.5.7 and 4.5.8. Thus $H_g^\Omega \in \mathcal{C}^0(\overline{\Omega})$. So, $g_\Omega(\cdot,x) = H_g^\Omega$ by Lemma 3.9.9. Easy calculation yields $g \in W^{1,p}(\partial\Omega)$ and

$$\|g\|_{W^{1,p}(\partial\Omega)} \leq C_1\,\mathrm{dist}(x,\partial\Omega)^{1-m},$$

where C_1 depends only on Ω and p. Since H_g^Ω is an L^p-solution of the Dirichlet problem with the boundary condition g, H_g^Ω is a regular L^p-solution of this problem (see Theorems 5.14.9 and 5.14.10). Moreover there exists a constant C_2 depending only on Ω and p such that

$$\|M_a(H_g^\Omega)\|_{L^p(\partial\Omega)} + \|M_a(\nabla H_g^\Omega)\|_{L^p(\partial\Omega)} \le C_2 \|g\|_{W^{1,p}(\partial\Omega)}.$$

Since $g_\Omega(\cdot, x) = H_g^\Omega$, we have

$$\|M_a(g_\Omega(\cdot, x))\|_{L^p(\partial\Omega)} + \|M_a(\nabla g_\Omega(\cdot, x))\|_{L^p(\partial\Omega)} \le C_2 C_1 \operatorname{dist}(x, \partial\Omega)^{1-m}.$$

Theorem 5.16.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \ge 2$, $1 < p < \infty$, $p' = p/(p-1)$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x-y)$ be Green's function of Ω. Fix $x \in \Omega$.*

- *If $f \in L^p(\partial\Omega)$, then the unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$ can be expressed as*

$$u(x) = -\int_{\partial\Omega} \frac{\partial G_\Omega(x,y)}{\partial n(y)} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y), \qquad x \in \Omega. \tag{5.55}$$

- *If $f \in L^p(\partial\Omega)$, then f is resolutive and H_f^Ω is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$.*
- *The harmonic measure μ_x^Ω and the surface measure $\mathcal{H}_{m-1}|\partial\Omega$ are mutually absolutely continuous,*

$$\mu_x^\Omega = -\frac{\partial G_\Omega(x,y)}{\partial n(y)} (\mathcal{H}_{m-1}|\partial\Omega), \tag{5.56}$$

  and $(-\partial G_\Omega(x,\cdot)/\partial n) \in L^{p'}(\partial\Omega))$.
- *$\partial G_\Omega(x,y)/\partial n(y)$ is finite and negative for almost all $y \in \partial\Omega$, and*

$$\int_{\partial\Omega} -\frac{\partial G_\Omega(x,y)}{\partial n(y)}\, \mathrm{d}\mathcal{H}_{m-1}(y) = 1. \tag{5.57}$$

Proof Suppose first that $f \in W^{1,p}(\partial\Omega)$. Then $M_a(\nabla u) \in L^p(\partial\Omega)$ by Theorems 5.14.9, 5.14.10 and Proposition 5.15.1. Let $\Omega(k)$ be a sequence of domains from Lemma 1.17.13. Since $u, g_\Omega(\cdot, x) \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega(k)})$, we have by second Green formula (Theorem 2.2.1)

$$0 = \int_{\partial\Omega(k)} \left[u \frac{\partial g_\Omega(\cdot, x)}{\partial n} - g_\Omega(\cdot, x) \frac{\partial u}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1}.$$

If $x \in \Omega(k)$, then Theorem on three potentials (Theorem 2.2.4) gives

$$u(x) = \mathcal{S}^{\Omega(k)} \frac{\partial u}{\partial n}(x) + \mathcal{D}^{\Omega(k)} u(x).$$

Adding, we get

$$u(x) = \int\limits_{\partial\Omega(k)} G_\Omega(\cdot, x) \frac{\partial u}{\partial n} \, \mathrm{d}\mathcal{H}_m - \int\limits_{\partial\Omega(k)} u \frac{\partial G_\Omega(\cdot, x)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}.$$

We have $M_a(g_\Omega(\cdot, x)), M_a(\nabla g_\Omega(\cdot, x) \in L^{p'}(\partial\Omega)$, where $p' = p/(p-1)$ (see Proposition 5.16.1). The limit of $G_\Omega(\cdot, x) = G_\Omega(x, \cdot)$ is zero on $\partial\Omega$ by Proposition 5.16.1. Letting $k \to \infty$, we get (5.55) by virtue of Lebesgue lemma (Lemma 1.13.2).

Let now f be general. Choose $f_k \in W^{1,p}(\partial\Omega)$ such that $f_k \to f$ in $L^p(\partial\Omega)$. Then there exists a unique L^p-solution u_k of the Dirichlet problem $\Delta u_k = 0$ in Ω, $u_k = f_k$ on $\partial\Omega$, and it holds $u_k(x) \to u(x)$. (See Theorems 5.15.2 and 5.15.3.) By virtue of Theorem 1.14.3,

$$u(x) = \lim_{k\to\infty} \int\limits_{\partial\Omega} -\frac{\partial G_\Omega(x, y)}{\partial n(y)} f_k(y) \, \mathrm{d}\mathcal{H}_{m-1} = -\int\limits_{\partial\Omega} \frac{\partial G_\Omega(x, y)}{\partial n(y)} f(y) \, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Let now $f \in \mathcal{C}^0(\partial\Omega)$. The function f is resolutive by Theorem 4.2.7. Since Ω has Lipschitz boundary, it is regular by Theorems 4.5.7 and 4.5.8. Thus $u = H_f^\Omega \in \mathcal{C}^0(\overline{\Omega})$. We have proved (5.55). This gives (5.56). The measure μ_x^Ω is nonnegative with $\mu_x^\Omega(\partial\Omega) = 1$ by Theorem 4.4.1. Thus $-\partial G_\Omega(x, y)/\partial n(y)$ is finite and nonnegative for almost all $y \in \partial\Omega$ and (5.57) holds. Moreover, μ_x^Ω is absolutely continuous with respect to the surface measure $\mathcal{H}_{m-1}|\partial\Omega)$. Proposition 5.16.1 gives that $\partial G_\Omega(x, \cdot)/\partial n \in L^{p'}(\partial\Omega)$.

If $f \in L^p(\partial\Omega)$, then $f[\partial G_\Omega(x, \cdot)\partial n] \in L^1(\partial\Omega)$ by Hölder's inequality (Proposition 1.14.1). Since $f \in L^1(\mu_x^\Omega)$, Corollary 4.4.7 gives that f is resolutive. According to Theorems 5.15.2 and 5.15.3, there exists a unique L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$. We have proved that u is given by (5.55). Theorem 4.4.5 gives that $H_f^\Omega = u$.

We now show that the surface measure $\mathcal{H}_{m-1}|\partial\Omega)$ is absolutely continuous with respect to the harmonic measure μ_x^Ω. Let now B be a Borel subset of $\partial\Omega$ such that $\mu_x^\Omega(B) = 0$. We have proved that χ_B is resolutive, and $u(y) = \int \chi_B \mathrm{d}\mu_y^\Omega$ is an L^2-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = \chi_B$ on $\partial\Omega$. Since $\mu_x^\Omega(B) = 0$, we have $\mu_y^\Omega(B) = 0$ for all $y \in \partial\Omega$ by Theorem 4.4.10. Thus $u(y) = \mu_y^\Omega(B) = 0$. Since $u \equiv 0$ converges non-tangentially to $\chi_B(z)$ for almost all $z \in \partial\Omega$, we deduce that $\mathcal{H}_{m-1}(B) = 0$.

Since μ_x^Ω and $\mathcal{H}_{m-1}$ are mutually absolutely continuous, we infer that $\partial G_\Omega(x, y)/\partial n(y)$ is negative at almost all $y \in \partial\Omega$.

Corollary 5.16.3 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. Suppose that $p = 2$ or $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $g \in L^\infty(\partial\Omega)$ and u be an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If Ω is unbounded and $m > 2$, suppose that $u(x) \to 0$ as $|x| \to \infty$. Then*

$$\sup_{x\in\Omega} |u(x)| \leq \|g\|_{L^\infty(\partial\Omega)}. \tag{5.58}$$

Proof Suppose first that Ω is bounded. Theorem 5.16.2 gives that u is a PWB-solution of the problem. If $x \in \omega$, then

$$u(x) = \int_{\partial\Omega} g \, \mathrm{d}\mu_x^\Omega$$

by Corollary 4.4.6. Since μ_x^Ω is absolutely continuous with respect to the surface measure on $\partial\Omega$ by Theorem 5.16.2, and $\mu_x^\Omega(\partial\Omega) = 1$ by Theorem 4.4.1, we obtain (5.58).

Let now Ω be unbounded. We can suppose that Ω is connected. Suppose that $M := \sup\{|u(x)|; x \in \Omega\} > \|g\|_{L^\infty(\partial\Omega)}$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$ and denote $K = \sup\{|u(x) : |x| = r\}$. If $K = M$, then there exists $x \in \partial B(0;r)$ such that $|u(x)| = M$. Theorem 2.4.2 gives that u is constant. Thus $|u| \equiv \|g\|_{L^\infty(\partial\Omega)}$. Let now $K < M$. Define $g = u$ in Ω and $\omega = \Omega \cap B(0;r)$. Since u is an L^p-solution of $\Delta u = 0$ in ω, $u = g$ on $\partial\omega$, we infer that $|u| \leq \max(K, \|g\|_{L^\infty(\partial\Omega)}) < M$ in ω. Since $M_a(u) \in L^p(\partial\Omega)$, u is bounded in a neighborhood of infinity. Thus u is bounded in $\Omega \setminus \omega$. Corollary 2.4.3 and Proposition 2.4.7 give that $|u| \leq K < M$ in $\Omega \setminus \omega$. Thus $M = \sup\{|u(x)|; x \in \Omega\} < M$, what is a contradiction.

5.17 L^p-Solution of the Dirichlet Problem on a Lipschitz Domain

In this section, we show that for a bounded open set $\Omega \subset R^m$ with Lipschitz boundary and $f \in L^p(\partial\Omega)$ with $2 \leq p < \infty$, there exists a unique L^p-solution of the Dirichlet problem with the boundary condition f.

Theorem 5.17.1 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $2 \leq p \leq \infty$. Suppose that Ω is bounded or $m = 2$. If $g \in L^p(\partial\Omega)$, then there exists a unique L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}$$

where a constant C does not depend on g.

Proof For $g \in L^2(\partial\Omega)$, there exists a unique L^2-solution Tg of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Moreover,

$$\|M_a(Tg)\|_{L^2(\partial\Omega)} \leq C_1 \|g\|_{L^2(\partial\Omega)}$$

where a constant C_1 does not depend on g. (See Theorems 5.15.2 and 5.15.3.) Since

$$|M_a(Tg) - M_a(Tf)| \leq M_a(Tg - Tf) = M_a(T(f-g)),$$

one has

$$\|M_a(Tg) - M_a(Tf)\|_{L^2(\partial\Omega)} \leq C_1 \|g - f\|_{L^2(\partial\Omega)}.$$

If $g \in L^\infty(\partial\Omega)$, then

$$\|M_a(Tg)\|_{L^\infty(\partial\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$$

by Corollary 5.16.3.

Let now $2 < p < \infty$. For $\theta = 1 - 2/p$, we have

$$L^p(\partial\Omega) = (L^2(\partial\Omega), L^\infty(\partial\Omega))_{\theta,p}$$

by Proposition 1.14.5. According to Lemma 1.10.6,

$$L^p(\partial\Omega) = (L^\infty(\partial\Omega), L^2(\partial\Omega))_{1-\theta,p}.$$

By virtue of Proposition 1.10.5 for $E_0 = F_0 = L^\infty(\partial\Omega)$, $E_1 = F_1 = L^2(\partial\Omega)$, and $Ag = M_a(Tg)$, we obtain

$$\|M_a(Tg)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}$$

where a constant C does not depend on g.

Theorem 5.17.2 *Let $\Omega \subset R^m$ be an unbounded open set with compact Lipschitz boundary, $m > 2$, $2 \leq p < \infty$. Fix $g \in L^p(\partial\Omega)$. If u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then there exists a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C\left[\|g\|_{L^p(\partial\Omega)} + |u_\infty|\right]$$

with a constant C dependent only on Ω, p, and a.

Proof Let u be an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Then u is an L^2-solution of the problem. According to Theorem 5.15.4, there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover, if $g \equiv 0$, $u_\infty = 0$, then $u \equiv 0$.

For $g \in L^2(\partial\Omega)$, there exists a unique L^2-solution Tg of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, $u(x) \to 0$ as $|x| \to \infty$. Moreover,

$$\|M_a(Tg)\|_{L^2(\partial\Omega)} \leq C_1 \|g\|_{L^2(\partial\Omega)}$$

where a constant C_1 does not depend on g. (See Theorem 5.15.4.) Since

$$|M_a(Tg) - M_a(Tf)| \leq M_a(Tg - Tf) = M_a(T(f - g)),$$

one has

$$\|M_a(Tg) - M_a(Tf)\|_{L^2(\partial\Omega)} \leq C_1 \|g - f\|_{L^2(\partial\Omega)}.$$

If $g \in L^\infty(\partial\Omega)$, then

$$\|M_a(Tg)\|_{L^\infty(\partial\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$$

by Corollary 5.16.3.

Let now $2 < p < \infty$. For $\theta = 1 - 2/p$, we have

$$L^p(\partial\Omega) = (L^2(\partial\Omega), L^\infty(\partial\Omega))_{\theta,p}$$

by Proposition 1.14.5. According to Lemma 1.10.6,

$$L^p(\partial\Omega) = (L^\infty(\partial\Omega), L^2(\partial\Omega))_{1-\theta,p}.$$

By virtue of Proposition 1.10.5 for $E_0 = F_0 = L^\infty(\partial\Omega)$, $E_1 = F_1 = L^2(\partial\Omega)$ and $Ag = M_a(Tg)$, we obtain

$$\|M_a(Tg)\|_{L^p(\partial\Omega)} \leq C_2 \|g\|_{L^p(\partial\Omega)}$$

where a constant C_2 does not depend on g.

Let now $g \in L^p(\partial\Omega)$ and $u_\infty \in R^1$ be given. Then $u = T(g - u_\infty) + u_\infty$ is a unique L^p solution of the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C_2 \|g\|_{L^p(\partial\Omega)} + |u_\infty|(1 + C_2 \mathcal{H}_{m-1}(\partial\Omega)^{1/p}).$$

5.18 Dirichlet Problem and a Single Layer Potential

In this section, we show that under certain conditions $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$, $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ are isomorphisms. As a consequence, we express an L^p-solution of the Dirichlet problem in the form of a single layer potential with density from $W^{-1,p}(\partial\Omega)$.

Proposition 5.18.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with* $0 < \alpha < 1$. *Define $Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$, $T = \mathcal{S}^\Omega(I-P)+P$.*

- *If $m > 2$ or $\partial\Omega \subset B(0;1)$, then $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ and $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms.*
- *If $m = 2$, then $T : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ and $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms.*

Proof Without loss of generality, we can suppose that Ω is bounded. $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are bounded operators by Lemma 5.4.9 and Corollary 5.4.11.

Let $\varphi \in L^p(\partial\Omega)$. If $m > 2$, define $u_\varphi = \mathcal{S}^\Omega\varphi$. If $m = 2$, define $u_\varphi = \mathcal{S}^\Omega(I-P)\varphi + P\varphi$. Put $\omega = R^m \setminus \overline{\Omega}$. Suppose that $u_\varphi = 0$ on $\partial\Omega$. Then u_φ is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$, and the Dirichlet problem $\Delta u = 0$ in ω, $u = 0$ on $\partial\omega$. (See Propositions 5.2.2 and 5.2.3.) If $m > 2$, then $u_\varphi(x) \to 0$ as $|x| \to \infty$. Proposition 5.15.1, Theorems 5.15.3 and 5.15.4 give that $u_\varphi \equiv 0$. According to Theorem 5.4.8,

$$0 = [\partial u_\varphi/\partial n]_+ - [\partial u_\varphi/\partial n]_- = (\varphi/2 - K'_\Omega\varphi) - (-\varphi/2 - K'_\Omega\varphi) = \varphi.$$

Fix $g \in W^{1,p}(\partial\Omega)$. According to Theorems 5.14.9 and 5.14.10, there exists an L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ such that $M_a(\nabla u) \in L^p(\Omega)$. According to Theorem 5.6.1, there exists the nontangential limit of ∇u at almost all points of $\partial\Omega$. Denote by f the nontangential limit $[n^\Omega \cdot \nabla u + u]_\Omega$. Then u is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = f$ on $\partial\Omega$. Theorems 5.10.4 and 5.11.5 show that there exists $\varphi \in L^p(\partial\Omega)$ such that $u = u_\varphi$. If $m > 2$ then $\mathcal{S}^\Omega$ is an injective operator from $L^p(\partial\Omega)$ onto $W^{1,p}(\partial\Omega)$; if $m = 2$, then T is an injective operator from $L^p(\partial\Omega)$ onto $W^{1,p}(\partial\Omega)$. These operators are isomorphisms by Theorem 1.6.3. The operator $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is the adjoint operator of $\mathcal{S}^\Omega : L^{p'}(\partial\Omega) \to W^{1,p'}(\partial\Omega)$ by Corollary 5.4.11. If $m > 2$ then $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism by Proposition 1.6.4. Let now $m = 2$. Then $T' = (I-P)\mathcal{S}^\Omega + P$ is an isomorphism form $W^{-1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Proposition 1.6.4. The operator $T - T'$ is an operator of finite rank and therefore compact. So, $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator with index 0 by Theorem 1.8.3. Since $T : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is an isomorphism, Lemma 1.8.4 gives that $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism.

Let now $m = 2$ and $\overline{\Omega} \subset B(0;1)$. Since $\mathcal{S}^\Omega - T$ is an operator of finite rank and therefore compact, Theorem 1.8.3 gives that $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$,

$\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are Fredholm operators with index 0. Let now $\varphi \in W^{-1,p}(\partial\Omega)$, $\mathcal{S}^\Omega\varphi = 0$ on $\partial\Omega$. Then $\varphi \in L^2(\partial\Omega)$ by Lemma 1.8.4. Since $\mathcal{S}^\Omega|\varphi| \in L^2(\partial\Omega)$ by Proposition 5.2.2, the Hölder inequality gives that the measure $\mu = \varphi\mathcal{H}_{m-1}|\partial\Omega$ has finite energy (see Proposition 1.14.1). Since

$$(\mu,\mu)_{\mathcal{E}} = \int\limits_{\partial\Omega} \varphi\mathcal{S}^\Omega\varphi \, \mathrm{d}\mathcal{H}_{m-1} = 0,$$

Proposition 3.41.5 and Theorem 3.39.3 give that $\varphi = 0$. Since $\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are one-to-one Fredholm operators with index 0, they are onto. They are isomorphisms by Theorem 1.6.3.

Proposition 5.18.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Fix $g \in L^p(\partial\Omega)$.*

- *Let Ω be bounded. If $m = 2$, suppose moreover that $\overline{\Omega} \subset B(0;1)$. Then there exists a unique $f \in W^{-1,p}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega f$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If $g \in W^{1,p}(\partial\Omega)$, then $f \in L^p(\partial\Omega)$.*
- *Let Ω be unbounded, $m > 2$. Then there exists a unique $f \in W^{-1,p}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega f$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, $u(x) \to 0$ as $|x| \to \infty$. If $g \in W^{1,p}(\partial\Omega)$, then $f \in L^p(\partial\Omega)$.*

Proof Choose $g_k \in W^{1,p}(\partial\Omega)$ such that $g_k \to g$ in $L^p(\partial\Omega)$. According to Proposition 5.18.1, there exist unique $f \in W^{-1,p}(\partial\Omega)$ and $f_k \in L^p(\partial\Omega)$ such that $\mathcal{S}^\Omega f = g$, $\mathcal{S}^\Omega f_k = g_k$ on $\partial\Omega$. Moreover, $f_k \to f$ in $W^{-1,p}(\partial\Omega)$. Then $u_k = \mathcal{S}^\Omega f_k$ is an L^p-solution of the Dirichlet problem $\Delta u_k = 0$ in Ω, $u_k = g_k$ on $\partial\Omega$ by Propositions 5.2.2 and 5.2.3. If Ω is unbounded, then $u_k(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. If Ω is bounded, then there exists a unique L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ (see Theorems 5.15.2 and 5.15.3). If Ω is unbounded, then there exists a unique L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, $u(x) \to 0$ as $|x| \to \infty$ (see Theorem 5.15.4). Integral representations of these solutions give that $u_k \to u$. Since $u_k = \mathcal{S}^\Omega f_k \to \mathcal{S} f$, we infer that $u = \mathcal{S}^\Omega f$.

Proposition 5.18.3 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Define $Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$, $T = \mathcal{S}^\Omega(I-P)+P$. Then $T : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ and $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms. Fix $g \in L^p(\partial\Omega)$. Put $f = T^{-1}g$, $u = \mathcal{S}^\Omega(I-P)f + Pf$. Then u is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$.*

Proof $T : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ and $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms by Proposition 5.18.1.

Choose $g_k \in W^{1,p}(\partial\Omega)$ such that $g_k \to g$ in $L^p(\partial\Omega)$. Put $f_k = T^{-1}g_k$. Then $f_k \in L^p(\partial\Omega)$ and $u_k = \mathcal{S}^\Omega(I-P)f_k + Pf_k$ is an L^p-solution of the Dirichlet problem $\Delta u_k = 0$ in Ω, $u_k = g_k$ on $\partial\Omega$ by Propositions 5.2.2 and 5.2.3. According to Theorem 5.15.3, there exists a unique L^p-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Integral representations of these solutions give that $u_k \to u$. Since $u_k = \mathcal{S}^\Omega(I-P)f_k + Pf_k \to \mathcal{S}(I-P)f + Pf$, we infer that $u = \mathcal{S}^\Omega(I-P)f + Pf$.

Corollary 5.18.4 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If $f \in W^{-1,p}(\partial\Omega)$, then $\mathcal{S}^\Omega f(x)$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all $x \in \partial\Omega$. If Ω is bounded or $m > 2$, put $X_p = W^{-1,p}(\partial\Omega)$. If $\Omega \subset R^2$ is unbounded, put $X_p = \{f \in W^{-1,p}(\partial\Omega); \langle f, 1\rangle = 0\}$. Then there exists a constant C such that*

$$\|M_a^\Omega(\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{-1,p}(\partial\Omega)} \qquad \forall f \in X_p.$$

Proof Suppose that $m > 2$ or $\langle f, 1\rangle = 0$. Define $u = \mathcal{S}^\Omega f$ in Ω, $g = \mathcal{S}^\Omega f$ on $\partial\Omega$. Then $g \in L^p(\partial\Omega)$ and

$$\|g\|_{L^p(\partial\Omega)} \leq C_1\|f\|_{W^{-1,p}(\partial\Omega)},$$

where C_1 does not depend on f (see Proposition 5.18.1). Propositions 5.18.1, 5.18.2, and 5.18.3 give that u is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If $m > 2$ and Ω is unbounded, then $u(x) \to 0$ as $|x| \to \infty$. Therefore $\mathcal{S}^\Omega f(x)$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all $x \in \partial\Omega$. According to Theorems 5.15.2, 5.15.3, and 5.15.4, there exists a constant C_2 independent of f such that

$$\|M_a^\Omega u\|_{L^p(\partial\Omega)} \leq C_2\|g\|_{L^p(\partial\Omega)}.$$

Hence

$$\|M_a^\Omega(\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \leq C_1C_2\|f\|_{W^{-1,p}(\partial\Omega)}.$$

Let now $\Omega \subset R^2$ be bounded. According to Lemma 5.11.7, there exist $g \in L^p(\partial\Omega) \subset W^{-1,p}(\partial\Omega)$ and a constant c such that $\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 1$ and $\mathcal{S}^\Omega g = c$ on $\overline{\Omega}$. Put $f_1 = f - \langle f, 1\rangle g$. Then $f_1 \in W^{-1,p}(\partial\Omega)$ and $\langle f_1, 1\rangle = 0$. Since $f = f_1 + \langle f, 1\rangle g$, $\mathcal{S}^\Omega f(x)$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all $x \in \partial\Omega$. (See Proposition 5.2.2.) There exists a constant C_3 independent of f such that $|\langle f, 1\rangle| \leq C\|f\|_{W^{-1,p}(\partial\Omega)}$. Thus

$$\|M_a^\Omega(\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \leq \|M_a^\Omega(\mathcal{S}^\Omega f_1)\|_{L^p(\partial\Omega)} + \|\langle f, 1\rangle M_a^\Omega(\mathcal{S}^\Omega g)\|_{L^p(\partial\Omega)}$$

$$\leq C_1C_2(1 + C_3\|g\|_{W^{-1,p}(\partial\Omega)})\|f\|_{W^{-1,p}(\partial\Omega)} + C_3c\mathcal{H}_{m-1}(\partial\Omega)^{1/p}\|f\|_{W^{-1,p}(\partial\Omega)}.$$

5.19 Bounded Harmonic Functions and Nontangential Limits

Proposition 5.19.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. If $u \in H(\Omega)$ is bounded, then there exists a nontangential limit $g(x)$ of u at almost all points x of $\partial\Omega$. If Ω is bounded, then u is a PWB-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$.*

Proof Suppose first that Ω is starshaped. We can suppose that Ω is starshaped with respect to 0 and $\overline{\Omega} \subset B(0;1)$. For $r \in (0,1)$, define $u_r(x) = u(rx)$. Then u_r is a classical solution of the Dirichlet problem for the Laplace equation with the boundary condition u_r. The operator $\mathcal{S}^\Omega : W^{-1,2}(\partial\Omega) \to L^2(\partial\Omega)$ is an isomorphism by Proposition 5.18.1. Put $f_r = [\mathcal{S}^\Omega]^{-1}u_r$. Then $u_r = \mathcal{S}^\Omega f_r$ by Proposition 5.18.2. Since $\{u_r\}$ is a bounded subset of $L^2(\partial\Omega)$, the set $\{f_r\}$ is also bounded in $W^{-1,2}(\partial\Omega)$. According to Theorem 1.5.9, there exists a sequence $r(j) \uparrow 1$ and $f \in W^{-1,2}(\partial\Omega)$ such that $f_{r(j)} \to f$ weakly in $W^{-1,2}(\partial\Omega)$. If $x \in \Omega$, then

$$u(x) = \lim_{j\to\infty} u_{r(j)}(x) = \lim_{j\to\infty} \mathcal{S}^\Omega f_{r(j)}(x) = \mathcal{S}^\Omega f(x).$$

Corollary 5.18.4 gives that there exists a nontangential limit of u at almost all points of $\partial\Omega$.

Let now Ω be general. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$. Put $\omega = \Omega \cap B(0;r)$. According to Lemma 1.17.17, there exist starshaped domains $\Omega_1, \dots, \Omega_k$ with Lipschitz boundary such that $\omega = \Omega_1 \cup \dots \cup \Omega_k$. Since there exist nontangential limits of u with respect to Ω_j at almost all points of $\partial\Omega_j$, there exists a nontangential limit of u with respect to Ω at almost all points of $\partial\Omega$.

If Ω is bounded, then u is an L^2-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. Theorem 5.16.2 gives that u is a PWB-solution of this problem.

5.20 Very Weak L^p-Solution of the Neumann Problem and the Robin Problem

In this section, we define and study a very weak L^p-solution of the Neumann problem and the Robin problem for the Laplace equation. For a boundary condition from $W^{-1,p}(\partial\Omega)$, we prove similar results like for a boundary condition from $L^p(\partial\Omega)$ and L^p-solutions of the problems.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$. *If F is a distribution supported on $\partial\Omega$, we say that u is a very weak L^p-solution of the Robin problem*

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = F \quad \text{on } \partial\Omega, \tag{5.59}$$

if $u \in H(\Omega)$, $M_a^\Omega(u) \in L^p(\partial\Omega)$, *there exists a nontangential limit* $u(x)$ *of* u *at almost all* $x \in \partial\Omega$, *and*

$$\int_\Omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi\rangle \quad \forall\varphi \in \mathcal{C}_c^\infty(R^m). \tag{5.60}$$

If $h \equiv 0$ *we call it a very weak* L^p*-solution of the Neumann problem.*

Lemma 5.20.1 *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary,* $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $f \in L^p(\partial\Omega)$. *Put* $F = f\mathcal{H}_{m-1}|\partial\Omega$. *If* u *is an* L^p*-solution of the Robin problem* $\Delta u = 0$ *in* Ω, $\partial u/\partial n + hu = f$ *on* $\partial\Omega$, *then* u *is a very weak* L^p*-solution of the Robin problem (5.59).*

Proof Let Ω be bounded. Then $u \in W^{1,p}(\Omega)$ by Lemma 1.28.1. The trace of u is equal to the nontangential limit of u. Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$. According to Lemma 5.8.1 and the Gauss-Green formula for Sobolev spaces (Proposition 1.24.4),

$$\langle F, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\varphi + \int_{\partial\Omega} hu\varphi = \int_\Omega u(-\Delta\varphi) + \int_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right).$$

Let now Ω be unbounded. Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$. Choose $r > 0$ such that $\partial\Omega \cup \operatorname{spt}\varphi \subset B(0;r)$. Put $\omega = \Omega \cap B(0;r)$, $h = 0$ and $f = \partial u/\partial n$ on $\partial B(0;r)$, $\tilde{F} = f\mathcal{H}_{m-1}|\partial\omega$. Then u is an L^p-solution of the Robin problem $\Delta u = 0$ in ω, $\partial u/\partial n + hu = f$ on $\partial\omega$. Thus

$$\begin{aligned}\langle F, \varphi\rangle = \langle \tilde{F}, \varphi\rangle &= \int_\omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\mathrm{d}\mathcal{H}_{m-1} \\ &= \int_\Omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\mathrm{d}\mathcal{H}_{m-1}.\end{aligned}$$

Lemma 5.20.2 *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary,* $1 < p < \infty$, $p' = p/(p-1)$. *Then* K'_Ω *can be extended as a bounded linear operator on* $W^{-1,p}(\partial\Omega)$. *This operator is the adjoint operator of* $K_\Omega : W^{1,p'}(\partial\Omega) \to W^{1,p'}(\partial\Omega)$.

Proof K_Ω is a bounded linear operator on $W^{1,p'}(\partial\Omega)$ by Lemma 5.5.4. Denote by T the adjoint operator of $K_\Omega : W^{1,p'}(\partial\Omega) \to W^{1,p'}(\partial\Omega)$. Then T is a bounded linear operator on $W^{-1,p}(\partial\Omega)$ by Proposition 1.5.11. If $f \in L^p(\partial\Omega)$, $g \in W^{1,p'}(\partial\Omega)$, then

$$\langle Tf, g\rangle = \langle f, K_\Omega g\rangle = \langle K'_\Omega f, g\rangle$$

by Lemma 5.4.6. So, T is an extension of K'_Ω.

Lemma 5.20.3 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Then $\mathcal{S}^\Omega$ is a bounded linear operator from $W^{-1,p}(\partial\Omega)$ to $L^p(\Omega)$.*

Proof Corollary 5.18.4 and Proposition 5.2.2 give

$$\|M_a^\Omega(\mathcal{S}^\Omega f)\|_{L^p(\partial\Omega)} \le C\|f\|_{W^{-1,p}(\partial\Omega)},$$

where C depends only on Ω and p. So, $\mathcal{S}^\Omega$ maps $W^{-1,p}(\partial\Omega)$ to $L^p(\Omega)$ by Lemma 1.28.1. If $x \in \Omega$, $f_k \to f$ in $W^{-1,p}(\partial\Omega)$, then $\mathcal{S}^\Omega f_k(x) \to \mathcal{S}^\Omega f(x)$. Since $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\Omega)$ is closed, it is bounded by Theorem 1.4.15.

Lemma 5.20.4 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \in L^\infty(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$. If $\Omega \subset R^2$ is unbounded, suppose moreover that $\langle f, 1\rangle = 0$. Denote $F = f/2 - K'_\Omega f + h\mathcal{S}^\Omega f$. Then $\mathcal{S}^\Omega f$ is a very weak solution of the Robin problem (5.59).*

Proof $\mathcal{S}^\Omega f \in H(\Omega)$ by Theorem 3.1.5. $\mathcal{S}^\Omega f(x)$ is the nontangential limit of $\mathcal{S}^\Omega f$ at almost all $x \in \partial\Omega$ and $M_a^\Omega(\mathcal{S}^\Omega f) \in L^p(\partial\Omega)$ by Corollary 5.18.4. So, it is enough to prove (5.60). Clearly, we can suppose that $h \equiv 0$. Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$.

Choose $r > 0$ such that $\partial\Omega \cup \operatorname{spt}\varphi \subset B(0; r)$. Denote $\omega = \Omega \cap B(0; r)$. We can choose $f_k \in L^p(\partial\Omega)$ such that $f_k \to f$ in $W^{-1,p}(\Omega)$. (See Theorems 1.23.1, 1.23.5, and 1.22.1.) Define $f_k = 0$ on $\partial\omega \setminus \partial\Omega$. Put $F_k = f_k/2 - K'_\Omega f_k$, $\tilde{F}_k = f_k/2 - K'_\omega f_k$. Then $\mathcal{S}^\Omega f_k$ is a very weak solution of the problem $\Delta u = 0$ in ω, $\partial u/\partial n = \tilde{F}_k$ on $\partial\omega$. (See Lemma 5.20.1, Theorems 5.4.7, 5.4.8, Propositions 5.2.2 and 5.2.8.) Clearly, $\tilde{F}_k = F_k$ on $\partial\Omega$. Corollary 5.4.11 and Lemma 5.20.3 give that $\mathcal{S}^\Omega f_k \to \mathcal{S}^\Omega f$ in $L^p(\partial\Omega)$ and in $L^p(\omega)$. Lemma 5.20.2 forces $F_k \to F$ in $W^{-1,p}(\partial\Omega)$. According to Theorem 1.14.3,

$$\int_\Omega (\mathcal{S}^\Omega f)(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} (\mathcal{S}^\Omega f)\frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_m = \int_\omega (\mathcal{S}^\Omega f)(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m$$

$$+\int_{\partial\omega} (\mathcal{S}^\Omega f)\frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_m = \lim_{k\to\infty}\left[\int_\omega (\mathcal{S}^\Omega f_k)(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int_{\partial\omega} (\mathcal{S}^\Omega f_k)\frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_m\right]$$

$$= \lim_{k\to\infty}\langle \tilde{F}_k, \varphi\rangle = \lim_{k\to\infty}\langle F_k, \varphi\rangle = \langle F, \varphi\rangle.$$

Proposition 5.20.5 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If $h \in L^\infty(\partial\Omega)$, then $\frac{1}{2}I + K'_\Omega + h\mathcal{S}^\Omega$, $\frac{1}{2}I - K'_\Omega + h\mathcal{S}^\Omega$ are Fredholm operators with index 0 on $W^{-1,p}(\partial\Omega)$.*

Proof Denote $p' = p/(p-1)$. Then $\frac{1}{2}I + K_\Omega$, $\frac{1}{2}I - K_\Omega$ are Fredholm operators with index 0 on $W^{1,p'}(\partial\Omega)$ by Propositions 5.14.4, 5.14.7, Corollary 5.4.10, and Theorem 1.8.3. Lemma 5.20.2 gives that $\frac{1}{2}I+K'_\Omega$ (the operator $\frac{1}{2}I-K'_\Omega$) is the adjoint operator of $\frac{1}{2}I + K_\Omega$ (of $\frac{1}{2}I - K_\Omega$), respectively. Thus $\frac{1}{2}I + K'_\Omega$, $\frac{1}{2}I - K'_\Omega$ are Fredholm operators with index 0 on $W^{-1,p}(\partial\Omega)$ by Theorem 1.8.2. Corollary 5.4.10 gives that $h\mathcal{S}^\Omega$ is a compact operator on $W^{-1,p}(\partial\Omega)$. Therefore $\frac{1}{2}I + K'_\Omega + h\mathcal{S}^\Omega$, $\frac{1}{2}I - K'_\Omega + h\mathcal{S}^\Omega$ are Fredholm operators with index 0 on $W^{-1,p}(\partial\Omega)$ by Theorem 1.8.3.

Theorem 5.20.6 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Denote by* T *the restriction of* $(1/2)I - K'_\Omega$ *onto* $X = \{f \in W^{-1,p}(\partial\Omega); \langle f, 1\rangle = 0\}$.

- *Then* T *is an isomorphism.*
- *If* F *is a distribution supported on* $\partial\Omega$, *then there exists a very weak* L^p*-solution of the Neumann problem*

$$\Delta u = 0 \text{ in } \Omega, \qquad \partial u/\partial n = F \text{ on } \partial\Omega \tag{5.61}$$

if and only if $F \in W^{-1,p}(\partial\Omega)$ *and* $\langle F, 1\rangle = 0$. *The general form of a solution of the problem (5.61) is* $\mathcal{S}^\Omega(T^{-1}F) + c$ *where* c *is an arbitrary constant. Moreover, there exists a unique solution* u *of the problem (5.61) such that*

$$\int_\Omega u \, d\mathcal{H}_m = 0.$$

This solution satisfies

$$\|M_a(u)\|_{L^p(\partial\Omega)} \le C\|F\|_{W^{-1,p}(\partial\Omega)} \tag{5.62}$$

where a constant C *depends only on* Ω, p, *and* a.

Proof Let F be a distribution supported on $\partial\Omega$ and u be a very weak L^p-solution of the problem (5.61). According to Propositions 5.18.2 and 5.18.3, there exist $\varphi \in W^{-1,p}(\partial\Omega)$ and a constant c such that $u = \mathcal{S}^\Omega\varphi + c$. The constant c is a very weak solution of the Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = 0$ on $\partial\Omega$ by Lemma 5.20.1. Lemma 5.20.4 gives $F = \frac{1}{2}\varphi - K'_\Omega\varphi$. So, $F \in W^{-1,p}(\partial\Omega)$ by Lemma 5.20.2. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on a neighborhood of $\overline{\Omega}$. Then

$$\langle F, 1\rangle = \int_\Omega u(-\Delta\varphi) \, d\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n} \, d\mathcal{H}_{m-1} = 0.$$

Hence $F \in X$.

$\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 on $W^{-1,p}(\partial\Omega)$ by Proposition 5.20.5. If $f \in W^{-1,p}(\partial\Omega)$, then $\mathcal{S}^\Omega f$ is a very weak solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \frac{1}{2}f - K'_\Omega f$ by Lemma 5.20.4. Hence $\frac{1}{2}f - K_\Omega f \in X$. Since $(\frac{1}{2}f - K'_\Omega)(X) \subset X$, Lemma 1.8.8 gives that T is a Fredholm operator with index 0. T is an isomorphism on $\{f \in L^p(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$ by Theorems 5.9.10 and 5.11.3. So, T is an isomorphism on X by Lemma 1.8.4. If $F \in X$ and c is a constant, then $\mathcal{S}^{\Omega} T^{-1} F + c$ is a very weak solution of the Neumann problem (5.61).

Suppose now that u is a very weak L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$. Then there exist $f \in W^{-1,p}(\partial\Omega)$ and a constant c such that $u = \mathcal{S}^{\Omega} f + c$. We have $\frac{1}{2} f - K'_{\Omega} f = 0$. The operator $\frac{1}{2} I - K'_{\Omega}$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$ by Propositions 5.9.9, 5.11.1, and Theorem 1.8.3. Since $\frac{1}{2} I - K'_{\Omega}$ is a Fredholm operator with index 0 on $W^{-1,p}(\partial\Omega)$, Lemma 1.8.4 gives that $f \in L^p(\partial\Omega)$. Thus u is an L^p-solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$ (see Lemma 5.2.1, Propositions 5.2.2, 5.2.8, Theorems 5.4.8 and 5.4.7). Theorems 5.9.10 and 5.11.3 give that u is constant.

$f \mapsto \fint_{\Omega} \mathcal{S}^{\Omega} f \, d\mathcal{H}_m$ is a bounded linear functional on $W^{-1,p}(\partial\Omega)$ by Lemma 5.20.3 and Theorem 1.14.3. If $F \in X$, then $u = \mathcal{S}^{\Omega}(T^{-1}F) - \fint_{\Omega} \mathcal{S}^{\Omega}(T^{-1}F) \, d\mathcal{H}_m$ is a unique very weak L^p-solution of the Neumann problem (5.61) such that $\fint_{\Omega} u \, d\mathcal{H}_m = 0$. Corollary 5.18.4 gives that there exists a constant C such that (5.62) holds.

Theorem 5.20.7 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

1. *Then $(1/2)I - K'_{\Omega}$ is an isomorphism on $W^{-1,p}(\partial\Omega)$.*
2. *Let F be a distribution supported on $\partial\Omega$.*
   - *If $m > 2$, then there exists a very weak L^p-solution of the Neumann problem (5.61) if and only if $F \in W^{-1,p}(\partial\Omega)$.*
   - *If $m = 2$, then there exists a very weak L^p-solution of the problem (5.61) if and only if $F \in W^{-1,p}(\partial\Omega)$ and $\langle F, 1\rangle = 0$.*
   - *Suppose that the Neumann problem (5.61) is solvable. If u is a solution of (5.61), then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^{\Omega}[(1/2)I - K'_{\Omega}]^{-1} F + u_\infty$ is a unique very weak L^p-solution of (5.61) such that $u(x) \to u_\infty$ as $|x| \to \infty$. This solution satisfies*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C(\|F\|_{L^p(\partial\Omega)} + |u_\infty|) \tag{5.63}$$

where a constant C depends only on Ω, p, and a.

Proof Let F be a distribution supported on $\partial\Omega$ and u be a very weak L^p-solution of (5.61). According to Propositions 5.18.2 and 5.18.3, there exist $\varphi \in W^{-1,p}(\partial\Omega)$ and a constant c such that $u = \mathcal{S}^{\Omega}\varphi + c$. If $m = 2$, then $\langle \varphi, 1\rangle = 0$. So, $u(x) \to c$ as $|x| \to \infty$. The constant c is a very weak solution of the Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = 0$ on $\partial\Omega$ by Lemma 5.20.1. Lemma 5.20.4 gives $F = \frac{1}{2}\varphi - K'_{\Omega}\varphi$. So, $F \in W^{-1,p}(\partial\Omega)$ by Lemma 5.20.2. Denote $X = \{f \in W^{-1,p}(\partial\Omega; \langle f, 1\rangle = 0\}$. If $f \in X$, then $\mathcal{S}^{\Omega} f$ is a very weak solution of the Neumann problem $\Delta v = 0$ in $R^m \setminus \overline{\Omega}$, $\partial v/\partial n = f/2 + K'_{\Omega} f$ on $\partial\Omega$ by Lemma 5.20.4. Theorem 5.20.6 gives that

$\langle f/2 + K'_\Omega f, 1\rangle = 0$. Thus

$$\langle f/2 - K'_\Omega \varphi, 1\rangle = \langle f, 1\rangle - \langle f/2 + K'_\Omega f, 1\rangle = 0.$$

If $m = 2$, then $\varphi \in X$ and thus $\langle F, 1\rangle = \langle \varphi/2 - K'_\Omega \varphi, 1\rangle = 0$.

$\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 on $W^{-1,p}(\partial\Omega)$ by Proposition 5.20.5. The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $L^p(\partial\Omega)$ by Theorems 5.9.11 and 5.11.4. Lemma 1.8.4 gives that $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $W^{-1,p}(\partial\Omega)$. Since $(\frac{1}{2}I - K'_\Omega)(X) \subset X$, we infer that $\frac{1}{2}I - K'_\Omega$ is an isomorphism on X.

Let now $F \equiv 0$ and u be a very weak solution of the problem (5.61), $u(x) \to 0$ as $|x| \to \infty$. Then there exists $\varphi \in W^{-1,p}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega \varphi$. If $m = 2$, then $\langle \varphi, 1\rangle = 0$. We have $\varphi/2 - K'_\Omega \varphi = 0$. Since $\frac{1}{2}I - K'_\Omega$ is an isomorphism, we deduce that $\varphi = 0$. Thus $u \equiv 0$.

Let now $F \in W^{-1,p}(\partial\Omega)$. If $m = 2$, suppose moreover that $\langle F, 1\rangle = 0$. Then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}F + u_\infty$ is a unique very weak L^p-solution of (5.61) such that $u(x) \to u_\infty$ as $|x| \to \infty$. (See Lemma 5.20.4.) The relation (5.63) is a consequence of Corollary 5.18.4.

Theorem 5.20.8 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0, \int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

1. *Define $Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$. If $m > 2$, put $Tf = f/2 - K'_\Omega f + h\mathcal{S}^\Omega f$. For $m = 2$, define $Tf = [(1/2)I - K'_\Omega](I - P)f + h[\mathcal{S}^\Omega(I - P)f + Pf]$. Then T is an isomorphism on $W^{-1,p}(\partial\Omega)$.*
2. *If F is a distribution supported on $\partial\Omega$, then there exists a very weak L^p-solution of the Robin problem*

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = F \quad \text{on } \partial\Omega \tag{5.64}$$

if and only if $F \in W^{-1,p}(\partial\Omega)$. Suppose that $F \in W^{-1,p}(\partial\Omega)$.

- *If Ω is bounded and $m > 2$, then $u = \mathcal{S}^\Omega(T^{-1}F)$ is a unique solution of (5.64). Moreover, there exists a constant C dependent only on Ω, h, p, and a such that the estimate (5.62) holds.*
- *If $m = 2$, $f = T^{-1}F$, then $\mathcal{S}^\Omega(I - P)f + Pf$ is a unique solution of (5.64). Moreover, there exists a constant C dependent only on Ω, h, p, and a such that the estimate (5.62) holds.*
- *If Ω is unbounded, $m > 2$, and u is a solution of (5.64), then there exists $u_\infty \in R$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^\Omega T^{-1}(F - hu_\infty) + u_\infty$ is a unique solution of (5.64) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover, there exists a constant C dependent only on Ω, h, p, and a such that the estimate (5.63) holds.*

Proof Let F be a distribution supported on $\partial\Omega$ and u be a very weak L^p-solution of the Robin problem (5.64). According to Propositions 5.18.2 and 5.18.3, there exist $\varphi \in W^{-1,p}(\partial\Omega)$ and a constant c such that $u = \mathcal{S}^\Omega\varphi + c$. If $m = 2$, then $\langle\varphi, 1\rangle = 0$. So, if Ω is unbounded, then $u(x) \to c$ as $|x| \to \infty$. According to Lemmas 5.20.1 and 5.20.4, we have $F = \frac{1}{2}\varphi - K'_\Omega\varphi + h(\mathcal{S}^\Omega f + c)$. So, $F \in W^{-1,p}(\partial\Omega)$ by Lemma 5.20.2 and Corollary 5.4.11.

Proposition 5.20.5, Corollary 5.4.11 and Theorem 1.8.3 give that T is a Fredholm operator with index 0 on $W^{-1,p}(\Omega)$. Since T is an isomorphism on $L^p(\partial\Omega)$ by Theorems 5.10.4 and 5.11.5, T is an isomorphism on $W^{-1,p}(\partial\Omega)$ by Lemma 1.8.4.

Fix $F \in W^{-1,p}(\partial\Omega)$. If Ω is bounded and $m > 2$, then $u = \mathcal{S}^\Omega(T^{-1}F)$ is a very weak L^p-solution of (5.64). If $m = 2, f = T^{-1}F$, then $\mathcal{S}^\Omega(I - P)f + Pf$ is a very weak L^p-solution of (5.64). If Ω is unbounded, $m > 2$, $u_\infty \in R^1$, then $u = \mathcal{S}^\Omega T^{-1}(F - hu_\infty) + u_\infty$ is a very weak L^p-solution of (5.64) such that $u(x) \to u_\infty$ as $|x| \to \infty$. (See Lemmas 5.20.1 and 5.20.4.)

Let now u be a very weak L^p-solution of (5.64) with $F \equiv 0$. If Ω is unbounded and $m > 2$ suppose, moreover that $u(x) \to 0$ as $|x| \to \infty$. Then there exists $f \in W^{-1,p}(\Omega)$ such that $u = \mathcal{S}^\Omega f$ for $m > 2$, $u = \mathcal{S}^\Omega(I - P)f + Pf$ for $m = 2$. (See Propositions 5.18.2 and 5.18.3.) Lemmas 5.20.1 and 5.20.4 give that $Tf = 0$. Since T is an isomorphism, we obtain $f = 0$. Hence $u = 0$.

The estimates (5.63) and (5.62) are consequences of Corollary 5.18.4.

5.21 Very Weak L^p-Solutions of a Transmission Problem

In this section, we define and study a very weak L^p-solution of the transmission problem for the Laplace equation. For a boundary condition from $L^p(\partial\Omega_+) \times W^{-1,p}(\partial\Omega_+)$, we prove similar results like for a boundary condition from $W^{1,p}(\partial\Omega_+) \times L^p(\partial\Omega_+)$ and L^p-solutions of the problem.

Let $\Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega_+)$, $g \in L^p(\partial\Omega_+)$, $1 < p < \infty$, F be a distribution supported on $\partial\Omega$. We say that $u_\pm$ is a very weak L^p-solution of the transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.65}$$

$$a_+u_+ - a_-u_- = g, \quad b_+\frac{\partial u_+}{\partial n} - b_-\frac{\partial u_-}{\partial n} + hu_+ = F \quad \text{on } \partial\Omega_+ \tag{5.66}$$

if $u_\pm \in H(\Omega_\pm)$, $M_a^{\Omega\pm}u_\pm \in L^p(\partial\Omega_\pm)$ for all $a > 0$, at almost all $x \in \partial\Omega_+$ there exist nontangential limits of $u_\pm$, $u_\pm$ in the boundary conditions means the nontangential limit of $u_\pm$ and $\partial u_\pm/\partial n$ in the boundary condition means the very weak L^p normal derivative of $u_\pm$ in the sense of a very weak L^p-solution of the Neumann problem.

If $u_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66), then $F \in W^{-1,p}(\partial\Omega)$ by Theorems 5.20.6 and 5.20.7.

If $\tilde{u}_\pm = a_\pm u_\pm$, $\tilde{b}_\pm = b_\pm/a_\pm$, $\tilde{h} = h/a_+$, then $u_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66) if and only if $\tilde{u}_\pm$ is a very weak L^p-solution of the transmission problem

$$\Delta \tilde{u}_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$\tilde{u}_+ - \tilde{u}_- = g, \quad \tilde{b}_+ \frac{\partial \tilde{u}_+}{\partial n} - \tilde{b}_- \frac{\partial \tilde{u}_-}{\partial n} + \tilde{h}\tilde{u}_+ = F \quad \text{on } \partial\Omega.$$

Suppose now that $a_\pm = 1$. If $u_\pm = v_\pm + \mathcal{D}^\Omega g$, then $u_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66) if and only if $v_\pm$ is an L^p-solution of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$v_+ - v_- = 0, \quad b_+ \frac{\partial v_+}{\partial n} - b_- \frac{\partial v_-}{\partial n} + h v_+ = \tilde{F}, \quad \text{on } \partial\Omega$$

where

$$\tilde{F} = F - b_+ \left[\frac{\partial \mathcal{D}^\Omega g}{\partial n}\right]_+ + b_- \left[\frac{\partial \mathcal{D}^\Omega g}{\partial n}\right]_- - h\left[\mathcal{D}^\Omega g\right]_+ .$$

Suppose now that $a_\pm = 1$, $g = 0$. For $m > 2$, we shall look for a solution of the problem (5.65), (5.66) in the form of a single layer potential $\mathcal{S}^\Omega \varphi$ with an unknown density $\varphi \in W^{-1,p}(\partial\Omega)$. The function $u_\pm = \mathcal{S}^\Omega \varphi$ is a very weak L^p-solution of the problem (5.65), (5.66) if and only if

$$\frac{b_+ + b_-}{2} \varphi - (b_+ - b_-) K'_\Omega \varphi + h \mathcal{S}^\Omega \varphi = F$$

(see Corollary 5.18.4 and Lemma 5.20.4).

If $m = 2$, define $Pf = \langle f, 1\rangle / \mathcal{H}_{m-1}(\partial\Omega)$. We shall look for a solution of the problem (5.65), (5.66) in the modified form

$$u_\pm = \mathcal{S}^\Omega (I - P)\varphi + P\varphi$$

with $\varphi \in W^{-1,p}(\partial\Omega)$. Then $u_\pm$ is a very weak L^p-solution of the problem (5.65), (5.66) if and only if

$$\frac{b_+ + b_-}{2}(I - P)\varphi - (b_+ - b_-)K'_\Omega (I - P)\varphi + h[\mathcal{S}^\Omega (I - P)\varphi + P\varphi] = F$$

(see Corollary 5.18.4, Lemmas 5.20.1 and 5.20.4).

Lemma 5.21.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary,* $1 < p < \infty$. *If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with* $0 < \alpha < 1$. *Define*

$Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$. Let $\lambda \in R^1$, $|\lambda| \geq 1/2$, $h \in L^\infty(\partial\Omega)$. Then $\lambda I - K'_\Omega + h\mathcal{S}^\Omega$, $\lambda(I-P) - K'_\Omega(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ are Fredholm operators with index 0 *on $W^{-1,p}(\partial\Omega)$.*

Proof Put $p' = p/(p-1)$. The operator $\lambda I - K_\Omega$ is Fredholm with index 0 on $W^{1,p'}(\partial\Omega)$ by Propositions 5.14.3, 5.14.4, 5.14.6 and Theorem 1.8.3. Theorem 1.8.2 gives that $\lambda I - K'_\Omega$ is a Fredholm operator with index 0 on $W^{-1,p}(\partial\Omega)$. So, $\lambda I - K'_\Omega + h\mathcal{S}^\Omega$, $\lambda(I-P) - K'_\Omega(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ are Fredholm operators with index 0 on $W^{-1,p}(\partial\Omega)$. (See Theorem 1.8.3 and Corollary 5.4.11.)

Theorem 5.21.2 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $m > 2$, $1 < p < \infty$. Denote $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$b_+/a_+ = b_-/a_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

Let $F \in W^{-1,p}(\partial\Omega)$, $g \in L^p(\partial\Omega)$. Then there exists a very weak L^p-solution of the transmission problem (5.65), (5.66). If $u_\pm$ is a very weak L^p-solution of this problem, then there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then there exists a unique very weak L^p-solution $u_\pm$ of the transmission problem (5.65), (5.66) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover,

$$\|M_a(u_\pm)\|_{L^p(\partial\Omega)} \leq C[\|g\|_{L^p(\partial\Omega)} + \|F\|_{W^{-1,p}(\partial\Omega)} + |u_\infty|], \tag{5.67}$$

where a constant C depends only on Ω, a_+, a_-, b_+, b_-, h, and a.

Proof According to the reasoning at the beginning of this section and Theorem 5.3.5, we can suppose that $a_+ = a_- = 1$ and $g \equiv 0$.

Let $u_\pm$ be a very weak L^p-solution of the transmission problem (5.65), (5.66). Since $M_a(u_-) \in L^p(\partial\Omega)$, there exists $r > 0$ such that u_- is bounded on $R^m \setminus B(0;r)$. (See Lemma 1.28.2.) Proposition 2.17.1 gives that there exists $u_\infty \in R$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Suppose now that $u_\infty = 0$, $F \equiv 0$. Denote $f = \partial u_+/\partial n^\Omega - \partial u_-/\partial n^\Omega$. Then $f \in W^{-1,p}(\partial\Omega)$ by Theorems 5.20.6 and 5.20.7. Corollary 5.6.7 gives $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$. Since $u_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66), we have $Tf = F = 0$, where

$$T\varphi = \frac{b_+ + b_-}{2}\varphi - (b_+ - b_-)K'_\Omega\varphi + h\mathcal{S}^\Omega\varphi.$$

If $b_+ \neq b_-$, then T is a Fredholm operator on $W^{-1,p}(\partial\Omega)$ with index 0 by Lemma 5.21.1. If $b_+ = b_-$, then T is a Fredholm operator on $W^{-1,p}(\partial\Omega)$ with index 0 by Theorem 1.8.3 and Corollary 5.4.11. T is an isomorphism on $L^p(\partial\Omega)$ by Propositions 5.12.2 and 5.12.4. Lemma 1.8.4 gives that T is an isomorphism on $W^{-1,p}(\partial\Omega)$. Since $Tf = 0$, we have $f = 0$. Thus $u_\pm = \mathcal{S}^\Omega f = 0$.

Let now $u_\infty \in R$, $F \in W^{-1,p}(\partial\Omega)$ be given. Then $u_\pm = \mathcal{S}^\Omega T^{-1}(F - hu_\infty) + u_\infty$ is a unique very weak L^p-solution of the transmission problem (5.65), (5.66) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. The estimate (5.67) is a consequence of Corollary 5.18.4.

Theorem 5.21.3 *Let $\Omega = \Omega_+ \subset R^2$ be a bounded open set with Lipschitz boundary, $1 < p < \infty$. Denote $\Omega_- = R^2 \setminus \overline{\Omega_+}$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_1 > 0$. Suppose that one of the following conditions is fulfilled:*

1. *$p = 2$;*
2. *$b_+/a_+ = b_-/a_-$;*
3. *Ω has boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

Let $F \in W^{-1,p}(\partial\Omega)$, $g \in L^p(\partial\Omega)$. Then there exists a unique very weak L^p-solution of the transmission problem (5.65), (5.66). Moreover,

$$\|M_a(u_\pm)\|_{L^p(\partial\Omega)} \leq C[\|g\|_{L^p(\partial\Omega)} + \|F\|_{W^{-1,p}(\partial\Omega)}], \tag{5.68}$$

where a constant C depends only on Ω, a_+, a_-, b_+, b_-, h, and a.

Proof According to the reasoning at the beginning of this section and Theorem 5.3.5, we can suppose that $a_+ = a_- = 1$ and $g \equiv 0$.

For $f \in W^{-1,p}(\partial\Omega)$, define $Pf = \langle f, 1\rangle / \mathcal{H}_{m-1}(\partial\Omega)$,

$$Tf = \frac{b_+ + b_-}{2}(I - P)f - (b_+ - b_-)K'_\Omega(I - P)f + h[\mathcal{S}^\Omega(I - P)f + Pf].$$

If $b_+ \neq b_-$, then T is a Fredholm operator on $W^{-1,p}(\partial\Omega)$ with index 0 by Lemma 5.21.1. If $b_+ = b_-$, then T is a Fredholm operator on $W^{-1,p}(\partial\Omega)$ with index 0 by Theorem 1.8.3 and Corollary 5.4.11. The operator T is an isomorphism on $L^p(\partial\Omega)$ by Proposition 5.12.5. So, Lemma 1.8.4 gives that T is an isomorphism on $W^{-1,p}(\partial\Omega)$.

Let $u_\pm$ be a very weak L^p-solution of the transmission problem (5.65), (5.66) with $F \equiv 0$. Since $M_a(u_-) \in L^p(\partial\Omega)$, there exists $r > 0$ such that u_- is bounded on $R^m \setminus B(0; r)$. (See Lemma 1.28.2.) Proposition 2.17.1 gives that there exists $u_\infty \in R$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Put $v_\pm = u_\pm - u_\infty$. Denote $f = \partial v_+/\partial n^\Omega - \partial v_-/\partial n^\Omega$. Then $f \in W^{-1,p}(\partial\Omega)$ by Theorems 5.20.6 and 5.20.7. Corollary 5.6.7 gives $v_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$. Since $v_-(x) \to 0$ as $|x| \to \infty$, we infer that $\langle f, 1\rangle = 0$. Set $\varphi = f + u_\infty$. Then $P\varphi = u_\infty$, $(I - P)\varphi = f$, and thus $u_\pm = \mathcal{S}^\Omega(I - P)\varphi + P\varphi$. Since $u_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66), we have $T\varphi = F = 0$. Since T is an isomorphism, we have $\varphi = 0$. Thus $u_\pm = \mathcal{S}^\Omega(I - P)\varphi + P\varphi = 0$.

Let now $F \in W^{-1,p}(\partial\Omega)$ be given. Then $u_\pm = \mathcal{S}^\Omega(I - P)T^{-1}F + PT^{-1}F$ is a unique very weak L^p-solution of the transmission problem (5.65), (5.66). The estimate (5.68) is a consequence of Corollary 5.18.4.

Theorem 5.21.4 *Let* $\Omega = \Omega_+ \subset R^2$ *be a bounded open set with Lipschitz boundary,* $1 < p < \infty$*. Denote* $\Omega_- = R^2 \setminus \overline{\Omega}_+$*. Let* $a_\pm$*,* $b_\pm$ *be positive constants,* $h \equiv 0$*. Suppose that one of the following conditions is fulfilled:*

1. $p = 2$;
2. $b_+/a_+ = b_-/a_-$;
3. Ω *has boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$.

Let $F \in W^{-1,p}(\partial\Omega)$*,* $g \in L^p(\partial\Omega)$*. Then there exists a very weak* L^p*-solution of the transmission problem (5.65), (5.66) if and only if* $\langle F, 1\rangle = 0$*. If* $u_\pm$ *is a very weak* L^p*-solution of the transmission problem (5.65), (5.66) then there exists a constant* u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$*. On the other hand, if* $\langle F, 1\rangle = 0$ *and* $u_\infty \in R^1$ *is given, then there exists a unique very weak* L^p*-solution* $u_\pm$ *of the transmission problem (5.65), (5.66) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$*. Moreover, the estimate (5.67) holds with a constant C dependent only on* Ω*,* a_+*,* a_-*,* b_+*,* b_-*, and a.*

Proof According to the reasoning at the beginning of this section and Theorem 5.3.5, we can suppose that $a_+ = a_- = 1$ and $g \equiv 0$.

For $f \in W^{-1,p}(\partial\Omega)$, define

$$Tf = \frac{b_+ + b_-}{2} f - (b_+ - b_-) K'_\Omega f.$$

Denote $X = \{f \in W^{-1,p}(\partial\Omega); \langle f, 1\rangle = 0\}$. The operator T is an isomorphism on $X \cap L^p(\partial\Omega)$ by Proposition 5.12.3. Since $X \cap L^p(\partial\Omega)$ is a dense subset of X, we infer that $T(X) \subset X$. The operator T is Fredholm with index 0 on $W^{-1,p}(\partial\Omega)$ (see Lemma 5.21.1). Hence, Lemma 1.8.8 gives that T is a Fredholm operator with index 0 on X. So, T is an isomorphism on X by Lemma 1.8.4.

Let $u_\pm$ be a very weak L^p-solution of the transmission problem (5.65), (5.66). Since $M_a(u_-) \in L^p(\partial\Omega)$, there exists $r > 0$ such that u_- is bounded on $R^m \setminus B(0; r)$. (See Lemma 1.28.2.) Proposition 2.17.1 gives that there exists $u_\infty \in R$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Put $v_\pm = u_\pm - u_\infty$. Then $v_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66) and $v_-(x) \to 0$ as $|x| \to \infty$. Denote $f = \partial v_+/\partial n^\Omega - \partial v_-/\partial n^\Omega$. Then $f \in W^{-1,p}(\partial\Omega)$ by Theorems 5.20.6 and 5.20.7. Corollary 5.6.7 gives $v_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$. Since $\mathcal{S}^\Omega f(x) = v_-(x) \to 0$ as $|x| \to \infty$, we infer that $f \in X$. Since $v_\pm$ is a very weak L^p-solution of the transmission problem (5.65), (5.66), we have $Tf = F$. Since $T(X) \subset X$ we deduce $F \in X$. Suppose now that $F = 0$. Since T is an isomorphism on X, we have $f = 0$. Thus $v_\pm = \mathcal{S}^\Omega f = 0$.

Let now $u_\infty \in R$, $F \in X$ be given. Denote by $\tilde{T}$ the restriction of T onto X. Then $u_\pm = \mathcal{S}^\Omega \tilde{T}^{-1} F + u_\infty$ is a unique very weak L^p-solution of the transmission problem (5.65), (5.66) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. The estimate (5.67) is a consequence of Corollary 5.18.4.

5.22 Very Weak L^p-Solution of the Jump Problem on Cracks

In this section, we define and study a very weak L^p-solution of the jump problem on cracks. For a boundary condition from $L^p(\Gamma) \times W^{-1,p}(\Gamma)$, we prove similar results like for a boundary condition from $W^{1,p}(\Gamma)\times L^p(\Gamma)$ and L^p-solutions of the problem.

Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. If v is a function defined on $R^m \setminus \partial\Omega$, denote by $[v]_\pm$ the nontangential limit of v with respect to $\Omega_\pm$ and $M_a^\pm v$ the nontangential maximal function of v with respect to $\Omega_\pm$. Denote by $n_\pm$ the exterior unit normal of $\Omega_\pm$, $n = n_+$. Let Γ be a closed subset of $\partial\Omega$, $h \in L^\infty(\partial\Omega)$, $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$, $1 < p < \infty$. We suppose that $h = 0$, $g = 0$ on $\partial\Omega \setminus \Gamma$ and f is supported on Γ. *We say that u is a very weak L^p-solution of the generalized jump problem*

$$\Delta u = 0 \quad \text{in } R^m \setminus \Gamma, \tag{5.69}$$

$$[u]_+ - [u]_- = g, \quad \left[\frac{\partial u}{\partial n}\right]_+ - \left[\frac{\partial u}{\partial n}\right]_- + h[u]_+ = f \quad \text{on } \Gamma \tag{5.70}$$

if $u \in H(R^m \setminus \Gamma)$, $M_a^+ u + M_a^- u \in L^p(\partial\Omega)$, there exist nontangential limits of u with respect to Ω_+ and with respect to Ω_- at almost all points of $\partial\Omega$, the jump condition $[u]_+ - [u]_- = g$ on the crack Γ is satisfied in the sense of a nontangential limit at $\mathcal{H}_{m-1}$-almost all points of Γ, and

$$\int_{R^m\setminus\Gamma} u(-\Delta\varphi)\, d\mathcal{H}_m + \int_\Gamma \left(g\frac{\partial\varphi}{\partial n} + h\varphi[u]_+\right) d\mathcal{H}_{m-1} = \langle f, \varphi\rangle \quad \forall \varphi \in \mathcal{C}_c^\infty(R^m).$$

Lemma 5.22.1 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let Γ be a closed subset of $\partial\Omega$, $h \in L^\infty(\partial\Omega)$, $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$, $1 < p < \infty$. Suppose that $h = 0$, $g = 0$ on $\partial\Omega \setminus \Gamma$, and f is supported in Γ.*

- *Suppose that there exists a very weak L^p-solution u of the generalized jump problem (5.69), (5.70). Define $u_\pm = u$ on $\Omega_\pm$. Then $u_\pm$ is a very weak L^p-solution of the transmission problem*

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{5.71}$$

$$u_+ - u_- = g, \quad \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} + hu_+ = f \quad \text{on } \partial\Omega. \tag{5.72}$$

- *Let $u_\pm$ be a very weak L^p-solution of the transmission problem (5.71), (5.72). Then there exist the nontangential limits $u_+(x) = u_-(x)$ at all $x \in \partial\Omega \setminus \Gamma$. Define $u = u_\pm$ in $\Omega_\pm$, $u = u_+$ on $\partial\Omega \setminus \Gamma$. Then u is a very weak L^p-solution of the generalized jump problem (5.69), (5.70).*

Proof Suppose first that u is a very weak L^p-solution of the generalized jump problem (5.69), (5.70). If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\left\langle \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} + hu_+, \varphi \right\rangle = \int\limits_{\Omega_+} u_+(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} u_+ \left(\frac{\partial \varphi}{\partial n} + h\varphi \right) \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int\limits_{\Omega_-} u_-(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m - \int\limits_{\partial\Omega} u_- \frac{\partial\varphi}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{R^m\setminus\Gamma} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m$$

$$+ \int\limits_{\Gamma} \left(g\frac{\partial\varphi}{\partial n} + hu_+\varphi \right) \mathrm{d}\mathcal{H}_{m-1} = \langle f, \varphi \rangle.$$

Thus $u_\pm$ is a very weak L^p-solution of the transmission problem (5.71), (5.72).

Let now $u_\pm$ be a very weak L^p-solution of the transmission problem (5.71), (5.72). First we prove that $u \in H(R^m \setminus \Gamma)$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $\omega = \Omega_- \cap B(0;r)$. Proposition 5.6.6 gives

$$\mathcal{D}^\Omega u_+ + \mathcal{S}^\Omega(\partial u_+/\partial n^\Omega) = \begin{cases} u \text{ in } \Omega, \\ 0 \text{ in } R^m \setminus \overline{\Omega}, \end{cases}$$

$$\mathcal{D}^\omega u_- + \mathcal{S}^\omega(\partial u_-/\partial n^\omega) = \begin{cases} u \text{ in } \omega, \\ 0 \text{ in } R^m \setminus \overline{\omega}. \end{cases}$$

Adding

$$u = \mathcal{D}^\Omega g + \mathcal{S}^\Omega(f - hu_+) + \mathcal{D}^{B(0;r)}u_- + \mathcal{S}^{B(0;r)}(n^{B(0;r)} \cdot \nabla u_-)$$

in $B(0;r) \setminus \partial\Omega$. Since the densities of these potentials are supported on $\Gamma \cup \partial B(0;r)$, these potentials are harmonic in $B(0;r) \setminus \Gamma$. So, $u \in H(B(0;r) \setminus \Gamma)$.

Fix $\varphi \in \mathcal{C}_c^\infty(R^m)$. Then

$$\langle f, \varphi \rangle = \left\langle \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} + hu_+, \varphi \right\rangle = \int\limits_{\Omega_+} u_+(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m$$

$$+ \int\limits_{\partial\Omega} u_+ \left(\frac{\partial\varphi}{\partial n} + h\varphi \right) \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\Omega_-} u_-(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m - \int\limits_{\partial\Omega} u_- \frac{\partial\varphi}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{R^m\setminus\Gamma} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\Gamma} \left(g\frac{\partial\varphi}{\partial n} + hu_+\varphi \right) \mathrm{d}\mathcal{H}_{m-1}.$$

So, u is a very weak L^p-solution of the generalized jump problem (5.69), (5.70).

Theorem 5.22.2 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let Γ be a closed subset of $\partial\Omega$. Let $1 < p < \infty$, $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that $g = 0$, $h = 0$ on $\partial\Omega \setminus \Gamma$, f is supported on Γ.*

- *If $m > 2$, then there exists a very weak L^p solution of the generalized jump problem (5.69), (5.70). If u is a very weak L^p-solution of this problem, then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then there exists a unique very weak L^p-solution u of the generalized jump problem (5.69), (5.70) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u)\|_{L^p} \leq C(\|f\|_{W^{-1,p}(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)} + |u_\infty|) \tag{5.73}$$

  with a constant C dependent only on Ω, p, h, and a. If $h = 0$, then $u = \mathcal{D}^\Omega g + \mathcal{S}^\Omega f$ is a unique L^p-solution of the generalized jump problem (5.69), (5.70) such that $u(x) \to 0$ as $|x| \to 0$.
- *If $m = 2$ and $h \not\equiv 0$ then there exists a unique very weak L^p-solution of the generalized jump problem (5.69), (5.70). Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C(\|f\|_{W^{-1,p}(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)})$$

  with a constant C dependent only on Ω, h, p, and a.
- *Let $m = 2$ and $h \equiv 0$. Then there exists a very weak L^p-solution of the generalized jump problem (5.69), (5.70) if and only if $\langle f, 1\rangle = 0$. If u is a very weak L^p-solution of the generalized jump problem (5.69), (5.70), then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $\langle f, 1\rangle = 0$ and $u_\infty \in R^1$ is given, then there exists a unique very weak L^p-solution u of the generalized jump problem (5.69), (5.70) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover, the estimate (5.73) holds with a constant C dependent only on Ω, p, and a. The function $u = \mathcal{D}^\Omega g + \mathcal{S}^\Omega f$ is a unique very weak L^p-solution of the generalized jump problem (5.69), (5.70) such that $u(x) \to 0$ as $|x| \to 0$.*

Proof The theorem is an easy consequence of Lemma 5.13.1, Theorems 5.21.2, 5.21.3, 5.21.4, and Corollary 5.6.7.

5.23 An L^2-Solution of the Oblique Derivative Problem

In this section, we study L^2-solutions of the derivative oblique problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \Theta \cdot \nabla u + hu = g \quad \text{on } \partial\Omega \tag{5.74}$$

where $\Omega \subset R^m$ is a domain with compact Lipschitz boundary. We look for a solution in the form of a single layer potential. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $f \in L^p(\partial\Omega)$, $1 < p < \infty$, $\Theta \in L^\infty(\partial\Omega, R^m)$, $h \in L^\infty(\partial\Omega)$. If $m = 2$, suppose moreover that $\int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0$. If c is a constant, then $\mathcal{S}^\Omega f + c$ is an L^p-solution of the problem (5.74) if and only if

$$T^\Omega_\Theta f + h(\mathcal{S}^\Omega f + c) = g,$$

where

$$T^\Omega_\Theta f = \Theta \cdot [\nabla \mathcal{S}^\Omega f]_\Omega. \tag{5.75}$$

Lemma 5.23.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $v \in \mathcal{C}^1(R^m, R^m)$. If $m > 2$, put $X = L^2(\partial\Omega)$; for $m = 2$, denote $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$. Let a and α be nonnegative numbers such that $a + \alpha > 0$. Suppose that $v \cdot n^\Omega \geq \alpha$ on $\partial\Omega$. Put $\Theta = an^\Omega + v$. Then there exists a positive constant C such that*

$$\|f\|_{L^2(\partial\Omega)} \leq C\left[\|T_\Theta f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}\right]$$

for all $f \in X$.

Proof Without loss of generality, we can suppose that v has compact support. Fix a constant C_1 such that $|\nabla v| \leq C_1$. According to Theorem 5.9.5, there exists a constant C_2 such that

$$\|f\|_{L^2(\partial\Omega)} \leq C_2[\|f/2 - K'_\Omega f\|_{L^2(\partial\Omega)} + \|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}], \quad f \in X. \tag{5.76}$$

Fix $f \in X$ and put $u = \mathcal{S}^\Omega f$ in Ω. According to Lemma 5.8.2, Theorem 5.4.7, Proposition 5.2.8, and Lemma 5.8.1,

$$\frac{\alpha}{2}\|\partial u/\partial n\|^2_{L^2(\partial\Omega)} \leq \frac{1}{2}\int\limits_{\partial\Omega} |\nabla u|^2 (n \cdot v) \, d\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(v \cdot \nabla u) \, d\mathcal{H}_{m-1}$$

$$+\frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 \sum_{j=1}^m \partial_j v_j \, d\mathcal{H}_m - \int\limits_{\Omega} \sum_{j,k=1}^m (\partial_j v_k)(\partial_j u)(\partial_k u) \leq \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(v \cdot \nabla u)$$

$$+C_1 3m^2 \int\limits_{\Omega} |\nabla u|^2 \leq \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(v \cdot \nabla u) \, d\mathcal{H}_{m-1} + 3C_1 m^2 \int\limits_{\partial\Omega} u \frac{\partial u}{\partial n} \, d\mathcal{H}_{m-1}.$$

Adding $a\|\partial u/\partial n\|^2_{L^2(\partial\Omega)}$ to the both sides, we obtain

$$(a+\alpha/2)\|\partial u/\partial n\|^2_{L^2(\partial\Omega)} \leq \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(\Theta\cdot\nabla u)\,\mathrm{d}\mathcal{H}_{m-1} + 3C_1 m^2 \int\limits_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1}.$$

By virtue of Hölder's inequality (Proposition 1.14.1),

$$(a+\alpha/2)\|\partial u/\partial n\|^2_{L^2(\partial\Omega)} \leq \|\partial u/\partial n\|_{L^2(\partial\Omega)}\left[\|T_\theta f\|_{L^2(\partial\Omega)} + 3C_1 m^2\|u\|_{L^2(\partial\Omega)}\right].$$

Put $C_3 = (1+3C_1m^2)/(a+\alpha/2)$. Then

$$\|\partial u/\partial n\|_{L^2(\partial\Omega)} \leq C_3\left[\|T_\theta f\|_{L^2(\partial\Omega)} + \|u\|_{L^2(\partial\Omega)}\right].$$

Since $\partial u/\partial n = f/2 - K'_\Omega f$ by Theorem 5.4.8, (5.76) gives

$$\|f\|_{L^2(\partial\Omega)} \leq C_2C_3\|T^\Omega_\Theta f\|_{L^2(\partial\Omega)} + C_2(1+C_3)\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)}.$$

Proposition 5.23.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $v \in \mathcal{C}^1(R^m, R^m)$. Let a and α be nonnegative numbers such that $a+\alpha > 0$. Suppose that $v\cdot n^\Omega \geq \alpha$ on $\partial\Omega$. Put $\Theta = an^\Omega + v$. Let $h \in L^\infty(\partial\Omega)$. Define $Pf = \oint_{\partial\Omega} f\,\mathrm{d}\mathcal{H}_{m-1}$ for $f \in L^2(\partial\Omega)$. Then $T_\Theta + h\mathcal{S}^\Omega$, $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ are Fredholm operators with index 0 in $L^2(\partial\Omega)$.*

Proof If $m > 2$, put $X = L^2(\partial\Omega)$. If $m = 2$, denote $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f\,\mathrm{d}\mathcal{H}_{m-1} = 0\}$. According to Lemma 5.23.1, there exists a constant C such that

$$\|f\|_{L^2(\partial\Omega)} \leq C\|T^\Omega_\Theta f\|_{L^2(\partial\Omega)} + C\|\mathcal{S}^\Omega f\|_{L^2(\partial\Omega)} \qquad \forall f \in X.$$

Since $\mathcal{S}^\Omega$ is a compact operator on $L^2(\partial\Omega)$ by Proposition 5.2.2, Lemma 1.7.9 gives that T_Θ is an upper semi-Fredholm from X to $L^2(\partial\Omega)$. If $m = 2$, then $T_\Theta(I-P)$ in an upper semi-Fredholm operator on $L^2(\partial\Omega)$. The operator T_Θ in an upper semi-Fredholm operator on $L^2(\partial\Omega)$ by Theorem 1.8.3.

For $a = 1$, $v \equiv 0$, we have $T_n = \frac{1}{2}I - K'_\Omega$ by Theorem 5.4.8. The operator T_n is a Fredholm operator with index 0 by Proposition 5.9.9. Theorem on the continuity of the index (Proposition 1.8.5) gives that $i(T_{t\Theta+(1-t)n}) = 0$ for $0 \leq t \leq 1$. Thus T_Θ is a Fredholm operator with index 0.

Since $T_\Theta + h\mathcal{S}^\Omega - T_\Theta$, $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P)+P] - T_\Theta$ are compact operators on $L^2(\partial\Omega)$, the operators $T_\Theta + h\mathcal{S}^\Omega$, $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ are Fredholm with index 0 by Theorem 1.8.3.

Lemma 5.23.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $v \in \mathcal{C}^1(R^m, R^m)$, $m \geq 2$. Let a, α be nonnegative numbers. Suppose that $v\cdot n^\Omega \geq \alpha$ on $\partial\Omega$. Put $\Theta = an^\Omega + v$. Then there exists a nonnegative number c_0 such that the following assertion holds: Let h be a constant, $h > c_0$, u be an L^2-solution of* *the derivative oblique problem (5.74) with $g = 0$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof Without loss of generality, we can suppose that v has a compact support. Fix a constant C_1 such that $|\nabla v| \le C_1$. Put $c_0 = 3C_1 m^2$. Let h be constant, $h > c_0$. Let u be an L^2-solution of the derivative oblique problem (5.74) with $g = 0$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to \infty$ as $|x| \to \infty$. According to Lemmas 5.8.2 and 5.8.1,

$$0 \le a\|\partial u/\partial n\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (n \cdot v)\, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial n}(\Theta \cdot \nabla u + hu)\, \mathrm{d}\mathcal{H}_{m-1} - h\int_{\partial\Omega} u\frac{\partial u}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} + \frac{1}{2}\int_{\Omega} |\nabla u|^2 \sum_{j=1}^{m} \partial_j v_j\, \mathrm{d}\mathcal{H}_m$$

$$- \int_{\Omega} \sum_{j,k=1}^{m} (\partial_j v_k)(\partial_j u)(\partial_k u) \le -h\int_{\Omega} |\nabla u|^2\, \mathrm{d}\mathcal{H}_m + c_0\int_{\Omega} |\nabla u|^2\, \mathrm{d}\mathcal{H}_m \le 0.$$

Thus $(h - c_0)|\nabla u|^2 = 0$ in Ω. Therefore u is constant on each component of Ω. Hence $0 = \Theta \cdot \nabla u + hu = hu$ on $\partial\Omega$. Since $h > 0$, we deduce that $u = 0$.

Theorem 5.23.4 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $v \in \mathcal{C}^1(R^m, R^m)$, $m > 2$. Let a, α be nonnegative numbers, $a + \alpha > 0$. Suppose that $v \cdot n^\Omega \ge \alpha$ on $\partial\Omega$. Put $\Theta = an^\Omega + v$. Then there exists a nonnegative number c_0 such that the following assertion holds: Fix a constant h such that $h > c_0$. Then $T_\Theta + h\mathcal{S}^\Omega$ is an isomorphism on $L^2(\partial\Omega)$. If $g \in L^2(\partial\Omega)$, then there exists an L^2-solution of the derivative oblique problem (5.74).*

- *If Ω is bounded, then $u = \mathcal{S}^\Omega[T_\Theta + h\mathcal{S}^\Omega]^{-1} g$ is a unique solution of the problem. Moreover,*

  $$\|M_b(u)\|_{L^2(\partial\Omega)} + \|M_b(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)} \tag{5.77}$$

  with a constant C that does not depend on g.
- *If Ω is unbounded and u is a solution of the problem, then there exists $u_\infty \in R$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if u_∞ is given, then $u = \mathcal{S}^\Omega[T_\Theta + h\mathcal{S}^\Omega]^{-1}(g - hu_\infty) + u_\infty$ is a unique solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

  $$\|M_b(u)\|_{L^2(\partial\Omega)} + \|M_b(\nabla u)\|_{L^2(\partial\Omega)} \le C\left[\|g\|_{L^2(\partial\Omega)} + |u_\infty|\right] \tag{5.78}$$

  with a constant C that does not depend on g and u_∞.

Proof Let c_0 be a constant from Lemma 5.23.3. Fix a constant h such that $h > c_0$. Then $T_\Theta + h\mathcal{S}^\Omega$ is a Fredholm operator with index 0 in $L^2(\partial\Omega)$ by

Propositions 5.23.2 and 5.2.3. Fix $\varphi \in \operatorname{Ker}[T_\Theta + h\mathcal{S}^\Omega]$. Then $u = \mathcal{S}^\Omega\varphi$ is a solution of the derivative oblique problem (5.74) with $g = 0$. If Ω is unbounded, then $u(x) \to 0$ as $|x| \to \infty$. Lemma 5.23.3 gives that $u = 0$ in Ω. Continuity of a single layer potential (Proposition 5.2.2) gives that $\mathcal{S}^\Omega\varphi = 0$ on $\partial\Omega$. Lemma 5.9.3 gives that $\varphi = 0$. Since $T_\Theta + h\mathcal{S}^\Omega$ is a Fredholm operator with index 0, it is surjective. Theorem 1.6.3 gives that $T_\Theta + h\mathcal{S}^\Omega$ is an isomorphism.

Let now $g \in L^2(\partial\Omega)$ be given. If Ω is bounded, then $\mathcal{S}^\Omega[T_\Theta + h\mathcal{S}^\Omega]^{-1}g$ is an L^2-solution of the problem (5.74). Lemma 5.23.3 gives that this solution is unique. Suppose now that Ω is unbounded. If $u_\infty \in R$ is given, then $u = \mathcal{S}^\Omega[T_\Theta + h\mathcal{S}^\Omega]^{-1}(g - hu_\infty) + u_\infty$ is an L^2-solution of the problem such that $u(x) \to u_\infty$ as $|x| \to \infty$. Let now v be an L^2-solution of the problem (5.74). Lemma 1.28.2 gives that v is bounded in a neighborhood of infinity. Hence there exists a constant v_∞ such that $v(x) \to v_\infty$ as $|x| \to \infty$ by Proposition 2.17.1. If $v_\infty = u_\infty$, then $u - v \equiv 0$ by Lemma 5.23.3.

The estimates (5.77), (5.78) are consequences of Propositions 5.2.2, 5.2.3, and 5.2.8.

Theorem 5.23.5 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary, $v \in \mathcal{C}^1(R^m, R^m)$. Let a, α be nonnegative numbers, $a+\alpha > 0$. Suppose that $v \cdot n^\Omega \geq \alpha$ on $\partial\Omega$. Put $\Theta = an^\Omega + v$. Define $Pf = \oint_{\partial\Omega} f \, d\mathcal{H}_1$ for $f \in L^2(\partial\Omega)$. Then there exists a nonnegative number c_0 such that the following assertion holds: Fix a constant h such that $h > c_0$. Then $T = T_\Theta(I - P) + h[\mathcal{S}^\Omega(I - P) + P]$ is an isomorphism on $L^2(\partial\Omega)$. If $g \in L^2(\partial\Omega)$, then*

$$u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$$

is a unique L^2-solution of the derivative oblique problem (5.74). Moreover, the estimate (5.77) holds with a constant C that does not depend on g.

Proof Let c_0 be a constant from Lemma 5.23.3. Fix a constant h such that $h > c_0$. Then T is a Fredholm operator with index 0 in $L^2(\partial\Omega)$ by Proposition 5.23.2. Fix $\varphi \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega(I - P)\varphi + P\varphi$ is a solution of the derivative oblique problem (5.74) with $g = 0$. Lemma 5.23.3 gives that $u = 0$ in Ω. Thus u is an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. Thus $[(1/2)I - K_\Omega](I - P)\varphi + P\varphi = 0$ (see Propositions 5.2.2, 5.2.3 and Theorem 5.4.8). Proposition 5.10.3 gives that $\varphi = 0$. Since T is a Fredholm operator with index 0, it is surjective. Theorem 1.6.3 gives that T is an isomorphism.

Let now $g \in L^2(\partial\Omega)$ be given. Clearly, $u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is an L^2-solution of the derivative oblique problem (5.74). The estimate (5.77) is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Lemma 5.23.6 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $p' = p/(p-1)$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetric (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^m)$ have compact support. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$.*

Let $\tilde{h} \in L^\infty(\partial\Omega)$. *Put*

$$h = \tilde{h} + \frac{1}{2}\sum_{i,j=1}^{m} n_j^\Omega(\partial_i a_{ij}).$$

Let u be an L^p-solution of the derivative oblique problem (5.74). Let $\varphi \in \mathcal{C}^\infty(\Omega)$ be such that $M_b(\varphi) \in L^{p'}(\partial\Omega)$, and there exists the nontangential limit of φ at almost all points of $\partial\Omega$. If Ω is unbounded, suppose moreover that $u(x) = O(|x|^{2-m})$, $\varphi(x) = O(|x|^{2-m})$, $|\nabla\varphi(x)| = O(|x|^{1-m})$ as $|x| \to \infty$. Then

$$\int_{\partial\Omega} \varphi g \, \mathrm{d}\mathcal{H}_{m-1} = \int_\Omega \{\nabla\varphi\cdot\nabla u + \sum_{i,j=1}^{m}[a_{ij}\partial_j\varphi\partial_i u + \frac{1}{2}(\partial_j a_{ij})\varphi\partial_i u$$
$$-\frac{1}{2}(\partial_j a_{ij})u\partial_i\varphi]\} \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \varphi\tilde{h}u \, \mathrm{d}\mathcal{H}_{m-1}.$$

If $M_b(u) \in L^{p'}(\partial\Omega)$, then

$$\int_{\partial\Omega} ug \, \mathrm{d}\mathcal{H}_{m-1} = \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \tilde{h}u^2 \, \mathrm{d}\mathcal{H}_{m-1}.$$

Proof We can suppose that $\tilde{h} \equiv 0$. If Ω is unbounded, then $|\nabla u(x)| = O(|x|^{1-m})$ as $|x| \to \infty$ by Lemma 1.28.2 and Proposition 2.17.3. If Ω is unbounded and $m = 2$, then $|\nabla u(x)| = O(|x|^{-2})$ as $|x| \to \infty$ by Proposition 2.17.3.

Suppose first that $a_{ij} \in \mathcal{C}^2(R^m)$. Let $\Omega(k)$ be an increasing sequence of open sets from Lemma 1.17.13. According to the Gauss-Green theorem (Theorem 1.19.1) and the Lebesgue lemma (Lemma 1.13.2),

$$\int_{\partial\Omega} \varphi g \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{k\to\infty} \int_{\partial\Omega(k)} \varphi \left\{ \frac{\partial u}{\partial n} + \sum_{i,j=1}^{m} n_j[a_{ij}\partial_i u + \frac{1}{2}u\partial_i a_{ij}] \right\} \mathrm{d}\mathcal{H}_{m-1}$$
$$= \lim_{k\to\infty} \int_{\Omega(k)} \{\nabla u\cdot\nabla\varphi + \sum_{i,j=1}^{m}[a_{ij}\partial_j\varphi\partial_i u + a_{ij}\varphi\partial_i\partial_j u + (\partial_j a_{ij})\varphi\partial_i u$$
$$+\frac{1}{2}\varphi(\partial_i a_{ij})\partial_j u + \frac{1}{2}u(\partial_i a_{ij})\partial_j\varphi + \frac{1}{2}\varphi u\partial_j\partial_i a_{ij}]\} \, \mathrm{d}\mathcal{H}_m.$$

Since $a_{ji} = -a_{ij}$, we have

$$\sum_{i,j=1}^{m}(\partial_i\partial_j a_{ij})u\varphi = 0, \quad \sum_{i,j=1}^{m} a_{ij}\varphi\partial_i\partial_j u = 0.$$

Thus

$$\int_{\partial\Omega} g\varphi = \lim_{k\to\infty} \int_{\Omega(k)} \{\nabla\varphi\cdot\nabla u + \sum_{i,j=1}^{m} [a_{ij}\partial_j\varphi\partial_i u + (\partial_j a_{ij})\varphi\partial_i u$$
$$-\frac{1}{2}\varphi(\partial_j a_{ij})\partial_i u - \frac{1}{2}u(\partial_j a_{ij})\partial_i\varphi]\}\,\mathrm{d}\mathcal{H}_m = \int_{\Omega} \{\nabla\varphi\cdot\nabla u$$
$$+ \sum_{i,j=1}^{m} [a_{ij}\partial_j\varphi\partial_i u + \frac{1}{2}(\partial_j a_{ij})\varphi\partial_i u - \frac{1}{2}(\partial_j a_{ij})u\partial_i\varphi]\}\,\mathrm{d}\mathcal{H}_m.$$

Let $a_{ij} \in \mathcal{C}^1(R^m)$. Choose a nonnegative function $\phi \in \mathcal{C}^\infty(R^m)$ supported in $B(0;1)$ such that

$$\int_{R^m} \phi(x)\, d\mathcal{H}_m(x) = 1.$$

For $\epsilon > 0$, put $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^m$, $a^\epsilon_{ij} = a_{ij} * \phi_\epsilon$. Then $a^\epsilon_{ij} \in \mathcal{C}^\infty_c(R^m)$ and $a^\epsilon_{ij} \to a_{ij}$ in $\mathcal{C}^{1,0}(\overline{\Omega})$. (See Sect. 1.15.) We have

$$\int_{\partial\Omega} g\varphi\,\mathrm{d}\mathcal{H}_{m-1} = \lim_{\epsilon\downarrow 0} \int_{\partial\Omega} \varphi \left\{ \frac{\partial u}{\partial n} + \sum_{i,j=1}^{m} n_j[a^\epsilon_{ij}\partial_i u + \frac{1}{2}u\partial_i a^\epsilon_{ij}] \right\} \mathrm{d}\mathcal{H}_{m-1}$$
$$= \lim_{\epsilon\downarrow 0} \int_{\Omega} \{\nabla\varphi\cdot\nabla u + \sum_{i,j=1}^{m} [a^\epsilon_{ij}\partial_j\varphi\partial_i u + \frac{1}{2}(\partial_j a^\epsilon_{ij})\varphi\partial_i u - \frac{1}{2}(\partial_j a^\epsilon_{ij})u\partial_i\varphi]\}$$
$$= \int_{\Omega} \{\nabla\varphi\cdot\nabla u + \sum_{i,j=1}^{m} [a_{ij}\partial_j\varphi\partial_i u + \frac{1}{2}(\partial_j a_{ij})\varphi\partial_i u - \frac{1}{2}(\partial_j a_{ij})u\partial_i\varphi]\}\,\mathrm{d}\mathcal{H}_m.$$

If $\varphi = u$, then $a_{ji} = -a_{ij}$ gives

$$\int_{\partial\Omega} ug\,\mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega} |\nabla u|^2\,\mathrm{d}\mathcal{H}_m.$$

Proposition 5.23.7 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $p' = p/(p-1)$. Let $A = (a_{ij})$ be a matrix function of the type $m\times m$ that is antisymmetric (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^m)$. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $\tilde{h} \in L^\infty(\partial\Omega)$, $\tilde{h} \geq 0$. Put*

$$h = \tilde{h} + \frac{1}{2}\sum_{i,j=1}^{m} n^\Omega_j(\partial_i a_{ij}).$$

Let u be an L^p-solution of the derivative oblique problem (5.74) with $g \equiv 0$. Suppose that $M_b(u) \in L^{p'}(\partial\Omega)$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then u is constant.

- *If $\int_{\partial\Omega} [|\tilde{h}| + |h|] \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then $u \equiv 0$.*
- *If Ω is unbounded and $m > 2$, then $u \equiv 0$.*

Proof We can suppose that a_{ij} have compact support. If Ω is unbounded, then $u(x) = O(|x|^{2-m})$ as $|x| \to \infty$ by Lemma 1.28.2 and Proposition 2.17.3. According to Lemma 5.23.6,

$$0 = \int_{\partial\Omega} ug \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \tilde{h} u^2 \, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus $\nabla u = 0$ in Ω, $\tilde{h} u^2 = 0$ on $\partial\Omega$. Since Ω is connected, there exists a constant c such that $u \equiv c$ (see Lemma 1.20.1). If $\int_{\partial\Omega} \tilde{h} \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then $c = 0$, because $\tilde{h} c^2 = 0$ on $\partial\Omega$. If $\int_{\partial\Omega} |h| \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then $c = 0$, because $0 = \Theta \cdot \nabla u + hu = hc$ a.e. on $\partial\Omega$. If Ω is unbounded and $m > 2$, then $c = 0$, because $u(x) \to 0$ as $|x| \to \infty$.

Lemma 5.23.8 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Let $a_{ij} \in \mathcal{C}^1(R^m)$ be functions with compact support such that $a_{ji} = -a_{ij}$. If $v \in \tilde{W}^{1,p}(\Omega)$, then*

$$\int_{\Omega} \sum_{i,j=1}^{m} (\partial_j a_{ij}) \partial_i v \, \mathrm{d}\mathcal{H}_m = -\int_{\partial\Omega} v \sum_{i,j=1}^{m} n_j (\partial_i a_{ij}) \, \mathrm{d}\mathcal{H}_{m-1}. \tag{5.79}$$

Proof Suppose first that $a_{ij} \in \mathcal{C}^2(R^m)$. According to the Gauss-Green formula for Sobolev spaces (Proposition 1.24.4), we have

$$\int_{\partial\Omega} v \sum_{i,j=1}^{m} n_j (\partial_i a_{ij}) = \int_{\Omega} \sum_{i,j=1}^{m} (\partial_i a_{ij}) \partial_j v + \int_{\Omega} v \sum_{i,j=1}^{m} (\partial_i \partial_j a_{ij}).$$

Since $a_{ji} = -a_{ij}$, we obtain (5.79).

Let $a_{ij} \in \mathcal{C}^1(R^m)$. Choose a nonnegative function $\phi \in \mathcal{C}^\infty(R^m)$ supported in $B(0;1)$ such that

$$\int_{R^m} \phi(x) \, d\mathcal{H}_m(x) = 1.$$

For $\epsilon > 0$, put $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^m$, $a_{ij}^\epsilon = a_{ij} * \phi_\epsilon$. Then $a_{ij}^\epsilon \in \mathcal{C}_c^\infty(R^m)$ and $a_{ij}^\epsilon \to a_{ij}$ in $\mathcal{C}^{1,0}(\overline{\Omega})$. (See Sect. 1.15.) We have proved (5.79) for a_{ij}^ϵ. Letting $\epsilon \downarrow 0$, we obtain (5.79) for a_{ij}.

Proposition 5.23.9 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. Suppose that Ω is bounded or $m = 2$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^m)$. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $g \in L^p(\partial\Omega)$,*

$$h = 0, \quad \sum_{i,j=1}^{m} n_j^\Omega \partial_i a_{ij} = 0 \qquad \text{on } \partial\Omega.$$

If there exists an L^p-solution of the derivative oblique problem (5.74), then

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Proof Suppose first that Ω is bounded. Define $\varphi \equiv 1$. Lemma 5.23.6 gives

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} g\varphi \, \mathrm{d}\mathcal{H}_{m-1} = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{m} (\partial_j a_{ij}) \partial_i u \, \mathrm{d}\mathcal{H}_m.$$

Since $u \in W^{1,p}(\Omega)$ by Lemma 1.28.1, we obtain by Lemma 5.23.8

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{m} (\partial_j a_{ij}) \partial_i u = -\frac{1}{2} \int_{\partial\Omega} u \sum_{i,j=1}^{m} n_j \partial_i a_{ij} = 0.$$

Let now $m = 2$ and Ω be unbounded. We can suppose that A has compact support. Since $u(x) = O(1)$ as $|x| \to \infty$ by Lemma 1.28.2, Proposition 2.17.3 gives that $|\nabla u(x)| = O(|x|^{-2})$ as $|x| \to \infty$. Thus

$$\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{r \to \infty} \int_{\partial[\Omega \cap B(0;r)]} \nabla u \cdot (n + An) \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Proposition 5.23.10 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^2$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^m)$. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Define $Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1}$ for $f \in L^2(\partial\Omega)$. If $h \in L^\infty(\partial\Omega)$, then $T_\Theta + h\mathcal{S}^\Omega$, $T_\Theta(I - P) + h[\mathcal{S}^\Omega(I - P) + P]$ are Fredholm operators with index 0 in $L^2(\partial\Omega)$.*

Proof First we show that there exists $u \in \mathcal{C}^1(R^m, R^m)$ such that $u = \tau$ on $\partial\Omega$. Fix $z \in \partial\Omega$. There exist a coordinate system centered in z, $r > 0$, and a function $\Phi \in \mathcal{C}^2(R^{m-1})$ such that $\Omega \cap B(0; r) = \{[x'; x_m]; x' \in R^{m-1}, x_m > \Phi(x')\} \cap B(0; r)$.

The vector function

$$v(x', x_m) = [-\nabla\Phi(x'), 1]/\sqrt{|\nabla\Phi(x')|^2 + 1} \in \mathcal{C}^1(R^m, R^m)$$

and $n^\Omega = v$ on $B(0; r) \cap \partial\Omega)$. So, for each $z \in \partial\Omega$, there exists $r(z) > 0$ and $v^z \in \mathcal{C}^1(R^m, R^m)$ such that $n^\Omega = v^z$ on $B(z; r(z)) \cap \partial\Omega$. Since $\partial\Omega$ is compact, there exist $z_1, \dots, z_k$ such that $\partial\Omega \subset B(z_1, r(z_1)) \cup \cdots \cup B(z_k, r(z_k))$. According to Lemma on partition of unity (Lemma 1.18.1), there exists $\varphi_j \in \mathcal{C}^\infty(B(z_j; r(z_j))$ such that $\varphi_1 + \cdots + \varphi_k = 1$ on $\partial\Omega$. Define

$$u = A\sum_{j=1}^{k} \varphi_j v^{z_j}.$$

Then $u \in \mathcal{C}^1(R^m, R^m)$ and $u = \tau$ on $\partial\Omega$.

Since $\tau(x)$ is a tangential vector function, we have $n^\Omega \cdot \tau = 0$ on $\partial\Omega$. Proposition 5.23.2 gives that $T_\Theta + h\mathcal{S}^\Omega$, $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P) + P]$ are Fredholm operators with index 0 in $L^2(\partial\Omega)$.

Theorem 5.23.11 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^2$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$), $a_{ij} \in \mathcal{C}^1(R^m)$, and*

$$\sum_{i,j=1}^{m} n_j^\Omega \partial_i a_{ij} = 0 \quad \text{on } \partial\Omega.$$

Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Denote $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_{m-1} = 0\}$. Then $T_\Theta(L^2(\partial\Omega)) = X$. Denote by T the restriction of T_Θ onto X. Then T is an isomorphism on X. Let $h \equiv 0$. If $g \in L^2(\partial\Omega)$, then there exists an L^2-solution of the derivative oblique problem (5.74) if and only if $\int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0$. The general form of a solution of this problem is $\mathcal{S}^\Omega(T^{-1}g) + c$ where c is an arbitrary constant. If u is an L^2-solution of the derivative oblique problem (5.74), then

$$\|M_b(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}; \tag{5.80}$$

if moreover

$$\int_\Omega u \, \mathrm{d}\mathcal{H}_m = 0, \tag{5.81}$$

then

$$\|M_b(u)\|_{L^2(\partial\Omega)} + \|M_b(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}. \tag{5.82}$$

Here a constant C does not depend on g.

Proof We can suppose that a_{ij} have compact support.

If there exists an L^2-solution of the derivative oblique problem (5.74), then $g \in X$ by Proposition 5.23.9.

Let $f \in L^2(\partial\Omega)$. Then $u = \mathcal{S}^\Omega f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = T_\theta f$. Thus $T_\theta f = g \in X$. So, $T_\theta(L^2(\partial\Omega)) \subset X$. The operator T_Θ is a Fredholm operator with index 0 in $L^2(\partial\Omega)$ by Proposition 5.23.10. Since $T_\Theta(X) \subset X$, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = 0$. So, there exists a constant c such that $\mathcal{S}^\Omega f = c$ in Ω (see Proposition 5.23.7). The nontangential limit gives that $\mathcal{S}^\Omega f = c$ on $\partial\Omega$ (see Proposition 5.2.2). Since $f \in X$, we obtain

$$\int\limits_{\partial\Omega} f\mathcal{S}^\Omega f \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} fc \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Lemma 5.9.3 gives that $f \equiv 0$. Since T is a Fredholm operator with index 0, we have $T(X) = X$. Thus T is an isomorphism by Theorem 1.6.3.

If $g \in X$ and c is a constant, then $\mathcal{S}^\Omega(T^{-1}g) + c$ is an L^2-solution of the derivative oblique problem (5.74). Proposition 5.23.7 gives that this is a general form of an L^2-solution of the derivative oblique problem (5.74).

Let $g \in X$. Then $u = \mathcal{S}^\Omega(T^{-1}g) - \oint_\Omega \mathcal{S}^\Omega(T^{-1}g) \, \mathrm{d}\mathcal{H}_m$ is a unique L^2-solution of the derivative oblique problem (5.74) such that (5.81) holds. Since $f \mapsto \oint_\Omega \mathcal{S}^\Omega f \, \mathrm{d}\mathcal{H}_m$ is a bounded linear functional on $L^2(\partial\Omega)$ (see Corollary 5.2.9 and Proposition 1.14.1), Propositions 5.2.2 and 5.2.8 give that there exists a constant C such that (5.82) holds. If v is another solution of the problem, then $v - u$ is constant and thus $\|M_b(\nabla v)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}$.

Theorem 5.23.12 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^2$, $m > 2$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^m)$. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$ on $\partial\Omega$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $\tilde{h} \in L^\infty(\partial\Omega)$, $\tilde{h} \ge 0$. Define*

$$h = \tilde{h} + \frac{1}{2}\sum_{i,j=1}^{m} n_j^\Omega \partial_i a_{ij}.$$

Suppose that

$$\int_{\partial\Omega} [|\tilde{h}| + |h|] \, \mathrm{d}\mathcal{H}_{m-1} > 0.$$

Define $Tf = T_\Theta f + h\mathcal{S}^\Omega f$. Then T is an isomorphism on $L^2(\partial\Omega)$. If $g \in L^2(\partial\Omega)$, then $u = \mathcal{S}^\Omega(T^{-1}g)$ is a unique L^2-solution of the derivative oblique problem (5.74). Moreover, the estimate (5.82) holds with a constant C that does not depend on g.

Proof Proposition 5.23.7 gives uniqueness.

If $f \in \operatorname{Ker} T$, then $u = \mathcal{S}^{\Omega} f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = 0$. So, $\mathcal{S}^{\Omega} f = 0$ in Ω. The nontangential limit gives that $\mathcal{S}^{\Omega} f = 0$ on $\partial\Omega$ (see Proposition 5.2.2). Lemma 5.9.3 forces $f \equiv 0$. Since T is a Fredholm operator with index 0 by Proposition 5.23.10, it is onto. Hence T is an isomorphism by Theorem 1.6.3.

Let now $g \in L^2(\partial\Omega)$. Then $u = \mathcal{S}^{\Omega}(T^{-1} g)$ is an L^2-solution of the derivative oblique problem (5.74). The estimate (5.82) is consequence of Propositions 5.2.2 and 5.2.8.

Theorem 5.23.13 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^2$. Let $A = (a_{ij})$ be a matrix function of the type 2×2 that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$), and $a_{ij} \in \mathcal{C}^1(R^2)$. Define the tangential vector function $\tau(x) = A(x) n^{\Omega}(x)$ on $\partial\Omega$, and put $\Theta(x) = n^{\Omega}(x) + \tau(x)$. Let $\tilde{h} \in L^{\infty}(\partial\Omega)$, $\tilde{h} \geq 0$. Define*

$$h = \tilde{h} + \frac{1}{2} \sum_{i,j=1}^{2} n_j^{\Omega} \partial_i a_{ij}.$$

Suppose that

$$\int_{\partial\Omega} [|\tilde{h}| + |h|] \, d\mathcal{H}_1 > 0.$$

Define $Pf = \oint_{\partial\Omega} f \, d\mathcal{H}_1$, $Tf = T_{\Theta}(I - P)f + h[\mathcal{S}^{\Omega}(I - P)f + Pf]$. Then T is an isomorphism on $L^2(\partial\Omega)$. If $g \in L^2(\partial\Omega)$, then

$$u = \mathcal{S}^{\Omega}[(I - P)T^{-1} g] + PT^{-1} g$$

is a unique L^2-solution of the derivative oblique problem (5.74). Moreover, the estimate (5.82) holds with a constant C that does not depend on g.

Proof Proposition 5.23.7 gives uniqueness.

If $f \in \operatorname{Ker} T$, then $u = \mathcal{S}^{\Omega}(I - P)f + Pf$ is an L^2-solution of the derivative oblique problem (5.74) with $g = 0$. Thus, $u = 0$ in Ω. Hence u is an L^2-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u / \partial n + u = 0$ on $\partial\Omega$. So, $\frac{1}{2}(I - P)f - K'_{\Omega}(I - P)f + \mathcal{S}^{\Omega}(I - P)f + Pf = 0$ by Propositions 5.2.2, 5.2.3 and Theorem 5.4.8. Therefore $f \equiv 0$ by Proposition 5.10.3. Since T is a Fredholm operator with index 0 by Proposition 5.23.10, it is onto. Thus T is an isomorphism by Theorem 1.6.3.

Let now $g \in L^2(\partial\Omega)$. Then $u = \mathcal{S}^{\Omega}[T^{-1}(I - P)g] + PT^{-1} g$ is an L^2-solution of the derivative oblique problem (5.74). The estimate (5.82) is consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Theorem 5.23.14 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$, $m > 2$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$ that is antisymmetrical (i.e., $a_{ij} = -a_{ji}$) and $a_{ij} \in \mathcal{C}^1(R^m)$. Define the tangential vector* *function* $\tau(x) = A(x)n^{\Omega}(x)$ *on* $\partial\Omega$, *and put* $\Theta(x) = n^{\Omega}(x) + \tau(x)$. *Let* $\tilde{h} \in L^{\infty}(\partial\Omega)$, $\tilde{h} \geq 0$. *Define*

$$h = \tilde{h} + \frac{1}{2}\sum_{i,j=1}^{m} n_j^{\Omega}\partial_i a_{ij}.$$

Put $Tf = T_{\Theta}f + h\mathcal{S}^{\Omega}f$. *Then* T *is an isomorphism on* $L^2(\partial\Omega)$. *Fix* $g \in L^2(\partial\Omega)$.

- *If* u *is an* L^2*-solution of the derivative oblique problem (5.74), then there exists* $u_{\infty} \in R$ *such that* $u(x) \to u_{\infty}$ *as* $|x| \to \infty$.
- *If* $u_{\infty} \in R$ *is given, then* $u = \mathcal{S}^{\Omega}T^{-1}(g - hu_{\infty}) + u_{\infty}$ *is a unique* L^2*-solution of the derivative oblique problem (5.74) such that* $u(x) \to u_{\infty}$ *as* $|x| \to \infty$. *Moreover,*

$$\|M_b(u)\|_{L^2(\partial\Omega)} + \|M_b(\nabla u)\|_{L^2(\partial\Omega)} \leq C\left[\|g\|_{L^2(\partial\Omega)} + |u_{\infty}|\right] \tag{5.83}$$

with a constant C *that does not depend on* g *and* u_{∞}.

Proof Let u be an L^2-solution of the derivative oblique problem (5.74). Since u is bounded in a neighborhood of infinity by Lemma 1.28.2, there exists a constant u_{∞} such that $u(x) \to u_{\infty}$ as $|x| \to \infty$ (see Proposition 2.17.1). If v is another L^2-solution of the derivative oblique problem (5.74) such that $v(x) \to u_{\infty}$ as $|x| \to \infty$, then $u - v \equiv 0$ by Proposition 5.23.7.

Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^{\Omega}f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = 0$ such that $u(x) \to 0$ as $|x| \to \infty$. So, $u = 0$ in Ω. Therefore u is an L^2-solution of the Neumann problem $-\Delta u = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$ and $[(1/2)I - K'_{\Omega}]f = 0$. Proposition 5.9.4 gives that $f = 0$. Since T is a Fredholm operator with index 0 by Proposition 5.23.10, it is onto. Hence T is an isomorphism by Theorem 1.6.3.

If $u_{\infty} \in R$, then $u = \mathcal{S}^{\Omega}T^{-1}(g - hu_{\infty}) + u_{\infty}$ is a unique L^2-solution of the derivative oblique problem (5.74) such that $u(x) \to u_{\infty}$ as $|x| \to \infty$. The estimate (5.83) is consequence of Propositions 5.2.3 and 5.2.8.

Theorem 5.23.15 *Let* $\Omega \subset R^2$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^2$. *Let* $A = (a_{ij})$ *be a matrix function of the type* 2×2 *that is antisymmetrical (i.e.,* $a_{ij} = -a_{ji}$*),* $a_{ij} \in \mathcal{C}^1(R^2)$*, and*

$$\sum_{i,j=1}^{2} n_j^{\Omega}\partial_i a_{ij} = 0 \quad \text{on } \partial\Omega$$

(i.e., A is constant on each component of $\partial\Omega$*). Define the tangential vector function* $\tau(x) = A(x)n^{\Omega}(x)$ *on* $\partial\Omega$, *and put* $\Theta(x) = n^{\Omega}(x) + \tau(x)$. *Denote* $X = \{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$. *Then* $T_{\Theta}(L^2(\partial\Omega)) = X$. *Denote by* T *the restriction of* T_{Θ} *onto* X. *Then* T *is an isomorphism on* X. *Let* $h \equiv 0$.

- *If* $g \in L^2(\partial\Omega)$*, then there exists an* L^2*-solution of the derivative oblique problem* (5.74) *if and only if* $g \in X$.
- *If* u *is an* L^2*-solution of the derivative oblique problem* (5.74)*, then there exists* $u_\infty \in R$ *such that* $u(x) \to u_\infty$ *as* $|x| \to \infty$.
- *If* $g \in X$ *and* $u_\infty \in R$ *are given, then*

$$u = \mathcal{S}^\Omega T^{-1} g + u_\infty$$

is a unique L^2*-solution of the derivative oblique problem* (5.74) *such that* $u(x) \to u_\infty$ *as* $|x| \to \infty$*. Moreover, the estimate* (5.83) *holds with a constant* C *that does not depend on* g *and* u_∞*.*

Proof If there exists an L^2-solution of the derivative oblique problem (5.74), then $g \in X$ by Proposition 5.23.9. Since u is bounded in a neighborhood of infinity by Lemma 1.28.2, there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$ (see Proposition 2.17.1). If v is another L^2-solution of the derivative oblique problem (5.74) such that $v(x) \to u_\infty$ as $|x| \to \infty$, then $u - v \equiv 0$ by Proposition 5.23.7.

Let $f \in L^2(\partial\Omega)$. Then $u = \mathcal{S}^\Omega f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = T_\theta f$. Thus $T_\theta f = g \in X$. So, $T_\theta(L^2(\partial\Omega)) \subset X$. The operator T_Θ is a Fredholm operator with index 0 in $L^2(\partial\Omega)$ by Proposition 5.23.10. Since $T_\Theta(X) \subset X$, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega f$ is an L^2-solution of the derivative oblique problem (5.74) with $g = 0$. So, there exists a constant c such that $\mathcal{S}^\Omega f = c$ in Ω (see Proposition 5.23.7). The nontangential limit gives that $\mathcal{S}^\Omega f = c$ on $\partial\Omega$ (see Proposition 5.2.3). Since $f \in X$, we obtain

$$\int_{\partial\Omega} f \mathcal{S}^\Omega f \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} fc \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Lemma 5.9.3 gives that $f \equiv 0$. Since T is a Fredholm operator with index 0, we have $T(X) = X$. Thus T is an isomorphism by Theorem 1.6.3.

If $g \in X$ and u_∞ is a constant, then $u = \mathcal{S}^\Omega(T^{-1}g) + u_\infty$ is an L^2-solution of the derivative oblique problem (5.74) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Propositions 5.2.3 and 5.2.8 give that there exists a constant C such that the estimate (5.83) holds.

5.24 L^p-Solution of the Derivative Oblique Problem in the Plane

In this section, we study L^p-solutions of the derivative oblique problem on a planar domain with compact Ljapunov boundary. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector on $\partial\Omega$. If A is an antisymmetrical matrix function, then there exists a function a such that $An^\Omega = a\tau$ on $\partial\Omega$. So, we shall study the derivative oblique problem (5.74) with $\Theta = n^\Omega + a\tau$.

Proposition 5.24.1 *Let $\Omega \subset R^2$ be a domain with connected compact Lipschitz boundary, $1 < p < \infty$. Let $a \in \mathcal{C}^1(R^2)$ and $c, h \in L^\infty(\partial\Omega)$. Define $\tau = (-n_2^\Omega, n_1^\Omega)$, $\Theta = n^\Omega + a\tau$. For $f \in W^{1,p}(\partial\Omega)$, define*

$$Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1,$$

$$u_f = \mathcal{S}^\Omega[\partial f/\partial\tau + c(f - Pf)] - \mathcal{D}^\Omega[a(f - Pf)] + Pf.$$

Then $u_f \in H(\Omega)$, $M_b(u_f), M_b(\nabla u_f) \in L^p(\partial\Omega)$, and there exist nontangential limits of u_f and ∇u_f at almost all points of $\partial\Omega$. Define

$$Tf = \Theta \cdot \nabla u_f + hu_f.$$

Then $T : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is a bounded linear operator. Moreover,

$$f \mapsto Tf - (1 + a^2)\left(\frac{1}{2}I - K'_\Omega\right)\frac{\partial f}{\partial\tau}$$

is a compact operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$.

Proof $u_f \in H(\Omega)$, there exist nontangential limits of u_f and ∇u_f at almost all points of $\partial\Omega$, and $M_b(u_f), M_b(\nabla u_f) \in L^p(\partial\Omega)$ by Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.3.5, 5.4.7, and 5.5.2. Denote $g = f - Pf$. According to Corollary 5.5.3, Propositions 5.2.2, 5.2.3, Theorems 5.3.5 and 5.4.7,

$$\begin{aligned} Tf = {} & \frac{1}{2}(\partial g/\partial\tau + cg) - K'_\Omega(\partial g/\partial\tau + cg) - \frac{\partial}{\partial\tau}\mathcal{S}^\Omega\left[\frac{\partial}{\partial\tau}(ag)\right] \\ & + a\frac{\partial}{\partial\tau}\mathcal{S}^\Omega(\partial g/\partial\tau + cg) + \frac{a}{2}\left[\frac{\partial}{\partial\tau}(ag)\right] - aK'_\Omega\left[\frac{\partial}{\partial\tau}(ag)\right] \\ & + h[\mathcal{S}^\Omega(\partial g/\partial\tau + cg) - \frac{1}{2}ag - K_\Omega ag + Pf]. \end{aligned}$$

By virtue of Theorem 5.4.7,

$$Tf = (1 + a^2)\left(\frac{1}{2}I - K'_\Omega\right)\frac{\partial f}{\partial\tau} + Lf,$$

where $L = L_1 + L_2 + L_3 + L_4 + hP$ with

$$L_1 f = \frac{c}{2}(I-P)f - K'_\Omega c(I-P)f - \frac{\partial}{\partial\tau}\mathcal{S}^\Omega\left[\frac{\partial a}{\partial\tau}(I-P)f\right] + a\frac{\partial}{\partial\tau}\mathcal{S}^\Omega[c(I-P)f]$$

$$+\frac{a}{2}\frac{\partial a}{\partial\tau}(I-P)f - aK'_\Omega\left[\frac{\partial a}{\partial\tau}(I-P)f\right] - \frac{1}{2}ah(I-P)f - hK_\Omega a(I-P)f,$$

$$L_2 f = h\mathcal{S}^\Omega[\partial f/\partial\tau + c(I-P)f],$$

$$L_3 f(x) = \int\limits_{\partial\Omega} \frac{\partial h_2(x-y)}{\partial\tau}[a(x)-a(y)]\frac{\partial f(y)}{\partial\tau}\, \mathrm{d}\mathcal{H}_1(y),$$

$$L_4 f(x) = a(x)\int\limits_{\partial\Omega} \frac{\partial h_2(x-y)}{\partial n}[a(x)-a(y)]\frac{\partial f(y)}{\partial\tau}\, \mathrm{d}\mathcal{H}_1(y).$$

It is enough to prove that L_1, L_2, L_3, and L_4 are compact operators from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ (see Theorem 1.7.4).

The identity is a compact linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Lemma 1.21.8. Since the operators K'_Ω, K_Ω, $\mathcal{S}^\Omega$, and $\frac{\partial}{\partial\tau}\mathcal{S}^\Omega$ are bounded linear operators on $L^p(\partial\Omega)$ (see Propositions 5.2.2, 5.3.3, 5.4.5, Theorem 5.4.7, and Proposition 5.2.8), the operator L_1 is a compact linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Theorem 1.7.4.

The mapping $f \mapsto [\partial f/\partial\tau + c(I-P)f]$ is a bounded linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$. Since $\mathcal{S}^\Omega$ is a compact linear operator on $L^p(\partial\Omega)$ by Proposition 5.2.2, the operator L_2 is a compact linear operator from $W^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega)$ by Theorem 1.7.4.

There exists a constant C such that $|\nabla h_2(x-y)[a(x)-a(y)]| \leq C$ for $x, y \in \partial\Omega$. Proposition 1.26.6 gives that L_3, L_4 are compact operators on $L^p(\partial\Omega)$.

Lemma 5.24.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m-1 < p < \infty$. Let $u \in H(\Omega)$, $M_b^\Omega(\nabla u) \in L^p(\partial\Omega)$. Then u can be extended as a function from $\mathcal{C}^0(\overline{\Omega})$.*

Proof According to Theorem 5.6.1, there exist nontangential limits $u(x)$, $\nabla u(x)$ of u, ∇u at almost all $x \in \partial\Omega$, $M_b(u) \in L^p(\partial\Omega)$,

$$u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(n^\Omega \cdot \nabla u),$$

and $u, \partial_j u \in L^p(\partial\Omega)$. Since $\partial_{\tau_{jk}} u = n_j^\Omega \partial_k u - n_k^\Omega \partial_j u \in L^p(\partial\Omega)$, we have $u \in W^{1,p}(\partial\Omega)$. Since $p > m-1$, Theorem 1.21.13 gives that $u \in \mathcal{C}^0(\partial\Omega)$.

Denote $g = u, f = n^\Omega \cdot \nabla u$ on $\partial\Omega$. Suppose first that $p \geq 2$. Then u is a PWB-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ by Theorem 5.16.2. The domain Ω is regular by Theorems 4.5.7 and 4.5.8. Since $g \in \mathcal{C}^0(\partial\Omega)$, we infer that $u \in \mathcal{C}^0(\overline{\Omega})$.

Let now $p < 2$. (Since $p > m-1$, we have $m = 2$.) Choose $g_k \in W^{1,2}(\partial\Omega)$, $f_k \in L^2(\partial\Omega)$ such that $g_k \to g$ in $W^{1,p}(\partial\Omega)$, $f_k \to f$ in $L^p(\partial\Omega)$. Denote

$$u_k = \mathcal{D}^\Omega g_k + \mathcal{S}^\Omega f_k.$$

Then $u_k \in H(\Omega)$, $M_b^\Omega(\nabla u_k) \in L^2(\partial\Omega)$ by Proposition 5.2.8 and Theorem 5.5.2. We have proved that u_k can be extended as functions from $\mathcal{C}^0(\overline{\Omega})$. According to Proposition 5.2.2 and Theorem 5.3.5, we have $u = g/2 + K^\Omega g + \mathcal{S}^\Omega f$, $u_k = g_k/2 + K^\Omega g_k + \mathcal{S}^\Omega f_k$ on $\partial\Omega$. This gives that $u_k \to u$ in $W^{1,p}(\partial\Omega)$ (see Lemmas 5.4.9 and 5.5.4). So, $u_k \to u$ in $\mathcal{C}^0(\partial\Omega)$ by Theorem 1.21.13. Since $u \in \mathcal{C}^0(\partial\Omega)$ and Ω is regular, there exists $v \in H(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ such that $v = u$ on $\partial\Omega$. Since $u_k \to u$ in $\mathcal{C}^0(\partial\Omega)$, the maximum principle (Theorem 2.4.2) gives that $u_k \to v$ in $\mathcal{C}^0(\overline{\Omega})$. If $x \in \Omega$, then

$$u(x) = \mathcal{D}^\Omega g(x) + \mathcal{S}^\Omega f(x) = \lim_{k\to\infty} [\mathcal{D}^\Omega g_k(x) + \mathcal{S}^\Omega f_k(x)] = \lim_{k\to\infty} u_k(x) = v(x).$$

Proposition 5.24.3 *Let $\Omega \subset R^2$ be a domain with connected compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $a \in \mathcal{C}^1(R^2)$ and $h \in L^\infty(\partial\Omega)$ be such that $h > 0$, $|a| > 0$ on $\partial\Omega$. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$, $\Theta = n^\Omega + a\tau$,*

$$c = \frac{3}{2a}\frac{\partial a}{\partial \tau} - a.$$

For $f \in W^{1,p}(\partial\Omega)$, define

$$Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1,$$

$$u_f = \mathcal{S}^\Omega[\partial f/\partial\tau + c(f - Pf)] - \mathcal{D}^\Omega[a(f - Pf)] + Pf.$$

Then $u_f \in H(\Omega)$, $M_b(u_f), M_b(\nabla u_f) \in L^p(\partial\Omega)$, and there exist nontangential limits of u_f and ∇u_f at almost all points of $\partial\Omega$. Define

$$Tf = \Theta \cdot \nabla u_f + \left[\frac{1}{2}\frac{\partial a}{\partial\tau} + h\right] u_f.$$

Then $T : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism. Let $g \in L^p(\partial\Omega)$. Put $f = T^{-1}g$. Then u_f is a unique L^p-solution of the derivative oblique problem

$$\Delta u = 0 \quad in\ \Omega, \qquad \Theta \cdot \nabla u + \left[\frac{1}{2}\frac{\partial a}{\partial\tau} + h\right] u = g \quad on\ \partial\Omega. \tag{5.84}$$

Moreover,

$$\|M_b(u_f)\|_{L^p(\partial\Omega)} + \|M_b(\nabla u_f)\|_{L^p(\partial\Omega)} \le C\|g)\|_{L^p(\partial\Omega)} \tag{5.85}$$

where a constant C does not depend on g.

Proof The uniqueness of an L^p-solution of the problem (5.84) follows from Lemma 5.24.2 and Proposition 5.23.7.

If $f \in W^{1,p}(\partial\Omega)$, then $u_f \in H(\Omega)$, $M_b(u_f), M_b(\nabla u_f) \in L^p(\partial\Omega)$, and there exist nontangential limits of u_f and ∇u_f at almost all points of $\partial\Omega$ (see Proposition 5.24.1).

$T_1 : f \mapsto \partial f/\partial\tau$ is a Fredholm operator with index 0 from $W^{1,p}(\partial\Omega)$ to $L^p(\Omega)$. The operator $(1 + a^2)[(1/2)I - K'_\Omega]$ is Fredholm with index 0 on $L^p(\partial\Omega)$ by Propositions 5.9.9, 5.11.1 and Theorem 1.8.3. So, $(1 + a^2)[(1/2)I - K'_\Omega] \circ T_1$ is a Fredholm operator with index 0 by Theorem 1.8.6. The operator $T - (1 + a^2)[(1/2)I - K'_\Omega] \circ T_1$ is compact by Proposition 5.24.1. Thus $T : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator with index 0 by Theorem 1.8.3.

Let $f \in \operatorname{Ker} T$. Then $f \in W^{1,2}(\partial\Omega)$ by Lemma 1.8.4. The function u_f is an L^2-solution of the derivative oblique problem (5.84) with $g \equiv 0$. So, $u_f = 0$ in Ω. Denote $\Omega_+ = \Omega$, $\Omega_- = R^2 \setminus \overline{\Omega}$. Define u_f on Ω_- by the same prescription. Denote $F = f - Pf$. By virtue of Corollary 5.5.3 and jump properties of boundary layer potentials and their derivatives (Propositions 5.2.2, 5.2.3, Theorems 5.3.5, 5.4.7, and 5.4.8), we obtain

$$[u_f]_- = [u_f]_- - [u_f]_+ = aF,$$

$$[\partial u_f/\partial n^{\Omega-}]_- = [\partial u_f/\partial n^{\Omega}]_+ - [\partial u_f/\partial n^{\Omega}]_- = \partial F/\partial\tau + cF,$$

$$[\partial u_f/\partial\tau]_- = [\partial u_f/\partial\tau]_- - [\partial u_f/\partial\tau]_+ = -\partial(aF)/\partial\tau.$$

Define $A = (1 - a)/a$ on a neighborhood of $\partial\Omega$. We can extend A in a such way that $A \in \mathcal{C}^1(R^2)$. Then u_f is an L^2-solution of the derivative oblique problem

$$\Delta u = 0 \quad \text{in } \Omega_-, \qquad \partial u/\partial n^{\Omega-} + A\partial u/\partial\tau + \frac{1}{2}(\partial A/\partial\tau)u + u = G \quad \text{on } \partial\Omega,$$

with

$$\begin{aligned} G &= [\partial u_f/\partial n^{\Omega-}]_- + A[\partial u_f/\partial\tau]_- + \left[\frac{1}{2}\frac{\partial A}{\partial\tau} + 1\right][u_f]_- \\ &= \frac{\partial F}{\partial\tau} + cF + \frac{a-1}{a}a\frac{\partial F}{\partial\tau} + \frac{a-1}{a}F\frac{\partial a}{\partial\tau} + aF\left[1 - \frac{1}{2a^2}\frac{\partial a}{\partial\tau}\right] \\ &= \frac{\partial(aF)}{\partial\tau} + F\left(c - \frac{1}{a}\frac{\partial a}{\partial\tau} + a - \frac{1}{2a}\frac{\partial a}{\partial\tau}\right) = \frac{\partial(aF)}{\partial\tau}. \end{aligned}$$

According to Lemma 5.23.6,

$$\int_{\Omega_-} |\nabla u_f|^2 + \int_{\partial\Omega} [u_f]_-^2 = \int_{\partial\Omega} u_f G = \int_{\partial\Omega} (aF)\frac{\partial(aF)}{\partial\tau} = \int_{\partial\Omega} \frac{1}{2}\frac{\partial(aF)^2}{\partial\tau} = 0.$$

Thus $0 = [u_f]_- = aF$ on $\partial\Omega$. Since $|a| > 0$ on $\partial\Omega$, we infer that $F \equiv 0$. Therefore $0 = u_f = Pf$ in Ω. Since $f = F + Pf$, we infer that $f \equiv 0$.

Since $T : W^{1,p}(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator with index 0 and trivial kernel, it is onto. Theorem 1.6.3 forces that T is an isomorphism.

Let now $g \in L^p(\partial\Omega)$. Put $f = T^{-1}g$. Then u_f is an L^p-solution of the derivative oblique problem (5.84). The estimate (5.85) is a consequence of Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.3.5 and 5.5.2.

Proposition 5.24.4 *Let $\Omega \subset R^2$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \in L^\infty(\partial\Omega)$, $a \in \mathcal{C}^1(R^2)$. For each component Λ of $\partial\Omega$, suppose that $|a| > 0$ on the whole Λ or $a = 0$ on the whole Λ. Define $\tau = (-n_2^\Omega, n_1^\Omega)$, $\Theta = n^\Omega + a\tau$. For $f \in L^p(\partial\Omega)$, define $Pf = ⨏_{\partial\Omega} f \, d\mathcal{H}_1$. Then $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P) + P]$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$.*

Proof Suppose first that $\partial\Omega$ is connected. Define

$$\tilde{h} = 1 + \frac{1}{2}\frac{\partial a}{\partial\tau} \quad \text{on } \partial\Omega.$$

If $a = 0$ on $\partial\Omega$, then $T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]$ is an isomorphism on $L^p(\partial\Omega)$ by Theorems 5.10.4 and 5.11.5. Let now $|a| > 0$ on $\partial\Omega$. If $f \in L^p(\partial\Omega)$, then $u = \mathcal{S}^\Omega(I-P)f + Pf$ is an L^p-solution of the derivative oblique problem $\Delta u = 0$ in Ω, $\Theta \cdot \nabla u + \tilde{h}u = g$ on $\partial\Omega$ if $T_\Theta(I-P)f + \tilde{h}[\mathcal{S}^\Omega(I-P)f + Pf] = g$. If $g \in L^p(\partial\Omega)$, then there exists an L^p-solution u of this derivative oblique problem by Proposition 5.24.3. Denote $\tilde{g} = \partial u/\partial n + u$. Then u is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = \tilde{g}$ on $\partial\Omega$. According to Theorems 5.11.5 and 5.10.4, there exists $f \in L^p(\partial\Omega)$ such that $u = \mathcal{S}^\Omega(I-P)f + Pf$. So, $T_\Theta(I-P)f + \tilde{h}[\mathcal{S}^\Omega(I-P)f + Pf] = \Theta\cdot\nabla u + \tilde{h}u = g$ and $\{T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]\}(L^p(\partial\Omega)) = L^p(\partial\Omega))$. If $f \in \mathrm{Ker}\{T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]\}$ then $u = \mathcal{S}^\Omega(I-P)f + Pf$ is an L^p-solution of the derivative oblique problem $\Delta u = 0$ in Ω, $\Theta\cdot\nabla u + \tilde{h}u = 0$ on $\partial\Omega$. Proposition 5.24.3 gives that $u = 0$ in Ω. Therefore u is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = 0$ on $\partial\Omega$. Theorems 5.11.5 and 5.10.4 force $f \equiv 0$. Thus $T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$. Since $\{T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]\} - \{T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P) + P]\}$ is a compact operator on $L^p(\partial\Omega)$ (see Proposition 5.2.2 and Theorem 1.7.4), Theorem 1.8.3 gives that $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P) + P]$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$.

Suppose now that $\partial\Omega$ is not connected. Let $C_1, \dots, C_k$ be components of $R^m \setminus \Omega$. Denote $\Omega(j) = R^m \setminus C_j$ for $j = 1, \dots, k$. For $f \in L^p(\partial\Omega(j))$ denote

$$T_j f = \Theta \cdot [\nabla \mathcal{S}^{\Omega(j)} f]_{\Omega(j)}$$

in the sense of the nontangential limit with respect to $\Omega(j)$. We have proved that $f \mapsto T_j(f - \oint_{\partial\Omega(j)} f)$ is a Fredholm operator with index 0 on $L^p(\partial\Omega(j))$. The operator T_j is a Fredholm operator with index 0 on $L^p(\partial\Omega(j))$ by Theorem 1.8.3. Define $Tf = T_j(f|\partial\Omega(j))$ on $\partial\Omega(j)$. Then T is a Fredholm operator with index 0 on $L^p(\partial\Omega)$. Since $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P)+P] - T$ is a compact operator on $L^p(\partial\Omega)$, Theorem 1.8.3 gives that $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ is a Fredholm operator with index 0 on $L^p(\partial\Omega)$.

Theorem 5.24.5 *Let $\Omega \subset R^2$ be a bounded domain with Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \equiv 0$, a be a function that is constant on each component of $\partial\Omega$. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector. Define $\Theta = n^\Omega + a\tau$. Denote $Y = \{g \in L^p(\partial\Omega); \int_{\partial\Omega} g \, d\mathcal{H}_1 = 0\}$. Then $T_\Theta(L^p(\partial\Omega)) = Y$. Denote by T the restriction of T_Θ onto Y. Then T is an isomorphism. If $g \in L^p(\partial\Omega)$, then there exists an L^p-solution of the derivative oblique problem (5.84) if and only if $g \in Y$. Let now $g \in Y$. Then the general form of an L^p-solution of the derivative oblique problem (5.84) is $\mathcal{S}^\Omega(T^{-1}g) + c$, where c is an arbitrary constant. If u is an L^p-solution of the derivative oblique problem (5.84), then*

$$\|M_b(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)};$$

if moreover

$$\int_\Omega u \, d\mathcal{H}_2 = 0 \tag{5.86}$$

then

$$\|M_b(u)\|_{L^p(\partial\Omega)} + \|M_b(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}. \tag{5.87}$$

Here a constant C does not depend on g.

Proof If u is an L^p-solution of the derivative oblique problem (5.84), then $g \in Y$ by Proposition 5.23.9. We have $u \in \mathcal{C}^0(\overline{\Omega})$ by Lemma 5.24.2. If $g \equiv 0$, then there exists a constant c such that $u = c$ in Ω (see Proposition 5.23.7).

Let $f \in L^p(\partial\Omega)$. Then $u = \mathcal{S}^\Omega f$ is an L^p-solution of the derivative oblique problem (5.84) with $g = T_\theta f$. Thus $T_\theta f = g \in Y$. So, $T_\theta(L^p(\partial\Omega)) \subset Y$. Denote $Pf = \oint_{\partial\Omega} f \, d\mathcal{H}_1$. The operator $T_\Theta(I-P)$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ by Proposition 5.24.4. Since $T_\Theta(Y) \subset Y$ and $T_\theta(I-P) = T_\Theta$ on Y, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega f$ is an L^p-solution of the derivative oblique problem (5.84) with $g = 0$. So, there exists a constant c such that $\mathcal{S}^\Omega f = c$ in Ω. Hence $0 = \partial\mathcal{S}^\Omega f/\partial n = [(1/2)I - K'_\Omega]f$ (see Theorem 5.4.8). Since $f \in Y$, Theorems 5.9.10 and 5.11.3 give that $f \equiv 0$. Since T is a Fredholm operator with index 0 we have $T(Y) = Y$. Thus T is an isomorphism by Theorem 1.6.3.

Let $g \in Y$. If c is a constant, then $\mathcal{S}^\Omega(T^{-1}g) + c$ is an L^p-solution of the derivative oblique problem (5.84). Moreover, $u = \mathcal{S}^\Omega(T^{-1}g) - \oint_\Omega \mathcal{S}^\Omega(T^{-1}g)\, d\mathcal{H}_2$ is a unique L^p-solution of the derivative oblique problem (5.84) such that (5.86) holds. Since $f \mapsto \oint_\Omega \mathcal{S}^\Omega f\, d\mathcal{H}_2$ is a bounded linear functional on $L^p(\partial\Omega)$ (see Corollary 5.2.9 and Proposition 1.14.1), Propositions 5.2.2 and 5.2.8 give that there exists a constant C such that (5.87) holds. Let now v be another L^p-solution of the derivative oblique problem (5.84). Since $v - u$ is constant, we obtain $\|M_b(\nabla v)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}$.

Theorem 5.24.6 *Let $\Omega \subset R^2$ be an unbounded domain with compact Lipschitz boundary, $h \equiv 0$, a be a function that is constant on each component of $\partial\Omega$. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector. Define $\Theta = n^\Omega + a\tau$. Let $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Denote $Y = \{g \in L^p(\partial\Omega); \int_{\partial\Omega} g\, d\mathcal{H}_1 = 0\}$. Then $T_\Theta(L^p(\partial\Omega)) = Y$. Denote by T the restriction of T_Θ onto Y. Then T is an isomorphism. If $g \in L^p(\partial\Omega)$, then there exists an L^p-solution of the derivative oblique problem (5.84) if and only if $g \in Y$. If u is an L^p-solution of the derivative oblique problem (5.84), then $u \in \mathcal{C}^{0,0}(\overline{\Omega})$, and there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $g \in Y$, $u_\infty \in R$, then $u = \mathcal{S}^\Omega T^{-1}g + u_\infty$ is a unique L^p-solution of the derivative oblique problem (5.84) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_b(u)\|_{L^p(\partial\Omega)} + \|M_b(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)} + |u_\infty|, \tag{5.88}$$

where a constant C does not depend on g and u_∞.

Proof Let u be an L^p-solution of the derivative oblique problem (5.84). Then $g \in Y$ by Proposition 5.23.9. Since $M_b(u) \in L^p(\partial\Omega)$, u is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1, there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. We have $u \in \mathcal{C}^0(\overline{\Omega \cap B(0;r)})$ by Lemma 5.24.2. Hence $u \in \mathcal{C}^{0,0}(\overline{\Omega})$. If $g \equiv 0$, then u is constant by Proposition 5.23.7.

Let $f \in L^p(\partial\Omega)$. Then $u = \mathcal{S}^\Omega f$ is an L^p-solution of the derivative oblique problem (5.84) with $g = T_\theta f$. Thus $T_\theta f = g \in Y$. So, $T_\theta(L^p(\partial\Omega)) \subset Y$. Denote $Pf = \oint_{\partial\Omega} f\, d\mathcal{H}_1$. The operator $T_\Theta(I - P)$ is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ by Proposition 5.24.4. Since $T_\Theta(I - P)(Y) \subset Y$ and $T_\Theta(I - P) = T_\Theta$ on Y, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega f$ is an L^p-solution of the derivative oblique problem (5.84) with $g = 0$. So, there exists a constant c such that $\mathcal{S}^\Omega f = c$ in Ω. Hence $0 = \partial\mathcal{S}^\Omega f/\partial n = [(1/2)I - K'_\Omega]f$ (see Theorem 5.4.8). Since $f \in Y$, Theorems 5.9.11 and 5.11.4 give that $f \equiv 0$. Since T is a Fredholm operator with index 0, we have $T(Y) = Y$. Thus T is an isomorphism by Theorem 1.6.3.

Let $g \in Y$, $u_\infty \in R$. Then $\mathcal{S}^\Omega(T^{-1}g) + u_\infty$ is a unique L^p-solution of the derivative oblique problem (5.84) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Propositions 5.2.3 and 5.2.8 give that there exists a constant C such that (5.88) holds.

Theorem 5.24.7 *Let $\Omega \subset R^2$ be a domain with compact Lipschitz boundary, $1 < p < \infty$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $a \in \mathcal{C}^1(R^2)$.*

For each component Λ *of* $\partial\Omega$, *suppose that* $|a| > 0$ *on the whole* Λ *or* $a = 0$ *on the whole* Λ. *Denote* $\tau = (-n_2^\Omega, n_1^\Omega)$ *the unit tangential vector. Define* $\Theta = n^\Omega + a\tau$. *Let* $h \in L^\infty(\partial\Omega)$, $h \geq 0$. *Suppose that*

$$\int_{\partial\Omega} \left[|h| + \left| \frac{\partial a}{\partial \tau} \right| \right] \, \mathrm{d}\mathcal{H}_1 > 0,$$

i.e., $h \not\equiv 0$ *or a is not locally constant. Define*

$$Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1,$$

$$Tf = T_\Theta(I-P)f + \left[h + \frac{1}{2}\frac{\partial a}{\partial \tau} \right] [\mathcal{S}^\Omega(I-P)f + Pf]$$

Then T *is an isomorphism on* $L^p(\partial\Omega)$. *If* $g \in L^p(\partial\Omega)$, *then* $\mathcal{S}^\Omega(I-P)T^{-1}g + PT^{-1}g$ *is a unique* L^p*-solution of the derivative oblique problem (5.84). Moreover, the estimate (5.87) holds with a constant C that does not depend on g.*

Proof Let u be an L^p-solution of the derivative oblique problem (5.84) with $g \equiv 0$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. We have $u \in \mathcal{C}^0(\overline{\Omega \cap B(0;r)})$ by Lemma 5.24.2. Since $M_b(u) \in L^p(\partial\Omega)$, u is bounded in a neighborhood of infinity. Hence u is bounded. Proposition 5.23.7 gives that $u \equiv 0$.

Proposition 5.24.4 gives that T is a Fredholm operator with index 0 in $L^p(\partial\Omega)$. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega(f - \oint f) + \oint f$ is an L^p-solution of the derivative oblique problem (5.84) with $g = 0$. So, $u = 0$ in Ω. Therefore u is an L^p-solution of the Robin problem $-\Delta u = 0$ in Ω, $\partial u/\partial n + u = 0$ on $\partial\Omega$ and $[(1/2)I - K'_\Omega](I-P)f + \mathcal{S}^\Omega(I-P)f + Pf = 0$. Proposition 5.10.3 and Theorem 5.11.5 give that $f = 0$. Since T is a Fredholm operator with index 0, it is onto. Hence T is an isomorphism by Theorem 1.6.3.

If $g \in L^p(\partial\Omega)$, then $\mathcal{S}^\Omega(I-P)T^{-1}g + PT^{-1}g$ is a unique L^p-solution of the derivative oblique problem (5.84). The estimate (5.87) is consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Example 5.24.8 If $\Theta = n^\Omega + a\tau$, where $\tau = (-n_2^\Omega, n_1^\Omega)$ and $a \in L^\infty(\partial\Omega)$, we cannot expect an analogy of Theorem 5.24.5.

Denote $\Omega = \{[x_1, x_2] \in R^2; |x_1|, |x_2| \leq 1\}$, $\tau = (-n_2^\Omega, n_1^\Omega)$

$$a(x) = \begin{cases} -1 & |x_1| = 1, |x_2| < 1, \\ 1 & |x_2| = 1, |x_1| < 1. \end{cases}$$

$\Theta = n^\Omega + a\tau$. Let $u(x_1, x_2) = x_1 + x_2$. Then $\nabla u = (1,1)$, $\Theta \cdot \nabla u = n_1 + a(-n_2) + n_2 + an_1 = 0$. So, $u \in \mathcal{C}^\infty(R^2)$ is a nonconstant solution of the derivative oblique problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \Theta \cdot \nabla u = 0 \quad \text{on } \partial\Omega.$$

5.25 L^p-Solution of the Robin-Dirichlet Problem

In this section, we study L^p-solutions and very weak L^p-solutions of the mixed problem for the Laplace equation in a domain $\Omega \subset R^m$ with compact Lipschitz boundary. A Dirichlet condition is given on a part of the boundary, and a Robin condition (or a Neumann condition) is prescribed on the rest of the boundary. If $p \neq 2$, we suppose that Ω has Ljapunov boundary. We prove the existence of a solution and discuss the uniqueness.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $f \in L^p(\Gamma_R)$, $h \in L^\infty(\Gamma_R)$, $g \in W^{1,p}(\Gamma_D)$. *We say that u is an L^p-solution of the mixed Robin-Dirichlet problem for the Laplace equation*

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = f \text{ on } \Gamma_R. \tag{5.89}$$

if $u \in H(\Omega)$, $M_a(u) + M_a(\nabla u) \in L^p(\partial\Omega)$, there exist nontangential limits of u and ∇u at almost all points of $\partial\Omega$, and the boundary conditions are fulfilled in the sense of nontangential limits at almost all points of $\partial\Omega$.

Let now $1 < p < \infty$, $f \in W^{-1,p}(\Gamma_R)$, $h \in L^\infty(\Gamma_R)$, $g \in L^p(\Gamma_D)$. *We say that u is a very weak L^p-solution of the mixed Robin-Dirichlet problem for the Laplace equation (5.89) if $u \in H(\Omega)$, $M_a(u) \in L^p(\partial\Omega)$, there exists a nontangential limit of u at almost all points of $\partial\Omega$, $u = g$ in the sense of a nontangential limit at almost all points of Γ_D, and*

$$\int_\Omega u(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right) \mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi\rangle \quad \forall\varphi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D).$$

Lemma 5.25.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $f \in L^p(\Gamma_R)$, $h \in L^\infty(\Gamma_R)$, $g \in W^{1,p}(\Gamma_D)$. If u is an L^p-solution of the Robin-Dirichlet problem (5.89), then u is a very weak L^p-solution of the Robin-Dirichlet problem (5.89).*

Proof Let Ω be bounded. Then $u \in W^{1,p}(\Omega)$ by Lemma 1.28.1. The trace of u is equal to the nontangential limit of u. Fix $\varphi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D)$. According to Lemma 5.8.1 and the Gauss-Green formula for Sobolev spaces (Proposition 1.24.4),

$$\langle f, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\varphi + \int_{\Gamma_R} hu\varphi = \int_{\Gamma_R} u(-\Delta\varphi) + \int_{\Gamma_R} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right).$$

Let now Ω be unbounded. Fix $\varphi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D)$. Choose $r > 0$ such that $\partial\Omega \cup \operatorname{spt}\varphi \subset B(0; r)$. Put $\omega = \Omega \cap B(0; r)$, $h = 0$, and $f = \partial u/\partial n$ on $\partial B(0; r)$. Then u is an L^p-solution of the problem $\Delta u = 0$ in ω, $u = g$ on Γ_D, $\partial u/\partial n + hu = f$

on $\partial\omega \setminus \Gamma_D$. Thus

$$\begin{aligned}\langle f, \varphi\rangle &= \int\limits_{\omega} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\omega\setminus\Gamma_D} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1} \\ &= \int\limits_{\Omega} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\Gamma_R} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1}.\end{aligned}$$

Proposition 5.25.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$, $h \geq 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let u be a very weak L^p-solution of the Robin-Dirichlet problem (5.89) with $f \equiv 0$, $g \equiv 0$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof Choose a bounded open set Ω_D with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_D \subset \Omega_D$ and $\Gamma_R \cap \Omega_D = \emptyset$. Put $\omega = \Omega \cap \Omega_D$, $\tilde{g} = u$ on $\partial\omega$. Then u is an L^p-solution of the Dirichlet problem

$$\Delta u = 0 \text{ in } \omega, \quad u = \tilde{g} \text{ on } \partial\omega.$$

Since $\tilde{g} \in W^{1,p}(\partial\omega) \cap W^{1,2}(\partial\omega)$, Theorems 5.15.2, 5.15.3, 5.14.9, 5.14.10, and 5.6.1 give that there exists a nontangential limit of ∇u at almost all points of $\partial\omega \supset \Gamma_D$ and $M_a^\omega(\nabla u) \in L^p(\partial\omega) \cap L^2(\partial\omega)$.

Choose a bounded open set Ω_R with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset \Omega_R$ and $\Gamma_D \cap \Omega_R = \emptyset$. Put $\tilde{\omega} = \Omega \cap \Omega_R$, $h = 0$ on $\partial\Omega_R$, $\tilde{f} = 0$ on Γ_R, $\tilde{f} = \partial u/\partial n$ on $\partial\tilde{\omega} \setminus \Gamma_R$. Let $\varphi \in \mathcal{C}_c^\infty(R^m)$. We can choose $\tilde{\varphi} \in \mathcal{C}_c^\infty(R^m)$ such that $\tilde{\varphi} = \varphi$ on $\tilde{\omega}$ and $\tilde{\varphi} = 0$ on a neighborhood of Γ_D. According to the Gauss-Green theorem (Theorem 1.19.1),

$$\begin{aligned}&\int\limits_{\tilde{\omega}} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\tilde{\omega}} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} u(-\Delta\tilde{\varphi})\,\mathrm{d}\mathcal{H}_m \\ &+ \int\limits_{\Gamma_R} u\left(\frac{\partial\tilde{\varphi}}{\partial n} + h\tilde{\varphi}\right)\,\mathrm{d}\mathcal{H}_{m-1} - \int\limits_{\Omega\setminus\tilde{\omega}} u(-\Delta\tilde{\varphi})\,\mathrm{d}\mathcal{H}_m - \int\limits_{\partial(\Omega\setminus\tilde{\omega})} u\frac{\partial\tilde{\varphi}}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} \\ &= 0 - \int\limits_{\Omega\setminus\tilde{\omega}} \nabla u \cdot \nabla\tilde{\varphi}\,\mathrm{d}\mathcal{H}_m = \int\limits_{\partial\tilde{\omega}\setminus\Gamma_R} \tilde{\varphi}\frac{\partial u}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\tilde{\omega}} \varphi\tilde{f}\,\mathrm{d}\mathcal{H}_{m-1}.\end{aligned}$$

So, u is a very weak L^p-solution of the Robin problem

$$\Delta u = 0 \text{ in } \tilde{\omega}, \quad \frac{\partial u}{\partial n} + hu = \tilde{f} \text{ on } \partial\tilde{\omega}.$$

Since $\tilde{f} \in L^p(\partial\tilde{\omega}) \cap L^2(\partial\tilde{\omega})$, Theorems 5.20.6, 5.20.8, 5.9.10, 5.10.4, 5.11.3, 5.11.5 and Lemma 5.20.1 give that u is an L^p-solution and also L^2-solution of this problem.

If Ω is unbounded, then $u(x) = O(1)$ as $|x| \to \infty$, because $M_a(u) \in L^p(\partial\Omega)$. According to Proposition 2.12.1, we have $\nabla u(x) \to 0$ as $|x| \to \infty$.

We have proved that there exists a nontangential limit of ∇u at almost all points of $\partial\Omega$, $M_a^{\omega}(\nabla u) \in L^2(\partial\omega)$, $M_a^{\tilde{\omega}}(\nabla u) \in L^2(\partial\tilde{\omega})$, and ∇u is bounded outside a neighborhood of $\partial\Omega$. These force $M_a^{\Omega}(\nabla u) \in L^2(\partial\Omega)$. So, u is an L^2-solution of some Robin-Dirichlet problem. Lemma 5.25.1 gives that u is an L^2-solution of (5.89). According to Lemma 5.8.1,

$$0 = \int\limits_{\partial\Omega} u\left(\frac{\partial u}{\partial n} + hu\right) \, d\mathcal{H}_{m-1} = \int\limits_{\Omega} |\nabla u|^2 \, d\mathcal{H}_m + \int\limits_{\partial\Omega} h|u|^2 \, d\mathcal{H}_{m-1}.$$

Thus $\nabla u \equiv 0$. Since Ω is connected, there exists a constant c such that $u \equiv c$. Since $c = u = 0$ on Γ_D, we infer that $u \equiv 0$.

Proposition 5.25.3 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$, $m > 2$. Let $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $\varphi \in W^{-1,p}(\partial\Omega)$, define*

$$T\varphi(x) = \begin{cases} \mathcal{S}^{\Omega}\varphi(x), & x \in \Gamma_D, \\ \frac{1}{2}\varphi(x) - K'_{\Omega}\varphi(x) + h\mathcal{S}^{\Omega}\varphi(x), & x \in \Gamma_R. \end{cases} \tag{5.90}$$

The operators

$$T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R),$$

$$T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$$

are bounded Fredholm operators with index 0. If $\varphi \in W^{-1,p}(\partial\Omega)$, then $\mathcal{S}^{\Omega}\varphi$ is a very weak L^p-solution of (5.89) if and only if $T\varphi = [g,f]$. If $\varphi \in L^p(\partial\Omega)$, then $\mathcal{S}^{\Omega}\varphi$ is an L^p-solution of (5.89) if and only if $T\varphi = [g,f]$.

Proof $\frac{1}{2}I - K'_{\Omega} + h\mathcal{S}^{\Omega}$ is a bounded Fredholm operator with index 0 on $L^p(\partial\Omega)$ and on $W^{-1,p}(\partial\Omega)$ by Propositions 5.11.1, 5.9.9, 5.2.2, Theorems 1.8.3, 1.7.4, and Proposition 5.20.5. Denote

$$T_1\varphi = \chi_{\Gamma_R}\left[\frac{1}{2}I - K'_{\Omega} + h\mathcal{S}^{\Omega}\right](\varphi\chi_{\Gamma_R}),$$

$$T_2\varphi = \chi_{\Gamma_D}\left[\frac{1}{2}I - K'_{\Omega} + h\mathcal{S}^{\Omega}\right](\varphi\chi_{\Gamma_D}).$$

Since $\frac{1}{2}I - K'_{\Omega} + h\mathcal{S}^{\Omega} - (T_1 + T_2)$ is a compact operator, the operators $T_1 + T_2 : L^p(\partial\Omega) \to L^p(\partial\Omega)$, $T_1 + T_2 : W^{-1,p}(\partial\Omega) \to W^{-1,p}(\partial\Omega)$ are Fredholm operators with index 0 by Theorem 1.8.3. Hence $T_1 : L^p(\Gamma_R) \to L^p(\Gamma_R)$, $T_1 : W^{-1,p}(\Gamma_R) \to W^{-1,p}(\Gamma_R)$ are Fredholm operators with index 0.

$\mathcal{S}^\Omega : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms by Proposition 5.18.1. Denote

$$T_3\varphi = \chi_{\Gamma_R}\mathcal{S}^\Omega(\varphi\chi_{\Gamma_R}),$$
$$T_4\varphi = \chi_{\Gamma_D}\mathcal{S}^\Omega(\varphi\chi_{\Gamma_D}).$$

Since $\mathcal{S}^\Omega - (T_3 + T_4)$ is a compact operator, the operators $T_3 + T_4 : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $T_3 + T_4 : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are Fredholm operators with index 0 by Theorem 1.8.3. Hence $T_4 : L^p(\Gamma_D) \to W^{1,p}(\Gamma_D)$, $T_4 : W^{-1,p}(\Gamma_D) \to L^p(\Gamma_D)$ are Fredholm operators with index 0. Clearly, $T_1 + T_4 : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$, $T_1 + T_4 : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ are Fredholm operators with index 0. Since $T - (T_1 + T_4)$ is a compact operator, the operators $T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$, $T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ are Fredholm operators with index 0 by Theorem 1.8.3.

Let $\varphi \in L^p(\partial\Omega)$. Then $\mathcal{S}^\Omega\varphi$ is an L^p-solution of (5.89) with $[g,f] = T\varphi$ by Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.

Fix $\varphi \in W^{-1,p}(\partial\Omega)$. Choose $\varphi_k \in L^p(\partial\Omega)$ such that $\varphi_k \to \varphi$ in $W^{-1,p}(\partial\Omega)$ as $k \to \infty$. Denote $[g,f] = T\varphi$, $[g_k,f_k] = T\varphi_k$. Fix $\psi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D)$. According to Lemma 5.25.1, Corollary 5.4.11, Proposition 5.2.2, Lemma 5.20.3, and Proposition 5.4.5,

$$\int_\Omega (-\Delta\psi)\mathcal{S}^\Omega\varphi \, d\mathcal{H}_m + \int_{\partial\Omega} \left(\frac{\partial\psi}{\partial n} + h\psi\right)\mathcal{S}^\Omega\varphi \, d\mathcal{H}_{m-1}$$
$$= \lim_{k\to\infty}\left[\int_\Omega (-\Delta\psi)\mathcal{S}^\Omega\varphi_k + \int_{\partial\Omega}\left(\frac{\partial\psi}{\partial n} + h\psi\right)\mathcal{S}^\Omega\varphi_k\right] = \lim_{k\to\infty}\langle f_k, \psi\rangle = \langle f, \psi\rangle.$$

Corollary 5.18.4 gives that $\mathcal{S}^\Omega\varphi$ is a very weak L^p-solution of the mixed problem (5.89).

Proposition 5.25.4 *Let $\Omega \subset R^2$ be a domain with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $\varphi \in W^{-1,p}(\partial\Omega)$, define*

$$P\varphi = \langle\varphi, 1\rangle/\mathcal{H}_1(\partial\Omega),$$
$$\tilde{T}\varphi = \begin{cases} \mathcal{S}^\Omega(I-P)\varphi + P\varphi & \text{on } \Gamma_D, \\ (1/2)(I-P)\varphi - K'_\Omega(I-P)\varphi + h[\mathcal{S}^\Omega(I-P)\varphi + P\varphi] & \text{on } \Gamma_R. \end{cases} \tag{5.91}$$

The operators

$$\tilde{T} : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R),$$

$$\tilde{T} : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$$

are bounded Fredholm operators with index 0*. If* $\varphi \in W^{-1,p}(\partial\Omega)$*, then* $\mathcal{S}^\Omega(I-P)\varphi + P\varphi$ *is a very weak* L^p*-solution of* (5.89) *if and only if* $\tilde{T}\varphi = [g,f]$*. If* $\varphi \in L^p(\partial\Omega)$*, then* $\mathcal{S}^\Omega(I-P)\varphi + P\varphi$ *is an* L^p*-solution of* (5.89) *if and only if* $\tilde{T}\varphi = [g,f]$*.*

Proof $[\frac{1}{2}I - K'_\Omega](I-P) + h[\mathcal{S}^\Omega(I-P)+P]$ is a bounded Fredholm operator with index 0 on $L^p(\partial\Omega)$ and on $W^{-1,p}(\partial\Omega)$ by Propositions 5.11.1, 5.9.9, 5.2.2, Theorems 1.8.3, 1.7.4, and Proposition 5.20.5. Denote

$$T_1\varphi = \chi_{\Gamma_R}\left\{\left[\frac{1}{2}I - K'_\Omega\right](I-P) + h[\mathcal{S}^\Omega(I-P)+P]\right\}(\varphi\chi_{\Gamma_R}),$$

$$T_2\varphi = \chi_{\Gamma_D}\left\{\left[\frac{1}{2}I - K'_\Omega\right](I-P) + h[\mathcal{S}^\Omega(I-P)+P]\right\}(\varphi\chi_{\Gamma_D}).$$

Since $[\frac{1}{2}I - K'_\Omega](I-P) + h[\mathcal{S}^\Omega(I-P)+P] - (T_1+T_2)$ is a compact operator, the operators $T_1 + T_2 : L^p(\partial\Omega) \to L^p(\partial\Omega)$, $T_1 + T_2 : W^{-1,p}(\partial\Omega) \to W^{-1,p}(\partial\Omega)$ are Fredholm operators with index 0 by Theorem 1.8.3. Hence $T_1 : L^p(\Gamma_R) \to L^p(\Gamma_R)$, $T_1 : W^{-1,p}(\Gamma_R) \to W^{-1,p}(\Gamma_R)$ are Fredholm operators with index 0.

$\mathcal{S}^\Omega(I-P) + P : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $\mathcal{S}^\Omega(I-P) + P : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are isomorphisms by Proposition 5.18.1. Denote

$$T_3\varphi = \chi_{\Gamma_R}[\mathcal{S}^\Omega(I-P)+P](\varphi\chi_{\Gamma_R}),$$

$$T_4\varphi = \chi_{\Gamma_D}[\mathcal{S}^\Omega(I-P)+P](\varphi\chi_{\Gamma_D}).$$

Since $\mathcal{S}^\Omega(I-P) + P - (T_3+T_4)$ is a compact operator, the operators $T_3 + T_4 : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$, $T_3 + T_4 : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$ are Fredholm operators with index 0 by Theorem 1.8.3. Hence $T_4 : L^p(\Gamma_D) \to W^{1,p}(\Gamma_D)$, $T_4 : W^{-1,p}(\Gamma_D) \to L^p(\Gamma_D)$ are Fredholm operators with index 0. Clearly, $T_1 + T_4 : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$, $T_1 + T_4 : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ are Fredholm operators with index 0. Since $\tilde{T} - (T_1 + T_4)$ is a compact operator, the operators $\tilde{T} : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$, $\tilde{T} : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ are Fredholm operators with index 0 by Theorem 1.8.3.

Let $\varphi \in L^p(\partial\Omega)$. Then $\mathcal{S}^\Omega(I-P)\varphi + P\varphi$ is an L^p-solution of (5.89) with $[g,f] = \tilde{T}\varphi$ by Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.

Fix $\varphi \in W^{-1,p}(\partial\Omega)$. Choose $\varphi_k \in L^p(\partial\Omega)$ such that $\varphi_k \to \varphi$ in $W^{-1,p}(\partial\Omega)$ as $k \to \infty$. Denote $[g,f] = \tilde{T}\varphi$, $[g_k,f_k] = \tilde{T}\varphi_k$. Fix $\psi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D)$. According to

Lemma 5.25.1, Corollary 5.4.11, Lemma 5.20.3, Propositions 5.4.5 and 5.2.2,

$$\int_{\Omega} (-\Delta\psi)[\mathcal{S}^{\Omega}(I-P)\varphi + P\varphi]\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \left(\frac{\partial\psi}{\partial n} + h\psi\right)[\mathcal{S}^{\Omega}(I-P)\varphi + P\varphi]\,\mathrm{d}\mathcal{H}_{m-1}$$

$$= \lim_{k\to\infty} \left\{ \int_{\Omega} (-\Delta\psi)[\mathcal{S}^{\Omega}(I-P)\varphi_k + P\varphi_k]\,\mathrm{d}\mathcal{H}_m \right.$$

$$\left. + \int_{\partial\Omega} \left(\frac{\partial\psi}{\partial n} + h\psi\right)[\mathcal{S}^{\Omega}(I-P)\varphi_k + P\varphi_k]\,\mathrm{d}\mathcal{H}_{m-1} \right\} = \lim_{k\to\infty} \langle f_k, \psi\rangle = \langle f, \psi\rangle.$$

Corollary 5.18.4 gives that $\mathcal{S}^{\Omega}\varphi$ is a very weak L^p-solution of the mixed problem (5.89).

Theorem 5.25.5 *Let $\Omega \subset R^2$ be a domain with compact Lipschitz boundary, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $h \in L^{\infty}(\Gamma_R)$, $h \geq 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $\varphi \in W^{-1,p}(\partial\Omega)$, define $P\varphi = \langle\varphi, 1\rangle/\mathcal{H}_1(\partial\Omega)$ and $\tilde{T}$ by (5.91). Then*

$$\tilde{T} : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R),$$

$$\tilde{T} : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$$

are isomorphisms. Fix $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$. Put

$$\varphi = \tilde{T}^{-1}[g,f], \qquad u = \mathcal{S}^{\Omega}(I-P)\varphi + P\varphi.$$

Then u is a unique very weak L^p-solution of (5.89) and

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C\left[\|g\|_{L^p(\Gamma_D)} + \|f\|_{W^{-1,p}(\Gamma_R)}\right] \tag{5.92}$$

where a constant C does not depend on g and f. If moreover $g \in W^{1,p}(\partial\Omega)$, $f \in L^p(\partial\Omega)$, then u is a unique L^p-solution of the mixed problem (5.89) and

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \leq C\left[\|g\|_{W^{1,p}(\Gamma_D)} + \|f\|_{L^p(\Gamma_R)}\right]. \tag{5.93}$$

Proof Lemma 5.25.1 and Proposition 5.25.2 give uniqueness of a very weak L^p-solution and an L^p-solution of the problem (5.89).

Let $\varphi \in W^{-1,p}(\partial\Omega)$, $\tilde{T}\varphi = 0$. Then $\mathcal{S}^{\Omega}(I-P)\varphi + P\varphi$ is a very weak L^p-solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on Γ_D, $\partial u/\partial n + hu = 0$ on Γ_R. (See Proposition 5.25.4.) Hence $\mathcal{S}^{\Omega}(I-P)\varphi + P\varphi = 0$ in Ω. The function $\mathcal{S}^{\Omega}(I-P)\varphi + P\varphi$ is a very weak L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = 0$ on $\partial\Omega$. Theorem 5.20.8 forces that $\varphi = 0$. Since

$\tilde{T} : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is a Fredholm operator with index 0 by Proposition 5.25.4, it is onto. Thus $\tilde{T} : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is an isomorphism by Theorem 1.6.3. $\tilde{T} : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ is a Fredholm operators with index 0 by Proposition 5.25.4. Lemma 1.8.4 forces that $\tilde{T} : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ is an isomorphism.

Fix $g \in L^p(\partial\Omega), f \in W^{-1,p}(\partial\Omega)$. Put $\varphi = \tilde{T}^{-1}[g,f]$, $u = \mathcal{S}^\Omega(I-P)\varphi + P\varphi$. Then u is a very weak L^p-solution of (5.89) by Proposition 5.25.4. Since P is a bounded linear functional on $W^{-1,p}(\partial\Omega)$, Corollary 5.18.4 gives that there exists a constant C such that (5.92) holds. If $g \in W^{1,p}(\partial\Omega), f \in L^p(\partial\Omega)$, then $\varphi \in L^p(\partial\Omega)$ and u is an L^p-solution of the mixed problem (5.89) by Proposition 5.25.4. The estimate (5.93) is a consequence of Proposition 5.2.8.

Theorem 5.25.6 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m > 2$, and Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$, $h \geq 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $\varphi \in W^{-1,p}(\partial\Omega)$, define $T\varphi$ by (5.90). Then*

$$T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R),$$

$$T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$$

are isomorphisms. Fix $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$. Then $u = \mathcal{S}^\Omega T^{-1}[g,f]$ is a unique very weak L^p-solution of (5.89) and the estimate (5.92) holds, where a constant C does not depend on g and f. If moreover $g \in W^{1,p}(\partial\Omega)$, $f \in L^p(\partial\Omega)$, then u is a unique L^p-solution of the mixed problem (5.89) and the estimate (5.93) holds.

Proof Lemma 5.25.1 and Proposition 5.25.2 give uniqueness of a very weak L^p-solution and an L^p-solution of the problem (5.89).

Let $\varphi \in W^{-1,p}(\partial\Omega)$, $T\varphi = 0$. Then $\mathcal{S}^\Omega\varphi$ is a very weak L^p-solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on Γ_D, $\partial u/\partial n + hu = 0$ on Γ_R. (See Proposition 5.25.3.) Hence $\mathcal{S}^\Omega\varphi = 0$ in Ω. The function $\mathcal{S}^\Omega\varphi$ is a very weak L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = 0$ on $\partial\Omega$. Theorem 5.20.8 forces that $\varphi = 0$. Since $T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is a Fredholm operator with index 0 by Proposition 5.25.3, it is onto. Thus $T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is an isomorphism by Theorem 1.6.3. The operator $T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ is Fredholm with index 0 by Proposition 5.25.3. Lemma 1.8.4 forces that $T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ is an isomorphism.

Fix $g \in L^p(\partial\Omega), f \in W^{-1,p}(\partial\Omega)$. Put $u = \mathcal{S}^\Omega T^{-1}[g,f]$. Then u is a very weak L^p-solution of (5.89) by Proposition 5.25.3. Corollary 5.18.4 gives that there exists a constant C such that (5.92) holds. If $g \in W^{1,p}(\partial\Omega), f \in L^p(\partial\Omega)$, then u is an L^p-solution of the mixed problem (5.89) by Proposition 5.25.3. The estimate (5.93) is a consequence of Proposition 5.2.8.

Theorem 5.25.7 *Let $\Omega \subset R^m$ be an unbounded domain with Lipschitz boundary, $m > 2$, and Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$, $h \geq 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $\varphi \in W^{-1,p}(\partial\Omega)$, define $T\varphi$ by (5.90).*

- *The following operators are isomorphisms:*

$$T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R),$$

$$T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R).$$

- *Fix $g \in L^p(\partial\Omega)$, $f \in W^{-1,p}(\partial\Omega)$. If u is a very weak L^p-solution of the mixed problem (5.89), then there exists u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then*

$$u = \mathcal{S}^\Omega T^{-1}[g - u_\infty, f - hu_\infty] + u_\infty \tag{5.94}$$

is a unique very weak L^p-solution of (5.89) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,

$$\|M_a(u)\|_{L^p(\partial\Omega)} \leq C\left(\|g\|_{L^p(\Gamma_D)} + \|f\|_{W^{-1,p}(\Gamma_R)} + |u_\infty|\right) \tag{5.95}$$

where a constant C does not depend on g, f, and u_∞.

- *Fix $g \in W^{1,p}(\partial\Omega)$, $f \in L^p(\partial\Omega)$. If u is an L^p-solution of the mixed problem (5.89), then there exists u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in R^1$ is given, then u given by (5.94) is a unique L^p-solution of (5.89) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \tag{5.96}$$
$$\leq C\left(\|g\|_{W^{1,p}(\Gamma_D)} + \|f\|_{L^p(\Gamma_R)} + |u_\infty|\right)$$

where a constant C does not depend on g, f, and u_∞.

Proof If $g = 0, f = 0$ and u is a very weak L^p-solution or an L^p-solution of the problem (5.89) such that $u(x) \to 0$ as $|x| \to \infty$, then $u \equiv 0$. (See Lemma 5.25.1 and Proposition 5.25.2.)

Let $\varphi \in W^{-1,p}(\partial\Omega)$, $T\varphi = 0$. Then $\mathcal{S}^\Omega\varphi$ is a very weak L^p-solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on Γ_D, $\partial u/\partial n + hu = 0$ on Γ_R. (See Proposition 5.25.3.) Moreover, $u(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. Hence $\mathcal{S}^\Omega\varphi = 0$ in Ω. The function $\mathcal{S}^\Omega\varphi$ is a very weak L^p-solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = 0$ on $\partial\Omega$. Theorem 5.20.8 forces that $\varphi = 0$. Since $T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is a Fredholm operator with index 0 by Proposition 5.25.3, it is onto. Thus $T : W^{-1,p}(\partial\Omega) \to L^p(\Gamma_D) \times W^{-1,p}(\Gamma_R)$ is an isomorphism by Theorem 1.6.3. The operator $T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$

is Fredholm with index 0 by Proposition 5.25.3. Lemma 1.8.4 forces that $T : L^p(\partial\Omega) \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R)$ is an isomorphism.

If u is a very weak L^p-solution or an L^p-solution of the mixed problem (5.89), then $u(x) = O(1)$ as $|x| \to \infty$, because $M_a(u) \in L^p(\partial\Omega)$. (See Lemma 1.28.2.) According to Proposition 2.17.1, there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$.

Fix $g \in L^p(\partial\Omega), f \in W^{-1,p}(\partial\Omega)$, $u_\infty \in R^1$. Then u given by (5.94) is a very weak L^p-solution of (5.89) by Proposition 5.25.3. Moreover, $u(x) \to u_\infty$ as $|x| \to \infty$ by Proposition 3.1.6. Corollary 5.18.4 gives that there exists a constant C such that (5.95) holds. If $g \in W^{1,p}(\partial\Omega), f \in L^p(\partial\Omega)$, then u is an L^p-solution of the mixed problem (5.89) by Proposition 5.25.3. The estimate (5.96) is a consequence of Proposition 5.2.8.

5.26 L^p-Solutions of the Transmission Problem for Several Domains

In this section, we study L^p-solutions of the transmission problem on the Euclidean space divided into several open sets. We prove the existence of a solution and discuss the uniqueness.

Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}, j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i, j\}$. We denote by n^j the unit exterior normal of Ω_j. So, $n^i = -n^j$ on $\partial\Omega_i \cap \partial\Omega_j$ for $i \neq j$.

Let $b_1, \dots, b_k$ be positive numbers, $1 < p < \infty$. For $i < j$, let $f_{ij} \in L^p(\partial\Omega_i \cap \partial\Omega_j)$, $g_{ij} \in W^{1,p}(\partial\Omega_i \cap \partial\Omega_j)$. *We say that* $u_1, \dots, u_k$ *is an* L^p*-solution of the transmission problem*

$$\Delta u_j = 0 \quad \text{in } \Omega_j, \tag{5.97}$$

$$u_i - u_j = g_{ij}, \qquad b_i \frac{\partial u_i}{\partial n^i} - b_j \frac{\partial u_j}{\partial n^i} = f_{ij} \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, i < j \tag{5.98}$$

if $u_j \in H(\Omega_j)$, $M_a^{\Omega_j}(u_j) + M_a^{\Omega_j}(\nabla u_j) \in L^p(\partial\Omega_j)$, *there exist nontangential limits of* u_j *and* ∇u_j *at almost all points of* $\partial\Omega_j$, *and the conditions (5.98) are fulfilled in the sense of a nontangential limit almost everywhere.*

Lemma 5.26.1 *Let* $\Omega_1, \dots, \Omega_k$ *be disjoint open sets with compact Lipschitz boundary such that* $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$, $m > 2$. *Suppose that for each component* B *of* $\partial\Omega_i$, *there exists* $j \in \{1, \dots, k\}$, $j \neq i$, *such that* $B \subset \partial\Omega_i \cap \partial\Omega_j$ *and* $B \cap \overline{\Omega}_l = \emptyset$ *for* $l \notin \{i, j\}$. *Let* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$ *for* $j = 1, \dots, k$. *If* $(u_1, \dots, u_k)$ *is an* L^p*-solution of the transmission problem (5.97), (5.98), then there exist unique* $\varphi_j \in L^p(\partial\Omega_j)$, $j = 1, \dots, k$, *and* $u_\infty \in R^1$ *such that* $u_j = \mathcal{S}^{\Omega_j}\varphi_j + u_\infty$ *for* $j = 1, \dots, k$.

Proof Suppose first that Ω_i is unbounded. Then u_i is an L^p-solution of some Neumann problem for the Laplace equation in Ω_j. According to Theorems 5.11.4 and 5.9.11, there exist unique $\varphi_i \in L^p(\partial\Omega_i)$ and $u_\infty \in R^1$ such that $u_i = \mathcal{S}^{\Omega_i}\varphi_i + u_\infty$.

Let now Ω_j be bounded. Define $f = \partial u_j/\partial n_j + u_j - u_\infty$ on $\partial\Omega_j$. Then $u_j - u_\infty$ is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω_j, $\partial u/\partial n + u = f$ on $\partial\Omega_j$. According to Theorems 5.11.5 and 5.10.4, there exists a unique $\varphi_j \in L^p(\partial\Omega_j)$ such that $u_j - u_\infty = \mathcal{S}^{\Omega_j}\varphi_j$.

Lemma 5.26.2 *Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^2 = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}$, $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i, j\}$. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. If $(u_1, \dots, u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98), then there exist unique $\varphi_j \in L^p(\partial\Omega_j)$, $j = 1, \dots, k$, such that*

$$u_j = \mathcal{S}^{\Omega_j}(\varphi_j - \oint_{\partial\Omega_j} \varphi_j \, \mathrm{d}\mathcal{H}_1) + \oint_{\partial\Omega_j} \varphi_j \, \mathrm{d}\mathcal{H}_1, \qquad j = 1, \dots, k. \tag{5.99}$$

Proof Define $f_j = \partial u_j/\partial n_j + u_j$ on $\partial\Omega_j$. Then u_j is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω_j, $\partial u/\partial n + u = f_j$ on $\partial\Omega_j$. According to Theorems 5.11.5 and 5.10.4, there exists a unique $\varphi_j \in L^p(\partial\Omega_j)$ such that (5.99) holds.

Proposition 5.26.3 *Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$, $m > 2$. Suppose that for each component B of $\partial\Omega_i$ there exists $j \in \{1, \dots, k\}$, $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i, j\}$. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Denote*

$$X_p = L^p(\partial\Omega_1) \times L^p(\partial\Omega_2) \times \dots \times L^p(\partial\Omega_k),$$

$$Y_p = \prod_{1 \leq i < j \leq k} [W^{1,p}(\partial\Omega_i \cap \partial\Omega_j) \times L^p(\partial\Omega_i \cap \partial\Omega_j)]$$

Let $b_1, \dots, b_k$ be positive numbers. For $[\varphi_1, \dots, \varphi_k] \in X_p$, define

$$T[\varphi_1, \dots, \varphi_k] = [\mathcal{S}^{\Omega_i}\varphi_i - \mathcal{S}^{\Omega_j}\varphi_j, \frac{b_i}{2}\varphi_i - b_i K'_{\Omega_i}\varphi_i + \frac{b_j}{2}\varphi_j - b_j K'_{\Omega_j}\varphi_j]$$

on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \leq i < j \leq k$. Then $T : X_p \to Y_p$ is a Fredholm operator with index 0. If $[\varphi_1, \dots, \varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\varphi_j$, then $(u_1, \dots, u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) if and only if $T[\varphi_1, \dots, \varphi_k] = [g_{ij}, f_{ij}]$ on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \leq i < j \leq k$.

Proof If $[\varphi_1, \dots, \varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\varphi_j$, then $(u_1, \dots, u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) if and only if $T[\varphi_1, \dots, \varphi_k] = [g_{ij}, f_{ij}]$ on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \leq i < j \leq k$. (See Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.)

$T : X_p \to Y_p$ is a bounded linear operator by Proposition 5.4.5 and Lemma 5.4.9. We now show that T is a Fredholm operator with index 0. Suppose first that $k = 2$. We can suppose that Ω_1 is unbounded. If $[g_{12}, f_{12}] \in Y_p$, then there exists a unique L^p-solution of the transmission problem (5.97), (5.98) such that $u_1(x) \to 0$ as $|x| \to \infty$. (See Theorem 5.12.7.) According to Lemma 5.26.1, there exist unique $\varphi_1, \varphi_2 \in L^p(\partial\Omega_1)$ and $u_\infty \in R^1$ such that $u_j = \mathcal{S}_j^{\Omega}\varphi_j + u_\infty$. Since $u_1(x) \to 0$ as $|x| \to \infty$, we infer that $u_\infty = 0$. We have proved that $T[\varphi_1, \varphi_2] = [g_{12}, f_{12}]$. So, T is onto. The uniqueness of the representation of u_1, u_2 forces that T is one-to-one. So, $T : X_p \to Y_p$ is an isomorphism.

Let now $k > 2$. We define an auxiliary operator $\tilde{T}$. If B is a component of $\partial(\Omega_1 \cup \cdots \cup \Omega_k)$, we define

$$\tilde{T}[\varphi_1, \dots, \varphi_k] = T[\chi_B\varphi_1, \dots, \chi_B\varphi_k] \quad \text{on } B.$$

We show that $\tilde{T} : X_p \to Y_p$ is an isomorphism. Let $i < j$ and B be a component of $\partial\Omega_i \cap \partial\Omega_j$. Then there exist disjoint open sets U_1, U_2 such that $R^m = \overline{U_1} \cup \overline{U_2}$, $B = \partial U_1 = \partial U_2$, $\Omega_i \subset U_1$, $\Omega_j \subset U_2$. Define

$$T_B[\psi_1, \psi_2] = [\mathcal{S}^{U_1}\psi_1 - \mathcal{S}^{U_2}\psi_2, \frac{b_i}{2}\psi_1 - b_iK'_{U_1}\psi_1 + \frac{b_j}{2}\psi_2 - b_jK'_{U_2}\psi_2].$$

We have proved that $T_B : L^p(B) \times L^p(B) \to W^{1,p}(B) \times L^p(B)$ is an isomorphism. Since $\tilde{T}[\varphi_1, \dots, \varphi_k] = T_B[\chi_B\varphi_i, \chi_B\varphi_j]$ on B, we deduce that $\tilde{T} : X_p \to Y_p$ is an isomorphism. Since $T - \tilde{T} : X_p \to \mathcal{C}^\infty(\partial(\Omega_1 \cup \cdots \cup \Omega_k))$ is bounded, we infer that $T - \tilde{T} : X_p \to Y_p$ is compact. (See Lemma 1.21.8.) Theorem 1.8.3 gives that $T : X_p \to Y_p$ is a Fredholm operator with index 0.

Proposition 5.26.4 *Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^2 = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \cdots \cup \overline{\Omega_k}$, $k \geq 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}$, $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega_l} = \emptyset$ for $l \notin \{i, j\}$. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let X_p, Y_p be defined like in Proposition 5.26.3. Let $b_1, \dots, b_k$ be positive numbers. For $[\varphi_1, \dots, \varphi_k] \in X_p$, define $P_j\varphi_j = \oint_{\partial\Omega_j} \varphi_j \, d\mathcal{H}_1$, $\tilde{P}_j\varphi_j = \varphi_j - P_j\varphi_j$, and*

$$G_{ij}[\varphi_i, \varphi_j] = \mathcal{S}^{\Omega_i}\tilde{P}_i\varphi_i + P_i\varphi_i - \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j - P_j\varphi_j,$$

$$F_{ij}[\varphi_i, \varphi_j] = \frac{b_i}{2}\tilde{P}_i\varphi_i - b_iK'_{\Omega_i}\tilde{P}_i\varphi_i + \frac{b_j}{2}\tilde{P}_j\varphi_j - b_jK'_{\Omega_j}\tilde{P}_j\varphi_j,$$

$$T[\varphi_1, \dots, \varphi_k] = [G_{ij}[\varphi_i, \varphi_j], F_{ij}[\varphi_i, \varphi_j]]$$

on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \leq i < j \leq k$. Then $T : X_p \to Y_p$ is a Fredholm operator with index 0. If $[\varphi_1, \dots, \varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j + P_j\varphi_j$, then $(u_1, \dots, u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) if and only if $T[\varphi_1, \dots, \varphi_k] = [g_{ij}, f_{ij}]$ on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \leq i < j \leq k$.

Proof If $[\varphi_1,\dots,\varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j + P_j\varphi_j$, then $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) if and only if $T[\varphi_1,\dots,\varphi_k] = [g_{ij},f_{ij}]$ on $\partial\Omega_j \cap \partial\Omega_i$ for $1 \le i < j \le k$. (See Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.)

$T : X_p \to Y_p$ is a bounded operator by Proposition 5.4.5, Lemma 5.4.9, and Proposition 1.5.11. We now show that T is a Fredholm operator with index 0. Suppose first that $k = 2$. We can suppose that Ω_1 is unbounded. If $[g_{12},f_{12}] \in Y_p$, then there exists an L^p-solution of the transmission problem (5.97), (5.98) if and only if

$$\int\limits_{\partial\Omega_1} f_{12}\,\mathrm{d}\mathcal{H}_1 = 0.$$

(See Theorem 5.12.7.) If (u_1,u_2) is a solution of this problem, then there exist $\varphi_1,\varphi_2 \in L^p(\partial\Omega_1)$ such that $u_j = \mathcal{S}_j^{\Omega}\tilde{P}_j\varphi_j + P_j\varphi_j$. (See Lemma 5.26.2.) Hence $T[\varphi_1,\varphi_2] = [g_{12},f_{12}]$. Thus $T(X_p)$ is a closed subspace of Y_p of codimension 1. Let now $T[\varphi_1,\varphi_2] = 0$. Put $u_j = \mathcal{S}_j^{\Omega}\tilde{P}_j\varphi_j + P_j\varphi_j$. We have proved that (u_1,u_2) is an L^p-solution of the problem (5.97), (5.98) with $g_{12} = 0, f_{12} = 0$. According to Theorem 5.12.7, there exists a constant c such that $u_j \equiv c$ in Ω_j. According to Lemma 5.26.2, there exist unique $\tilde{\varphi}_j \in L^p(\partial\Omega_j)$ such that $\mathcal{S}_j^{\Omega}\tilde{P}_j\tilde{\varphi}_j + P_j\tilde{\varphi}_j = 1$ in Ω_j and $(\varphi_1,\varphi_2) = c(\tilde{\varphi}_1,\tilde{\varphi}_2)$. So, $\dim\operatorname{Ker} T = 1$ and $T : X_p \to Y_p$ is a Fredholm operator with index 0.

Let now $k > 2$. We define an auxiliary operator $\tilde{T}$. If B is a component of $\partial(\Omega_1 \cup \dots \cup \Omega_k)$, we define

$$\tilde{T}[\varphi_1,\dots,\varphi_k] = T[\chi_B\varphi_1,\dots,\chi_B\varphi_k] \quad \text{on } B.$$

We show that $\tilde{T} : X_p \to Y_p$ is a Fredholm operator with index 0. Let $i < j$ and B be a component of $\partial\Omega_i \cap \partial\Omega_j$. Then there exist disjoint open sets U_1, U_2 such that $R^2 = \overline{U}_1 \cup \overline{U}_2$, $B = \partial U_1 = \partial U_2$, $\Omega_i \subset U_1$, $\Omega_j \subset U_2$. Define $P\varphi = \fint_{\partial U_1}\varphi\,\mathrm{d}\mathcal{H}_1$, $\tilde{P}\varphi = \varphi - P\varphi_j$, and

$$G[\psi_1,\psi_2] = \mathcal{S}^{U_1}\tilde{P}\psi_1 + P\psi_1 - \mathcal{S}^{U_1}\tilde{P}\psi_1 + P\psi_1,$$

$$F[\psi_1,\psi_2] = \frac{b_i}{2}P\psi_1 - b_iK'_{U_1}\tilde{P}\psi_1 + \frac{b_i}{2}P\psi_2 - b_iK'_{U_2}\tilde{P}\psi_2,$$

$$T_B[\psi_1,\psi_2] = [G[\psi_1,\psi_2], F[\psi_1,\psi_2]].$$

We have proved that $T_B : L^p(B) \times L^p(B) \to W^{1,p}(B) \times L^p(B)$ is a Fredholm operator with index 0. Since $\tilde{T}[\varphi_1,\dots,\varphi_k] = T_B[\chi_B\varphi_i, \chi_B\varphi_j]$ on B, we deduce that $\tilde{T} : X_p \to Y_p$ is a Fredholm operator with index 0. Since $T - \tilde{T} : X_p \to \mathcal{C}^\infty(\partial(\Omega_1 \cup \dots \cup \Omega_k))$ is bounded, we infer that $T - \tilde{T} : X_p \to Y_p$ is compact. (See Lemma 1.21.8.) Theorem 1.8.3 gives that $T : X_p \to Y_p$ is a Fredholm operator with index 0.

Proposition 5.26.5 *Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}$, $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i,j\}$. Suppose that Ω_1 is unbounded. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let $b_1, \dots, b_k$ be positive numbers and $(u_1, \dots, u_k)$ be an L^p-solution of the transmission problem (5.97), (5.98). If $g_{ij} = f_{ij} = 0$ and $u_1(x) \to 0$ as $|x| \to \infty$, then $u_j \equiv 0$ for $j = 1, \dots, k$.*

Proof Let X_p, Y_p be defined like in Proposition 5.26.3. If $m > 2$, then T is defined in Proposition 5.26.3. If $m = 2$, then T is defined in Proposition 5.26.4.

If $m > 2$, then there exist unique $(\varphi_1, \dots, \varphi_k) \in X_p$ and $u_\infty \in R^1$ such that $u_j = \mathcal{S}^{\Omega_j}\varphi_j + u_\infty$. (See Lemma 5.26.1.) If $m = 2$, then there exist $(\varphi_1, \dots, \varphi_k) \in X_p$ such that u_j are given by (5.99). (See Lemma 5.26.2.) Propositions 5.26.3 and 5.26.4 give $T(\varphi_1, \dots, \varphi_k) = 0$. The operators $T : X_p \to Y_p$, $T : X_2 \to Y_2$ are Fredholm with index 0 by Propositions 5.26.3 and 5.26.4. So, $(\varphi_1, \dots, \varphi_k) \in X_2$ by Lemma 1.8.4. Thus $(u_1, \dots, u_k)$ is an L^2-solution of the transmission problem (5.97), (5.98) by Propositions 5.26.3 and 5.26.4.

By virtue of Lemma 5.8.1,

$$0 = \sum_{1 \leq i < j \leq k} \int_{\partial\Omega_i \cap \partial\Omega_j} u_i \left(b_i \frac{\partial u_i}{\partial n^i} - b_j \frac{\partial u_j}{\partial n^i} \right) \mathrm{d}\mathcal{H}_{m-1}$$

$$= \sum_{i=1}^{k} \int_{\partial\Omega_i} u_i b_i \frac{\partial u_i}{\partial n^i} \, \mathrm{d}\mathcal{H}_{m-1} = \sum_{i=1}^{k} \int_{\Omega_i} b_i |\nabla u_i|^2 \, \mathrm{d}\mathcal{H}_m.$$

Since $\nabla u_i = 0$, the function u_i is constant on each component of Ω_i. Since $u_i = u_j$ on $\partial\Omega_i \cap \partial\Omega_j$, there exists a constant c such that $u_j \equiv c$ for $j = 1, \dots, k$. Since $u_1(x) \to 0$ as $|x| \to \infty$, we infer that $u_j \equiv 0$.

Theorem 5.26.6 *Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact Lipschitz boundary such that $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$, $m > 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}$, $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i,j\}$. Suppose that Ω_1 is unbounded. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let $b_1, \dots, b_k$ be positive numbers. Let T, X_p, and Y_p be as in Proposition 5.26.3.*

- *$T : X_p \to Y_p$ is an isomorphism.*
- *If $(u_1, \dots, u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98), then there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.*
- *Let $[g_{ij}, f_{ij}] \in Y_p$ and $u_\infty \in R^1$ be given. Put*

$$[\varphi_1, \dots, \varphi_k] = T^{-1}[g_{ij}, f_{ij}], \tag{5.100}$$

$$u_j = \mathcal{S}^{\Omega_j}\varphi_j + u_\infty, \quad j = 1, \dots, k. \tag{5.101}$$

Then $(u_1,\dots,u_k)$ *is a unique* L^p*-solution of the transmission problem (5.97), (5.98) such that* $u_1(x)\to u_\infty$ *as* $|x|\to\infty$. *Moreover,*

$$\|M_a(u_l)\|_{L^p(\partial\Omega_l)}+\|M_a(\nabla u_l)\|_{L^p(\partial\Omega_l)}$$
$$\le C\sum_{1\le i<j\le k}\Big(\|g_{ij}\|_{W^{1,p}(\partial\Omega_i\cap\partial\Omega_j)}+\|f_{ij}\|_{L^p(\partial\Omega_i\cap\partial\Omega_j)}\Big)+|u_\infty|$$

where C *does not depend on* $[g_{ij},f_{ij}]$, *and* u_∞.

Proof Let $(\varphi_1,\dots,\varphi_k)\in\operatorname{Ker}T$. Put $u_j=\mathcal{S}^{\Omega_j}\varphi_j$, $j=1,\dots,k$. Then $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) with $g_{ij}=0, f_{ij}=0$. (See Proposition 5.26.3.) Moreover, $u_1(x)\to 0$ as $|x|\to\infty$ by Proposition 3.1.6. Proposition 5.26.5 gives that $u_j\equiv 0$ for $j=1,\dots,k$. Hence $\varphi_j\equiv 0$ by Lemma 5.26.1. $T:X_p\to Y_p$ is a Fredholm operator with index 0 by Proposition 5.26.3. Thus $T(X_p)=Y_p$. Hence $T:X_p\to Y_p$ is an isomorphism by Theorem 1.6.3.

Let $(u_1,\dots,u_k)$ be an L^p-solution of the transmission problem (5.97), (5.98). Since $M_a(u_1)\in L^p(\partial\Omega_1)$, the function u_1 is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1, there exists $u_\infty\in R^1$ such that $u_1(x)\to u_\infty$ as $|x|\to\infty$.

Let $[g_{ij},f_{ij}]\in Y_p$ and $u_\infty\in R^1$ be fixed. Let φ_j and u_j be defined by (5.100) and (5.101), respectively. Proposition 5.26.3 gives that $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) such that $u_1(x)\to u_\infty$ as $|x|\to\infty$. The uniqueness follows from Proposition 5.26.5. The rest is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Theorem 5.26.7 *Let* $\Omega_1,\dots,\Omega_k$ *be disjoint open sets with compact Lipschitz boundary such that* $R^2=\overline{\Omega}_1\cup\overline{\Omega}_2\cup\dots\cup\overline{\Omega}_k$, $k\ge 2$. *Suppose that for each component* B *of* $\partial\Omega_i$, *there exists* $j\in\{1,\dots,k\}$, $j\ne i$, *such that* $B\subset\partial\Omega_i\cap\partial\Omega_j$ *and* $B\cap\overline{\Omega}_l=\emptyset$ *for* $l\notin\{i,j\}$. *Suppose that* Ω_1 *is unbounded. Let* $1<p<\infty$. *If* $p\ne 2$, *suppose moreover that* $\partial\Omega_j$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0<\alpha<1$ *for* $j=1,\dots,k$. *Let* $b_1,\dots,b_k$ *be positive numbers. Let* X_p *and* Y_p *be defined as in Proposition 5.26.3. Let* T *be the operator from Proposition 5.26.4. Define*

$$\tilde T(\varphi_1,\dots,\varphi_k)=\left[T(\varphi_1,\dots,\varphi_k),\oint_{\partial\Omega_1}\varphi_1\,\mathrm{d}\mathcal{H}_1\right].$$

Denote by Z_p *the set of all* $[g_{ij},f_{ij}]\in Y_p$ *satisfying*

$$\sum_{1\le i<j\le k}\int_{\partial\Omega_i\cap\partial\Omega_j}f_{ij}\,\mathrm{d}\mathcal{H}_1=0.$$

- $\tilde T:X_p\to Z_p\times R^1$ *is an isomorphism.*

- *If* $(u_1,\dots,u_k)$ *is an* L^p*-solution of the transmission problem (5.97), (5.98), then* $[g_{ij},f_{ij}]\in Z_p$*, and there exists* $u_\infty\in R^1$ *such that* $u_1(x)\to u_\infty$ *as* $|x|\to\infty$.
- *Let* $[g_{ij},f_{ij}]\in Z_p$ *and* $u_\infty\in R^1$ *be given. Put*

$$[\varphi_1,\dots,\varphi_k]=\tilde{T}^{-1}\{[g_{ij},f_{ij}];1\le i<j\le k\}\times u_\infty, \tag{5.102}$$

$$u_j=\mathcal{S}^{\Omega_j}(\varphi_j-\oint_{\partial\Omega_j}\varphi_j\,\mathrm{d}\mathcal{H}_1)+\oint_{\partial\Omega_1}\varphi_1\,\mathrm{d}\mathcal{H}_1,\quad j=1,\dots,k. \tag{5.103}$$

Then $(u_1,\dots,u_k)$ *is a unique* L^p*-solution of the transmission problem (5.97), (5.98) such that* $u_1(x)\to u_\infty$ *as* $|x|\to\infty$*. Moreover,*

$$\|M_a(u_l)\|_{L^p(\partial\Omega_l)}+\|M_a(\nabla u_l)\|_{L^p(\partial\Omega_l)}$$
$$\le C\sum_{1\le i<j\le k}\left(\|g_{ij}\|_{W^{1,p}(\partial\Omega_i\cap\partial\Omega_j)}+\|f_{ij}\|_{L^p(\partial\Omega_i\cap\partial\Omega_j)}\right)+|u_\infty|$$

where C does not depend on $[g_{ij},f_{ij}]$ *and* u_∞.

Proof Let $(u_1,\dots,u_k)$ be an L^p-solution of the transmission problem (5.97), (5.98). Since $M_a(u_1)\in L^p(\partial\Omega_1)$, the function u is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1, there exists $u_\infty\in R^1$ such that $u_1(x)\to u_\infty$ as $|x|\to\infty$. Since

$$\int_{\partial\Omega_j}\frac{\partial u_j}{\partial n}\,\mathrm{d}\mathcal{H}_1=0,\qquad j=1,\dots,k$$

by Theorems 5.9.10, 5.9.11, 5.11.3, and 5.11.4, we obtain

$$0=\sum_{j=1}^k b_j\int_{\partial\Omega_j}\frac{\partial u_j}{\partial n^{\Omega_j}}\,\mathrm{d}\mathcal{H}_1=\sum_{1\le i<j\le k}f_{ij}.$$

Hence $[g_{ij},f_{ij}]\in Z_p$.

$T:X_p\to Y_p$ is a Fredholm operator with index 0 by Proposition 5.26.4. Define $\hat{T}(\varphi_1,\dots,\varphi_k)=[T(\varphi_1,\dots,\varphi_k),0]$. Then $\hat{T}:X_p\to Y_p\times R^1$ is a Fredholm operator and $\alpha(\hat{T})=\alpha(T)$, $\beta(\hat{T})=\beta(T)+1$. The operator $\tilde{T}-\hat{T}$ is one-dimensional and therefore compact. So, $\tilde{T}:X_p\to Y_p\times R^1$ is a Fredholm operator with $i(\tilde{T})=i(\hat{T})=i(T)-1=-1$.

If $(\varphi_1,\dots,\varphi_k)\in X_p$ and u_j is given by (5.103) for $j=1,\dots,k$, then $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) with $[g_{ij},f_{ij}]=\tilde{T}(\varphi_1,\dots,\varphi_k)$ by Proposition 5.26.4. We have proved that $[g_{ij},f_{ij}]\in Z_p$. Thus $\tilde{T}:X_p\to Z_p\times R^1$ is a Fredholm operator and $i(\tilde{T}:X_p\to Z_p\times R^1)=i(\tilde{T}:X_p\to Y_p\times R^1)-1=0$.

Let $(\varphi_1,\dots,\varphi_k) \in \operatorname{Ker}\tilde{T}$. Let u_j be given by (5.103) for $j = 1,\dots,k$. Then $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) with $g_{ij} = 0$, $f_{ij} = 0$ and $u_1(x) \to 0$ as $|x| \to \infty$. (See Propositions 5.26.4 and 3.1.6.) Proposition 5.26.5 gives that $u_j \equiv 0$ for $j = 1,\dots,k$. Hence $\varphi_j \equiv 0$ by Lemma 5.26.2. Since $\tilde{T} : X_p \to Z_p \times R^1$ is a Fredholm operator with index 0, we deduce $\tilde{T}(X_p) = Z_p \times R^1$. Hence $\tilde{T} : X_p \to Z_p \times R^1$ is an isomorphism by Theorem 1.6.3.

Let $[g_{ij}, f_{ij}] \in Z_p$ and $u_\infty \in R^1$ be fixed. Let φ_j and u_j be defined by (5.102) and (5.103), respectively. Propositions 5.26.4 and 3.1.6 give that $(u_1,\dots,u_k)$ is an L^p-solution of the transmission problem (5.97), (5.98) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. The uniqueness follows from Proposition 5.26.5. The rest is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

5.27 L^p-Solutions of the Mixed Transmission Problem

This section is devoted to L^p-solutions of the mixed transmission problem. Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary. Suppose that Ω is divided to open sets $\Omega_1,\dots,\Omega_k$ with compact Lipschitz boundary. We study a boundary value problem for the Laplace equation, where the transmission conditions are given on the interfaces of Ω_i and Ω_j, the Dirichlet condition is given on the part of $\partial\Omega$, and the Robin condition or the Neumann condition is prescribed on the rest of $\partial\Omega$. We find a necessary and sufficient condition for the existence of a solution and study the uniqueness.

Let $\Omega \subset R^m$ be a domain with nonempty compact Lipschitz boundary. Let Γ_D, Γ_R be disjoint closed subset of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $\Omega_1,\dots,\Omega_k$ be disjoint domains with compact Lipschitz boundary such that $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$. Suppose that for each component T of $\partial\Omega_i$, one of the following conditions is fulfilled:

1. $T \subset \Gamma_D$ and $T \cap \overline{\Omega}_j = \emptyset$ for $j \neq i$.
2. $T \subset \Gamma_R$ and $T \cap \overline{\Omega}_j = \emptyset$ for $j \neq i$.
3. There exists $j \in \{1,\dots,k\}$, $j \neq i$, such that $T \subset \partial\Omega_i \cap \partial\Omega_j$, $T \cap \partial\Omega = \emptyset$, $T \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i,j\}$.

We denote by n^j the unit exterior normal of Ω_j. So, $n^i = -n^j$ on $\partial\Omega_i \cap \partial\Omega_j$ for $i \neq j$.

Let $b_1,\dots,b_k$ be positive numbers, $1 < p < \infty$, $h \in L^\infty(\Gamma_R)$, $f \in L^p(\Gamma_R)$, $g \in W^{1,p}(\Gamma_D)$. For $i < j$, let $f_{ij} \in L^p(\partial\Omega_i \cap \partial\Omega_j)$, $g_{ij} \in W^{1,p}(\partial\Omega_i \cap \partial\Omega_j)$. *We say that* $(u_1,\dots,u_k)$ *is an* L^p*-solution of the mixed transmission problem*

$$\Delta u_j = 0 \quad \text{in } \Omega_j, \tag{5.104}$$

$$u_j = g \quad \text{on } \Gamma_D \cap \partial\Omega_j, \qquad \frac{\partial u_j}{\partial n^\Omega} + h u_j = f \quad \text{on } \Gamma_R \cap \partial\Omega_j, \tag{5.105}$$

$$u_i - u_j = g_{ij}, \qquad b_i \frac{\partial u_i}{\partial n^i} - b_j \frac{\partial u_j}{\partial n^i} = f_{ij} \quad \text{on } \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j,\ i < j \tag{5.106}$$

if $u_j \in H(\Omega_j)$, $M_a^{\Omega_j}(u_j) + M_a^{\Omega_j}(\nabla u_j) \in L^p(\partial\Omega_j)$, *there exist nontangential limits of* u_j *and* ∇u_j *at almost all points of* $\partial\Omega_j$, *and the conditions (5.105) and (5.106) are fulfilled in the sense of a nontangential limit almost everywhere.*

Proposition 5.27.1 *Let* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $h \geq 0$, *and* $(u_1, \dots, u_k)$ *be an* L^p*-solution of the mixed transmission problem (5.104), (5.105), (5.106). If* Ω_j *is unbounded and* $m > 2$, *suppose moreover that* $u_j(x) \to 0$ *as* $|x| \to \infty$. *If* $g \equiv 0$, $f \equiv 0$, $g_{ij} \equiv 0$, $f_{ij} \equiv 0$, *then there exists a constant c such that* $u_j \equiv c$ *for* $j = 1, \dots, k$.

Proof Choose a bounded open set ω_D with smooth boundary such that $\Gamma_D \subset \omega_D$ and $\partial\omega_D \cap \partial\Omega_j = \emptyset$. Put $g_j = u_j$ on $\partial(\Omega_j \cap \omega_D)$. Since $g_j \in W^{1,2}(\partial(\Omega_j \cap \omega_D))$, the function u_j is an L^2-solution of the Dirichlet problem $\Delta u_j = 0$ in $\Omega_j \cap \omega_D$, $u_j = g_j$ on $\partial(\Omega_j \cap \omega_D)$ such that $M_a(\nabla u) \in L^2(\partial(\Omega_j \cap \omega_D))$. (See Theorems 5.14.9 and 5.14.10.) In particular, $u_j \in W^{1,2}(\partial\Omega_j \cap \Gamma_D)$ and $\partial u_j/\partial n \in L^2(\partial\Omega_j \cap \Gamma_D)$.

Choose a bounded open set ω_R with smooth boundary such that $\Gamma_R \subset \omega_R$ and $\partial\omega_R \cap \partial\Omega_j = \emptyset$. Put $h = 1$ on $\partial\omega_R$, $f_j = \partial u_j/\partial n + hu_j$ on $\partial(\Omega_j \cap \omega_R)$. Since $f_j \in L^2(\partial(\Omega_j \cap \omega_R))$, the function u_j is an L^2-solution of the Robin problem $\Delta u_j = 0$ in $\Omega_j \cap \omega_R$, $\partial u_j/\partial n + hu_j = f_j$ on $\partial(\Omega_j \cap \omega_R)$. (See Theorems 5.10.4 and 5.11.5.) In particular, $u_j \in W^{1,2}(\partial\Omega_j \cap \Gamma_R)$ and $\partial u_j/\partial n \in L^2(\partial\Omega_j \cap \Gamma_R)$.

Define $\Omega_{k+1} = R^m \setminus \overline{\Omega}$, $u_{k+1} = 0$ in Ω_{k+1}, $b_{k+1} = 1$, $g_{i,k+1} = u_i$, $f_{i,k+1} = b_i \partial u_i/\partial n_i$. Then $(u_1, \dots, u_{k+1})$ is an L^p-solution of the transmission problem

$$\Delta u_j = 0 \qquad \text{on } \Omega_j, \quad j = 1, \dots, k+1,$$

$$u_i - u_j = g_{ij},\ b_i \frac{\partial u_i}{\partial n^i} - b_j \frac{\partial u_j}{\partial n^i} = f_{ij} \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \quad 1 \leq i < j \leq k+1.$$

Since $g_{ij} \in W^{1,2}(\partial\Omega_i \cap \partial\Omega_j)$, $f_{ij} \in L^2(\partial\Omega_i \cap \partial\Omega_j)$, Theorems 5.26.6 and 5.26.7 give that $(u_1, \dots, u_{k+1})$ is an L^2-solution of this problem.

Since $u_i = u_j$, $b_i(\partial u_i/\partial n^i) = b_j(\partial u_j/\partial n^j)$ on $\partial\Omega_i \cap \partial\Omega_j$, we have by Lemma 5.8.1

$$0 = \sum_{j=1}^{k} b_j \int_{\partial\Omega_j} u_j \left[\frac{\partial u_j}{\partial n^j} + h u_j \chi_{\Gamma_R} \right] \mathrm{d}\mathcal{H}_{m-1}$$

$$= \sum_{j=1}^{k} b_j \int_{\Omega_j} |\nabla u_j|^2 \,\mathrm{d}\mathcal{H}_{m-1} + \sum_{j=1}^{k} b_j \int_{\Gamma_R \cap \partial\Omega_j} h u_j^2 \,\mathrm{d}\mathcal{H}_{m-1}.$$

Since $b_j > 0$, $h \geq 0$, we infer that $\nabla u_j = 0$ in Ω_j. Thus u_j is constant on each component of Ω_j. Since $u_j = u_i$ on $\partial\Omega_i \cap \partial\Omega_j$, we deduce that there exists a constant c such that $u_j \equiv c$ for $j = 1, \dots, k$.

Proposition 5.27.2 *Let* $m > 2$, $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega_j$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$ *for* $j = 1, \dots, k$. *Let* X_p, Y_p *be the spaces from*

Proposition 5.26.3. For $[\varphi_1,\dots,\varphi_k] \in X_p$, *define*

$$F_{ij}(\varphi_1,\dots,\varphi_k) = \frac{b_i}{2}\varphi_i - b_i K'_{\Omega_i}\varphi_i + \frac{b_j}{2}\varphi_j - b_j K'_{\Omega_j}\varphi_j \quad \text{on } \Gamma_{ij},$$

$$G_{ij}(\varphi_1,\dots,\varphi_k) = \mathcal{S}^{\Omega_i}\varphi_i - \mathcal{S}^{\Omega_j}\varphi_j \quad \text{on } \Gamma_{ij},\ 1 \le i < j \le k,$$

$$T(\varphi_1,\dots,\varphi_k) = \begin{cases} \mathcal{S}^{\Omega_i}\varphi_i & \text{on } \Gamma_D \cap \partial\Omega_i, \\ \frac{1}{2}\varphi_i - K'_{\Omega_i}\varphi_i + h\mathcal{S}^{\Omega_i}\varphi_i & \text{on } \Gamma_R \cap \partial\Omega_i, \\ (G_{ij}(\varphi_1,\dots,\varphi_k), F_{ij}(\varphi_1,\dots,\varphi_k)) & \text{on } \Gamma_{ij}. \end{cases}$$

Then $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ *is a Fredholm operator with index* 0. *If* $[\varphi_1,\dots,\varphi_k] \in X_p$ *and* $u_j = \mathcal{S}^{\Omega_j}\varphi_j$, *then* $(u_1,\dots,u_k)$ *is an* L^p*-solution of the mixed transmission problem (5.104), (5.105), (5.106) if and only if*

$$T[\varphi_1,\dots,\varphi_k] = [g,f,\{[g_{ij},f_{ij}]; 1 \le i < j \le k\}].$$

Proof If $[\varphi_1,\dots,\varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\varphi_j$, then $(u_1,\dots,u_k)$ is an L^p-solution of the mixed transmission problem (5.104), (5.105), (5.106) if and only if $T[\varphi_1,\dots,\varphi_k] = [g,f,\{[g_{ij},f_{ij}]; 1 \le i < j \le k\}]$. (See Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.)

$T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is a Fredholm operator with index 0 by Propositions 5.26.3, 5.11.1, 5.2.2, 5.9.9, 5.18.1, Theorems 1.7.4 and 1.8.3.

Proposition 5.27.3 *Let* $m = 2$, $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega_j$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$ *for* $j = 1,\dots,k$. *Let* X_p, Y_p *be the spaces from Proposition 5.26.3. For* $[\varphi_1,\dots,\varphi_k] \in X_p$, *define* $P_j\varphi_j = \oint_{\partial\Omega_j} \varphi_j \, d\mathcal{H}_1$, $\tilde{P}_j\varphi_j = \varphi_j - P_j\varphi_j$, *and*

$$F_{ij}(\varphi_1,\dots,\varphi_k) = \frac{b_i}{2}\tilde{P}_i\varphi_i - b_i K'_{\Omega_i}\tilde{P}_i\varphi_i + \frac{b_j}{2}\tilde{P}_j\varphi_j - b_j K'_{\Omega_j}\tilde{P}_j\varphi_j \quad \text{on } \Gamma_{ij},$$

$$G_{ij}(\varphi_1,\dots,\varphi_k) = \mathcal{S}^{\Omega_i}\tilde{P}_i\varphi_i + P_i\varphi - \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j - P_j\varphi_j \text{ on } \Gamma_{ij},\ i < j,$$

$$T(\varphi_1,\dots,\varphi_k) = \begin{cases} \mathcal{S}^{\Omega_i}\tilde{P}_i\varphi_i + P_i\varphi_i & \text{on } \Gamma_D \cap \partial\Omega_i, \\ \frac{1}{2}\varphi_i - K'_{\Omega_i}\varphi_i + h(\mathcal{S}^{\Omega_i}\tilde{P}_i\varphi_i + P_i\varphi_i) & \text{on } \Gamma_R \cap \partial\Omega_i, \\ (G_{ij}(\varphi_1,\dots,\varphi_k), F_{ij}(\varphi_1,\dots,\varphi_k)) & \text{on } \Gamma_{ij}. \end{cases}$$

Then $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ *is a Fredholm operator with index* 0. *If* $[\varphi_1,\dots,\varphi_k] \in X_p$ *and* $u_j = \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j + P_j\varphi_j$, *then* $(u_1,\dots,u_k)$ *is an* L^p*-solution of the mixed transmission problem (5.104), (5.105), (5.106) if and only if* $T[\varphi_1,\dots,\varphi_k] = [g,f,\{[g_{ij},f_{ij}]; 1 \le i < j \le k\}]$.

Proof If $[\varphi_1,\dots,\varphi_k] \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}\tilde{P}_j\varphi_j + P_j\varphi_j$, then $(u_1,\dots,u_k)$ is an L^p-solution of the mixed transmission problem (5.104), (5.105), (5.106) if and only if $T[\varphi_1,\dots,\varphi_k] = [g,f,\{[g_{ij},f_{ij}]; 1 \le i < j \le k\}]$. (See Proposition 3.1.6, Lemma 5.2.1, Propositions 5.2.2, 5.2.3, 5.2.8, Theorems 5.4.7 and 5.4.8.)

$T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is a Fredholm operator with index 0 by Propositions 5.26.4, 5.11.1, 5.2.2, 5.9.9, 5.18.1, Theorems 1.7.4 and 1.8.3.

Theorem 5.27.4 *Let Ω be bounded or $m = 2$. Suppose that $\Gamma_D \neq \emptyset$ or $\mathcal{H}_{m-1}(\{x \in \Gamma_R; h(x) > 0\}) > 0$. Let $h \geq 0$, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let X_p, Y_p be the spaces from Proposition 5.26.3. For $m = 2$, the operators T, P_j are defined in Proposition 5.27.3. For $m > 2$, the operator T is defined in Proposition 5.27.2 and $P_j = 0$. Then $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is an isomorphism. Fix $[g,f,\{[g_{ij},f_{ij}]\}] \in W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$. Denote $(\varphi_1, \dots, \varphi_k) = T^{-1}[g,f,\{[g_{ij},f_{ij}]\}]$, $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$. Then $(u_1, \dots, u_k)$ is a unique L^p-solution of the mixed transmission problem (5.104), (5.105), (5.106). Moreover,*

$$\|M_a(u_l)\|_{L^p(\partial\Omega_l)} + \|M_a(\nabla u_l)\|_{L^p(\partial\Omega_l)} \leq C\Bigg[\|g\|_{W^{1,p}(\Gamma_D)} + \|f\|_{L^p(\Gamma_R)} + \sum_{1 \leq i<j \leq k} \Big(\|g_{ij}\|_{W^{1,p}(\Gamma_{ij})} + \|f_{ij}\|_{L^p(\Gamma_{ij})} \Big) \Bigg]$$

where C does not depend on $[g_{ij},f_{ij}]$, g, and f.

Proof Suppose first that $(u_1, \dots, u_k)$ is an L^p-solution of (5.104), (5.105), (5.106) with $g \equiv 0, f \equiv 0, g_{ij} \equiv 0, f_{ij} \equiv 0$. According to Proposition 5.27.1, there exists $c \in R^1$ such that $u_j = c$ for $j = 1, \dots, k$. Thus $0 = u_j = c$ on Γ_D, $0 = \partial u_j/\partial n + hu_j = hc$ on Γ_R. This forces $c = 0$.

$T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is a Fredholm operator with index 0 by Propositions 5.27.2 and 5.27.3. Let $(\varphi_1, \dots, \varphi_k) \in \operatorname{Ker} T$. Put $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$. Then $(u_1, \dots, u_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106) with $g \equiv 0, f \equiv 0, g_{ij} \equiv 0, f_{ij} \equiv 0$. (See Propositions 5.27.2 and 5.27.3.) So, $u_j \equiv 0, j = 1, \dots, k$. Define $\Omega_{k+1} = R^m \setminus \overline{\Omega}$, $u_{k+1} \equiv 0$. Then $(u_1, \dots, u_{k+1})$ is an L^p-solution of the transmission problem

$$\Delta u_j = 0 \text{ in } \Omega_j, \quad u_i - u_j = 0, \ \frac{\partial u_i}{\partial n^i} - \frac{\partial u_j}{\partial n^i} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j, \quad 1 \leq i,j \leq k+1.$$

So, $\varphi_j = 0$ for $j = 1, \dots, k$ by Lemmas 5.26.1 and 5.26.2. Since T is a Fredholm operator with index 0 and trivial kernel, we deduce that $T(X_p) = W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$. Thus $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is an isomorphism by Theorem 1.6.3.

Fix $[g,f,\{[g_{ij},f_{ij}]\}] \in W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$. Denote $(\varphi_1, \dots, \varphi_k) = T^{-1}[g,f,\{[g_{ij},f_{ij}]\}]$, $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$. Then $(u_1, \dots, u_k)$ is a unique L^p-solution of the mixed transmission problem (5.104), (5.105), (5.106) by Propositions 5.27.2 and 5.27.3. The rest is a consequence of Propositions 5.2.2 and 5.2.8.

Theorem 5.27.5 *Let Ω be bounded, $h \equiv 0$, $\Gamma_D = \emptyset$. Let $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let Y_p be the space from Proposition 5.26.3. Fix $[f, \{[g_{ij}, f_{ij}]\}] \in L^p(\partial\Omega) \times Y_p$. Then there exists an L^p-solution of the mixed transmission problem (5.104), (5.105), (5.106) if and only if*

$$\sum_{1 \leq i < j \leq k} \int_{\partial\Omega_i \cap \partial\Omega_j} f_{ij} \, \mathrm{d}\mathcal{H}_{m-1} + \sum_{i=1}^{k} \int_{\partial\Omega_i \cap \partial\Omega} b_i f \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{5.107}$$

If $(v_1, \dots, v_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106), then the general form of an L^p-solution of the problem (5.104), (5.105), (5.106) is $u_j = v_j + c$, $j = 1, \dots, k$, where c is a given constant.

Proof If $(v_1, \dots, v_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106), then

$$\sum_{1 \leq i < j \leq k} \int_{\partial\Omega_i \cap \partial\Omega_j} f_{ij} + \sum_{i=1}^{k} \int_{\partial\Omega_i \cap \partial\Omega} b_i f = \sum_{i=1}^{k} \int_{\partial\Omega_i} b_i \frac{\partial u_i}{\partial n^i} = 0$$

by Theorems 5.9.10 and 5.11.3. If c is a constant, then $(v_1 + c, \dots, v_k + c)$ is an L^p-solution of the problem (5.104), (5.105), (5.106), too. If $(u_1, \dots, u_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106), then there exists $c \in R^1$ such that $u_j = v_j + c, j = 1, \dots, k$. (See Proposition 5.27.1.)

Let X_p be the space from Proposition 5.26.3. Denote by Z_p the set of all $[f, \{[g_{ij}, f_{ij}]\}] \in L^p(\partial\Omega) \times Y_p$ satisfying (5.107). For $m = 2$, the operators T, P_j are defined in Proposition 5.27.3. For $m > 2$, the operator T is defined in Proposition 5.27.2 and $P_j = 0$. If $(\varphi_1, \dots, \varphi_k) \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$, then $(u_1, \dots, u_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106) with $[f, \{[g_{ij}, f_{ij}]\}] = T(\varphi_1, \dots, \varphi_k)$. (See Propositions 5.27.2 and 5.27.3.) Thus $T(X_p) \subset Z_p$. Suppose now that $T(\varphi_1, \dots, \varphi_k) = 0$. Then there exists a constant c such that $u_j = c, j = 1, \dots, k$. (See Proposition 5.27.1.) Suppose that $c = 0$. Define $\Omega_{k+1} = R^m \setminus \overline{\Omega}$, $u_{k+1} = 0$. Then $(u_1, \dots, u_{k+1})$ is an L^p-solution of the transmission problem

$$\Delta u_j = 0 \text{ in } \Omega_j, \quad u_i - u_j = 0, \ \frac{\partial u_i}{\partial n^i} - \frac{\partial u_j}{\partial n^i} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j.$$

Lemmas 5.26.1 and 5.26.2 force that $\varphi_j = 0, j = 1, \dots, k$. So $\dim \operatorname{Ker} T \leq 1$. Since $T : X_p \to L^p(\partial\Omega) \times Y_p$ is a Fredholm operator with index 0 by Propositions 5.27.2 and 5.27.3, the codimension of $T(X_p)$ is at most 1. Since $T(X_p) \subset Z_p$, we infer that $T(X_p) = Z_p$.

Theorem 5.27.6 *Let Ω_1 be unbounded, $m > 2$, $h \geq 0$, $1 < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega_j$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ for $j = 1, \dots, k$. Let X_p, Y_p be the spaces from Proposition 5.26.3. The operator T is defined in*

Proposition 5.27.2. Then $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ *is an isomorphism. If* $(u_1, \dots, u_k)$ *is an* L^p*-solution of the problem (5.104), (5.105), (5.106), then there exists* $u_\infty \in R^1$ *such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *Fix* $[g, f, \{[g_{ij}, f_{ij}]\}] \in W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$, $u_\infty \in R^1$. *Denote* $(\varphi_1, \dots, \varphi_k) = T^{-1}[g - u_\infty, f - hu_\infty, \{[g_{ij}, f_{ij}]\}]$, $u_j = \mathcal{S}^{\Omega_j}\varphi_j + u_\infty$. *Then* $(u_1, \dots, u_k)$ *is a unique* L^p*-solution of the mixed transmission problem (5.104), (5.105), (5.106) such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\|M_a(u_l)\|_{L^p(\partial\Omega_l)} + \|M_a(\nabla u_l)\|_{L^p(\partial\Omega_l)} \le C\Bigg[\|g\|_{W^{1,p}(\Gamma_D)} + \|f\|_{L^p(\Gamma_R)} + \sum_{1 \le i < j \le k} \Big(\|g_{ij}\|_{W^{1,p}(\Gamma_{ij})} + \|f_{ij}\|_{L^p(\Gamma_{ij})} \Big) + |u_\infty| \Bigg]$$

where C *does not depend on* $[g_{ij}, f_{ij}]$, g, f, *and* u_∞.

Proof Let $(u_1, \dots, u_k)$ be an L^p-solution of the problem (5.104), (5.105), (5.106). Since $M_a(u_1) \in L^p(\partial\Omega_1)$, the function u_1 is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1 there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

Let $(u_1, \dots, u_k)$ be an L^p-solution of (5.104), (5.105), (5.106) with $g \equiv 0, f \equiv 0$, $g_{ij} \equiv 0, f_{ij} \equiv 0$ such that $u_1(x) \to 0$ as $|x| \to \infty$. According to Proposition 5.27.1, there exists $c \in R^1$ such that $u_j = c$ for $j = 1, \dots, k$. Clearly, $c = 0$.

$T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is a Fredholm operator with index 0 by Proposition 5.27.2. Let $(\varphi_1, \dots, \varphi_k) \in \operatorname{Ker} T$. Put $u_j = \mathcal{S}^{\Omega_j}\varphi_j$. Then $(u_1, \dots, u_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106) with $g \equiv 0, f \equiv 0$, $g_{ij} \equiv 0, f_{ij} \equiv 0$. (See Proposition 5.27.2.) Moreover, $u_1(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. So, $u_j \equiv 0, j = 1, \dots, k$. Define $\Omega_{k+1} = R^m \setminus \overline{\Omega}$, $u_{k+1} \equiv 0$. Then $(u_1, \dots, u_{k+1})$ is an L^p-solution of the transmission problem

$$\Delta u_j = 0 \text{ in } \Omega_j, \quad u_i - u_j = 0, \ \frac{\partial u_i}{\partial n^i} - \frac{\partial u_j}{\partial n^i} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j, \quad 1 \le i, j \le k+1.$$

So, $\varphi_j = 0$ for $j = 1, \dots, k$ by Lemma 5.26.1. Since T is a Fredholm operator with index 0 and trivial kernel, we deduce that $T(X_p) = W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$. Thus $T : X_p \to W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$ is an isomorphism by Theorem 1.6.3.

Fix $[g, f, \{[g_{ij}, f_{ij}]\}] \in W^{1,p}(\Gamma_D) \times L^p(\Gamma_R) \times Y_p$, $u_\infty \in R^1$. Denote $(\varphi_1, \dots, \varphi_k) = T^{-1}[g - u_\infty, f - hu_\infty, \{[g_{ij}, f_{ij}]\}]$, $u_j = \mathcal{S}^{\Omega_j}\varphi_j + u_\infty$. Proposition 5.27.2 gives that $(u_1, \dots, u_k)$ is a unique L^p-solution of the problem (5.104), (5.105), (5.106) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. The rest is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Theorem 5.27.7 *Let* $m = 2$, Ω_1 *be unbounded,* $h \equiv 0$, $\Gamma_D = \emptyset$. *Let* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega_j$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$ *for* $j = 1, \dots, k$. *Let* Y_p *be the space from Proposition 5.26.3. Denote by* Z_p *the space of all* $[f, \{[g_{ij}, f_{ij}]\}] \in L^p(\partial\Omega) \times Y_p$ *satisfying (5.107). If* $(u_1, \dots, u_k)$ *is an* L^p*-solution of the* *problem (5.104), (5.105), (5.106), then* $[f, \{[g_{ij}, f_{ij}]\}] \in Z_p$ *and there exists* $u_\infty \in R^1$ *such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$*. If* $[f, \{[g_{ij}, f_{ij}]\}] \in Z_p$ *and* $u_\infty \in R^1$*, then there exists a unique* L^p*-solution* $(u_1, \dots, u_k)$ *of the mixed transmission problem (5.104), (5.105), (5.106) such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$*. Moreover,*

$$\|M_a(u_l)\|_{L^p(\partial\Omega_l)} + \|M_a(\nabla u_l)\|_{L^p(\partial\Omega_l)} \le C\Bigg[\|f\|_{L^p(\Gamma_R)}$$
$$+ \sum_{1\le i<j\le k}\left(\|g_{ij}\|_{W^{1,p}(\Gamma_{ij})} + \|f_{ij}\|_{L^p(\Gamma_{ij})}\right)\Bigg] + |u_\infty|$$

where C *does not depend on* $[g_{ij}, f_{ij}]$*,* g*,* f*, and* u_∞*.*

Proof Let $(u_1, \dots, u_k)$ be an L^p-solution of the problem (5.104), (5.105), (5.106). Then

$$\sum_{1\le i<j\le k}\int_{\partial\Omega_i\cap\partial\Omega_j} f_{ij} + \sum_{i=1}^{k}\int_{\partial\Omega_i\cap\partial\Omega} b_i f = \sum_{i=1}^{k}\int_{\partial\Omega_i} b_i\frac{\partial u_i}{\partial n^i} = 0$$

by Theorems 5.9.10, 5.9.11, 5.11.3, and 5.11.4. Since $M_a(u_1) \in L^p(\partial\Omega_1)$, the function u_1 is bounded in a neighborhood of infinity by Lemma 1.28.2. According to Proposition 2.17.1, there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. If $f \equiv 0, f_{ij} \equiv 0$, $g_{ij} \equiv 0$, $u_\infty = 0$, then $u_j \equiv 0$ for $j = 1, \dots, k$ by Proposition 5.27.1.

Let X_p be the space from Proposition 5.26.3. The operators T, P_j are defined in Proposition 5.27.3. Then $T : X_p \to L^p(\partial\Omega) \times Y_p$ is a Fredholm operator with index 0 by Proposition 5.27.3. Define $\hat{T}(\varphi_1, \dots, \varphi_k) = [T(\varphi_1, \dots, \varphi_k), 0]$, $\tilde{T}(\varphi_1, \dots, \varphi_k) = [T(\varphi_1, \dots, \varphi_k), P_1\varphi_1]$. Then $\hat{T} : X_p \to L^p(\partial\Omega) \times Y_p \times R^1$ is a Fredholm operator with index -1. The operator $\tilde{T} - \hat{T}$ is one-dimensional and therefore compact. Theorem 1.8.3 gives that $\tilde{T} : X_p \to L^p(\partial\Omega) \times Y_p \times R^1$ is a Fredholm operator with index -1.

If $(\varphi_1, \dots, \varphi_k) \in X_p$ and $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$, then $(u_1, \dots, u_k)$ is an L^p-solution of the problem (5.104), (5.105), (5.106) with $[f, \{[g_{ij}, f_{ij}]\}] = T(\varphi_1, \dots, \varphi_k)$, and $u_1(x) \to P_1\varphi_1$ as $|x| \to \infty$. (See Propositions 5.27.3 and 3.1.6.) Thus $\tilde{T}(X_p) \subset Z_p \times R^1$. Suppose now that $\tilde{T}(\varphi_1, \dots, \varphi_k) = 0$, $u_j = \mathcal{S}^{\Omega_j}(I - P_j)\varphi_j + P_j\varphi_j$. We have proved that $u_j \equiv 0$. Define $\Omega_{k+1} = R^m \setminus \overline{\Omega}$, $u_{k+1} = 0$. Then $(u_1, \dots, u_{k+1})$ is an L^p-solution of the transmission problem

$$\Delta u_j = 0 \text{ in } \Omega_j, \quad u_i - u_j = 0, \ \frac{\partial u_i}{\partial n^i} - \frac{\partial u_j}{\partial n^i} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j.$$

Lemma 5.26.2 forces that $\varphi_j = 0, j = 1, \dots, k$. So, $\alpha(\tilde{T}) = 0$, $\beta(\tilde{T}) = 1$. Since $\tilde{T}(X_p) \subset Z_p \times R^1$, we infer that $\tilde{T}(X_p) = Z_p \times R^1$. The rest is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

Chapter 6
Classical Solutions of BVP

Abstract Chapter 6 studies classical solutions of boundary value problems for the Poisson equation.

This chapter studies classical solutions of boundary value problems on bounded and unbounded domains. It is shown that there exists a solution of the Dirichlet problem for the Poisson equation in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for continuous data. Solutions in $\mathcal{C}^{0,\lambda}(\overline{\Omega})$ and in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ are also studied. In general, there is not a solution in $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ of the Neumann problem or the Robin problem for a continuous boundary condition. So, we define classical solutions of these problems in a bit weaker sense. Classical solutions of the Neumann problem and of the Robin problem are functions from $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ for which there exists a normal derivative on the boundary. We find a necessary and sufficient condition for the existence of a classical solution of the Neumann problem and the Robin problem. We also show that solutions of these problems are from $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ for boundary conditions from $\mathcal{C}^{0,\lambda}(\partial\Omega)$. We obtain the maximum principle for the Robin problem of the Laplace equation. Further, we study classical solutions of the derivative oblique problem, the transmission problem, the jump problem on cracks, and mixed problems. At the end of this chapter, we show symmetry of solutions of the Dirichlet problem and the Robin problem for symmetric domains and symmetric boundary conditions.

6.1 Classical Boundary Value Problems

Definition 6.1.1 Let $\Omega \subset R^m$ be an open set, $\Gamma_D \subset \partial\Omega$, $\theta = (\theta_1, \dots, \theta_m)$ be a nonzero vector function on $\partial\Omega \setminus \Gamma_D$, and $f \in \mathcal{C}^0(\Omega)$, g, h be functions on $\partial\Omega$. We say that u is a classical solution of the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_D, \quad \partial u/\partial\theta + hu = g \quad \text{on } \partial\Omega \setminus \Gamma_D. \tag{6.1}$$

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If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, there exists $\partial u/\partial \theta$ on $\partial\Omega \setminus \Gamma_D$ and (6.1) holds true. If $\Gamma_D = \partial\Omega$, we call it the Dirichlet problem (or the first problem) for the Poisson equation. If $\Gamma_D = \emptyset$, $\theta = n^\Omega$, and $h \equiv 0$, we call it the Neumann problem (the second problem) for the Poisson equation. If $\Gamma_D = \emptyset$, $\theta = n^\Omega$, and $h \not\equiv 0$, we call it the Robin problem (the third problem) for the Poisson equation. If $\Gamma_D = \emptyset$ and $\theta \not\equiv n^\Omega$, we call it the oblique derivative problem for the Poisson equation. If $\emptyset \neq \Gamma_D \neq \partial\Omega$, we call it the mixed problem for the Poisson equation. If $f \equiv 0$, we say about boundary value problems for the Laplace equation.

Definition 6.1.2 Let $\Omega_+ \subset R^m$ be an open set with boundary of class $\mathcal{C}^1$, $m \geq 2$, $\Omega_- = R^m \setminus \overline{\Omega_+}$; g, f, and h be functions on $\partial\Omega_+$; and $a_\pm$ and $b_\pm$ be positive constants. Denote by $n(y)$ the outward unit normal of Ω_+ at $y \in \partial\Omega_+$. We say that u_+ and u_- are classical solutions of the transmission problem

$$-\Delta u_+ = 0 \quad \text{in } \Omega_+, \qquad -\Delta u_- = 0 \quad \text{in } \Omega_-, \tag{6.2}$$

$$a_+u_+ - a_-u_- = g \quad \text{on } \partial\Omega, \qquad b_+\frac{\partial u_+}{\partial n} - b_-\frac{\partial u_-}{\partial n} + hu_+ = f \quad \text{on } \partial\Omega. \tag{6.3}$$

If $u_+ \in \mathcal{C}^2(\Omega_+) \cap \mathcal{C}^0(\overline{\Omega}_+)$ and $u_- \in \mathcal{C}^2(\Omega_-) \cap \mathcal{C}^0(\overline{\Omega}_-)$, there exist $\partial u_+(y)/\partial n$ and $\partial u_-(y)/\partial n$ at each point $y \in \partial\Omega_+$, and (6.2) and (6.3) hold true.

6.2 Monotony

In this section, we prove the monotony principle for the problem $\Delta u = f(x, u)$ in Ω, $u = g$ on Γ_D, and $\partial u/\partial\Theta + hu = g$ on $\partial\Omega \setminus \Gamma_D$. As a consequence, we obtain the maximum principle for the Robin problem of the Laplace equation.

Lemma 6.2.1 *Let $\Omega \subset R^m$ be a domain, $m \geq 2$. Suppose that f is a measurable function on $\Omega \times R^1$ such that $f(x, \cdot)$ is non-decreasing for each $x \in \Omega$. Let $u, v \in \mathcal{C}^0(\Omega)$, $-\Delta u(x) + f(x, u) \geq -\Delta v(x) + f(x, v)$ in Ω in the sense of distributions. If there exists $z \in \Omega$ such that $u(z) - v(z) = \inf\{u(y) - v(y); y \in \Omega\} < 0$, then $u - v$ is constant.*

Proof According to Lemma 1.20.3, there exists a nonnegative Radon measure μ such that $-\Delta(u - v) + f(\cdot, u) - f(\cdot, v) = \mu$. Put $K = \{x \in \overline{\Omega}; u(x) - v(x) = u(z) - v(z)\}$. If $x \in K$, then $u(x) - v(x) < 0$, and there exists $r > 0$ such that $u(y) < v(y)$, and therefore, $f(y, u(y)) \leq f(y, v(y))$ on $B(x; r)$. Since $-\Delta(u - v) = \mu - f(\cdot, u) + f(\cdot, v) \geq 0$ on $B(x; r)$, the function $u - v$ is superharmonic in $B(x; r)$ (see Theorem 3.19.4). Since $u - v$ attains its minimum in x, the function $u - v$ must be constant in $B(x;r)$ by Proposition 3.3.5. Thus, K is open. Since K is a nonempty open-closed subset of the connected metric space Ω, we deduce $K = \Omega$.

Proposition 6.2.2 *Let $\Omega \subset R^m$ be a domain, $m \geq 2$, $\Gamma_D \subset \partial\Omega$, and $\theta = (\theta_1, \dots, \theta_m)$ be an outward-pointing vector function on $\partial\Omega \setminus \Gamma_D$. Let g be a function on $\partial\Omega$, h be a nonnegative function on $\partial\Omega \setminus \Gamma_D$, f be a measurable function on $\Omega \times R^1$, and $f(x, \cdot)$ be non-decreasing for each $x \in \Omega$. Suppose that $\Gamma_D \neq \emptyset$ or $h \not\equiv 0$. Suppose moreover that for each $x \in \partial\Omega \setminus \overline{\Gamma_D}$, we have $h(x) > 0$ or Ω has boundary of class C^2 in a neighborhood of x. Let $u, v \in C^0(\overline{\Omega})$ be such that there exist $\partial u/\partial\theta$ and $\partial v/\partial\theta$ on $\partial\Omega \setminus \Gamma_D$. Suppose that $-\Delta u(x) + f(x,u) \geq -\Delta v(x) + f(x,v)$ in Ω in the sense of distributions, $u \geq v$ on Γ_D, and $\partial u/\partial\theta + hu \geq \partial v/\partial\theta + hv$ on $\partial\Omega \setminus \Gamma_D$. If Ω is unbounded, suppose moreover*

$$\liminf_{\substack{|x| \to \infty \\ x \in \Omega}} (u(x) - v(x)) \geq 0. \tag{6.4}$$

Then $u \geq v$ in Ω.

Proof Suppose that $M = \inf(u - v) < 0$. Suppose first that there exist $z \in \Omega$ such that $u(z) - v(z) = M$. Then $u - v \equiv M$ by Lemma 6.2.1. If $x \in \Gamma_D$, then $u(x) - v(z) = M < 0$ what is a contradiction. Thus, $\Gamma_D = \emptyset$. Choose $x \in \partial\Omega$ such that $h(x) > 0$. Then $\partial[u(x) - v(x)]/\partial\theta + h(x)[u(x) - v(x)] = h(x)M < 0$. That is a contradiction.

Suppose now that $u - v > M$ in Ω. According to (6.4), there exist $z \in \partial\Omega$ such that $u(z) - v(z) = M < 0$. The continuity of u and v gives $z \in \partial\Omega \setminus \overline{\Gamma_D}$. Since $u - v \geq u(z) - v(z)$, we have $\partial[u(z) - v(z)]/\partial\theta \leq 0$. If $h(z) > 0$, then $\partial[u(z) - v(z)]/\partial\theta + h(z)[u(z) - v(z)] < 0$, what is a contradiction. Suppose now that $h(z) = 0$ and Ω has boundary of class C^2 in a neighborhood of z. Choose $r > 0$ such that $u < v$ on $\Omega \cap B(z;r)$. Then $-\Delta(u(x) - v(x)) \geq f(x,v) - f(x,u) \geq 0$ in $\Omega \cap B(z;r)$, and therefore, $u - v$ is a superharmonic function in $\Omega \cap B(z;r)$ by Theorem 3.19.4. Since Ω satisfies the interior ball condition at z (see Lemma 1.17.11), Proposition 3.5.1 gives that $\partial[u(z) - v(z)]/\partial\theta < 0$. That contradicts $\partial u/\partial\theta + hu \geq \partial v/\partial\theta + hv$ on $\partial\Omega \setminus \Gamma_D$.

Proposition 6.2.3 *Let $\Omega \subset R^2$ be an unbounded domain, $\overline{\Omega} \neq R^2$, $\Gamma_D \subset \partial\Omega$, $\theta = (\theta_1, \theta_2)$ be a bounded outward-pointing vector function on $\partial\Omega \setminus \Gamma_D$. Let g and h be functions on $\partial\Omega$, $\inf h > 0$, f be a measurable function on $\Omega \times R^1$, and $f(x, \cdot)$ be non-decreasing for each $x \in \Omega$. Let $u, v \in C^0(\overline{\Omega})$ be such that there exist $\partial u/\partial\theta$ and $\partial v/\partial\theta$ on $\partial\Omega \setminus \Gamma_D$. Suppose that $-\Delta u(x) + f(x,u) \geq -\Delta v(x) + f(x,v)$ in Ω in the sense of distributions, $u \geq v$ on Γ_D, $\partial u/\partial\theta + hu \geq \partial v/\partial\theta + hv$ on $\partial\Omega \setminus \Gamma_D$,* *and*

$$\liminf_{\substack{|x| \to \infty \\ x \in \Omega}} (u(x) - v(x)) > -\infty.$$

Then $u \geq v$ *in* Ω.

Proof Fix $z \in R^2$ and $r > 0$ such that $B(z; 2r) \subset R^2 \setminus \overline{\Omega}$. Choose $C > 0$ such that $|\theta| \leq C$ and $h > 1/C$. Put $w(x) = \ln(|x - z|/r) + C^2/r$. Then $w > 0$ in $\overline{\Omega}$, $\partial w(x)/\partial\theta + h(x)w(x) = \theta \cdot (x - z)/|x - z|^2 + h(x)w(x) \geq -C/r + \ln 2/C + C/r > 0$ on $\partial\Omega$. For $\epsilon > 0$, put $u_\epsilon = u + \epsilon w$. Then $u_\epsilon(x) - v(x) \to \infty$ as $|x| \to \infty$, $u_\epsilon \geq u \geq v$ on Γ_D, and $\partial u_\epsilon/\partial\theta + hu_\epsilon \geq \partial v/\partial\theta + hv$ on $\partial\Omega \setminus \Gamma_D$. Since $w \in H(\Omega)$ (see Remark 2.1.3), we have $-\Delta u_\epsilon(x) + f(x, u_\epsilon(x)) = -\Delta u(x) + f(x, u_\epsilon(x)) \geq -\Delta u(x) + f(x, u(x)) \geq -\Delta v(x) + f(x, v(x))$. Proposition 6.2.2 gives that $u_\epsilon \geq v$. For $\epsilon \to 0$, we get $u \geq v$.

Corollary 6.2.4 *Let* $\Omega \subset R^m$ *be an open set,* $m \geq 2$*,* $\Gamma_D \subset \partial\Omega \neq \emptyset$*,* $\theta = (\theta_1, \ldots, \theta_m)$ *be an outward-pointing vector function on* $\partial\Omega \setminus \Gamma_D$*, and* g *be a function on* $\partial\Omega$*. Suppose that* u *is a classical solution of the problem* $\Delta u = 0$ *in* Ω*,* $u = g$ *on* Γ_D*, and* $\partial u/\partial\theta + u = g$ *on* $\partial\Omega \setminus \Gamma_D$*. If* Ω *is unbounded and* $m > 2$ *suppose moreover that* $u(x) \to 0$ *as* $|x| \to \infty$*. If* Ω *is unbounded and* $m = 2$*, suppose that* u *is bounded. Then*

$$\inf_{x \in \partial\Omega} g(x) \leq \inf_{x \in \Omega} u(x) \leq \sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} g(x),$$

$$\operatorname{dist}(x, \partial\Omega)|\nabla u(x)| \leq m \sup_{y \in \partial\Omega} |g(y)|, \qquad x \in \Omega.$$

Proof If $\inf g$ is finite, put $v \equiv \inf g$. Then $\Delta v \equiv 0$, $v \leq g = u$ on Γ_D, and $\partial v/\partial\theta + v \leq g = \partial u/\partial\theta + u$ on $\partial\Omega \setminus \Gamma_D$. Propositions 6.2.2 and 6.2.3 give that $\inf g = v \leq u$. If $\sup g$ is finite, put $w \equiv \sup g$. We obtain $u \leq \sup g$ by the same argument.

Suppose now that $\sup |g| < \infty$. Fix $x \in \Omega$. Choose a unit vector Θ such that $|\nabla u(x)| = \Theta \cdot \nabla u(x)$. We can choose a coordinate system such that Θ is the direction of the first coordinate (see Lemma 2.1.2). Theorem 2.9.2 gives

$$\operatorname{dist}(x, \partial\Omega)|\nabla u(x)| = \operatorname{dist}(x, \partial\Omega)|\partial_1 u(x)| \leq m \sup_{y \in \Omega} |u(y)| \leq m \sup_{y \in \partial\Omega} |g(y)|.$$

Proposition 6.2.5 *Suppose that* $\Omega \subset R^2$ *is a domain,* $\overline{\Omega} \neq R^2$*, and* $\partial\Omega$ *is compact. Let* $\Gamma_D \subset \partial\Omega$*,* $\theta = (\theta_1, \theta_2)$ *be an outward-pointing vector function on* $\partial\Omega \setminus \Gamma_D$*. Let* g *be a function on* $\partial\Omega$*,* h *be a nonnegative function on* $\partial\Omega \setminus \Gamma_D$*,* f *be a measurable function on* $\Omega \times R^1$*, and* $f(x, \cdot)$ *be non-decreasing for each* $x \in \Omega$*. Suppose that* $\Gamma_D \neq \emptyset$ *or* $h \not\equiv 0$*. Suppose moreover that for each* $x \in \partial\Omega \setminus \overline{\Gamma_D}$*, we have* $h(x) > 0$

or Ω *has boundary of class* $\mathcal{C}^2$ *in a neighborhood of* x. *Let* $u, v \in \mathcal{C}^0(\overline{\Omega})$ *be such that there exist* $\partial u/\partial\theta$ *and* $\partial v/\partial\theta$ *on* $\partial\Omega \setminus \Gamma_D$. *Suppose that* $-\Delta u(x) + f(x,u) \geq -\Delta v(x) + f(x,v)$ *in* Ω *in the sense of distributions,* $u \geq v$ *on* Γ_D, *and* $\partial u/\partial\theta + hu \geq \partial v/\partial\theta + hv$ *on* $\partial\Omega \setminus \Gamma_D$. *If* Ω *is unbounded, suppose moreover*

$$\liminf_{\substack{|x| \to \infty \\ x \in \Omega}} (u(x) - v(x)) > -\infty.$$

Then $u \geq v$ *in* Ω.

Proof Define $w = u - v$, $G = \{x \in \Omega; w(x) < 0\}$. If $G = \emptyset$ then $u \geq v$. Suppose that $G \neq \emptyset$. Denote

$$m = \inf_{x \in \partial\Omega} w(x), \quad \tilde{m} = \inf_{x \in \partial G} w(x).$$

If $x \in \Omega \cap \partial G$, then $w(x) = 0$. If $x \in \partial G \setminus \Omega$, then $x \in \partial\Omega$ and $w(x) \geq m$. Thus, $\tilde{m} \geq \min(0, m)$. We show that $w \geq \tilde{m}$ in G. Since $-\Delta w(x) \geq f(x,v) - f(x,u) \geq 0$ in G, the function w is superharmonic in G by Theorem 3.19.4. Fix $z \in R^2$ and $r \in (0,\infty)$ such that $\overline{B(z;r)} \cap \overline{\Omega} = \emptyset$. Put $h(x) = \ln(|x - z|/r)$. Then h is a positive harmonic function in Ω by Remark 2.1.3. Moreover

$$\liminf_{G \ni x \to y} \frac{w(x) - \tilde{m}}{h(y)} \geq 0, \quad \forall y \in \partial G,$$

$$\liminf_{G \ni x, |x| \to \infty} \frac{w(x) - \tilde{m}}{h(y)} \geq 0.$$

Proposition 3.4.2 gives that $w \geq \tilde{m}$ in G.

Since $w \geq \tilde{m} \geq \min(0, m)$ in G and $w \geq 0$ in $\Omega \setminus G$, we have that $w \geq \min(0, m)$. Since $G \neq \emptyset$, we deduce that $m < 0$. Hence $w \geq m$ in $\overline{\Omega}$. Thus, there exists $x \in \partial\Omega$ such that

$$0 > m = w(x) = \min_{y \in \partial\Omega} w(y) = \min_{y \in \overline{\Omega}} w(y).$$

If $x \in \overline{\Gamma}_D$, then $w(x) \geq 0$, what is a contradiction. Thus, $x \in \partial\Omega \setminus \overline{\Gamma}_D$. Since w achieves its minimum in x, we have $\partial w(x)/\partial\theta \leq 0$. If $h(x) > 0$, then $\partial w(x)/\partial\tau + hw(x) \leq h(x)w(x) = h(x)m < 0$. That contradicts with the assumption $\partial w(x)/\partial\tau + h(x)w(x) \geq 0$. Therefore, $h(x) = 0$. If there exists $y \in \Omega$ such that $w(y) = m$, then w is constant by Lemma 6.2.1. But in this case, we can choose $x \in \Gamma_D$ or $x \in \partial\Omega \setminus \Gamma_D$ with $h(x) > 0$. So, we can suppose that $w < m$ in Ω. According to the assumption, Ω has a boundary of class $\mathcal{C}^2$ in a neighborhood of x. Since Ω satisfies the interior ball condition at x (see Lemma 1.17.11), Proposition 3.5.1 gives that $\partial w(x)/\partial\theta < 0$. That contradicts the assumption $\partial w/\partial\theta + hw \geq 0$ on $\partial\Omega \setminus \Gamma_D$.

6.3 Uniqueness of Boundary Value Problems

In this section, we study the uniqueness of a classical solution of the Dirichlet problem, the Robin problem, the mixed problem, and the transmission problem.

Theorem 6.3.1 *Let* $\Omega \subset R^m$ *be a domain,* $m \geq 2$, $\Gamma_D \subset \partial\Omega$, $\theta = (\theta_1, \dots, \theta_m)$ *be an outward-pointing vector function on* $\partial\Omega \setminus \Gamma_D$. *Let* g *be a function on* $\partial\Omega$, h *be a nonnegative function on* $\partial\Omega \setminus \Gamma_D$, f *be a measurable function on* $\Omega \times R^1$, *and* $f(x, \cdot)$ *be non-decreasing for each* $x \in \Omega$, $c \in R^1$. *Suppose that* $\Gamma_D \neq \emptyset$ *or* $h \not\equiv 0$ *on* $\partial\Omega \setminus \Gamma_D$. *Suppose moreover that for each* $x \in \partial\Omega \setminus \overline{\Gamma}_D$, *we have* $h(x) > 0$ *or* Ω *has boundary of class* $\mathcal{C}^2$ *in a neighborhood of* x.

- *If* Ω *is bounded then there exists at most one classical solution of the problem*

$$\Delta u = f(x, u) \text{ in } \Omega\,,\; u = g \text{ on } \Gamma_D,\; \partial u/\partial\theta + hu = g \text{ on } \partial\Omega \setminus \Gamma_D. \tag{6.5}$$

- *If* Ω *is unbounded then there exists at most one classical solution* u *of the problem (6.5) such that* $u(x) \to c$ *as* $|x| \to \infty$.
- *If* $\Omega \subset R^2$ *is an unbounded domain,* $\overline{\Omega} \neq R^2$, $\inf h > 0$, θ *is a bounded vector function, then there exists at most one bounded classical solution of the problem (6.5).*
- *If* $\Omega \subset R^2$ *is an unbounded domain with compact boundary,* $\overline{\Omega} \neq R^2$, *then there exists at most one bounded classical solution of the problem (6.5).*

Proof The proof immediately follows from Propositions 6.2.2, 6.2.3 and 6.2.5.

Proposition 6.3.2 *Let* $\Omega_+ \subset R^m$ *be a bounded open set with boundary of class* $\mathcal{C}^1$, $m \geq 2$, $\Omega_- = R^m \setminus \overline{\Omega}_+$. *Let* g, f, h *be functions on* $\partial\Omega_+$, $h \geq 0$ *and* a_+, *and* a_- *be positive constants,* $d \in R^1$. *Suppose, moreover, that for each* $x \in \partial\Omega_+$, *we have* $h(x) > 0$ *or* Ω_+ *has boundary of class* $\mathcal{C}^2$ *in a neighborhood of* x. *Then there exists at most one classical solution of the transmission problems (6.2)–(6.3) such that* $u_-(x) \to d$ *as* $|x| \to \infty$.

Proof If we put $v_\pm = u_\pm a_\pm$, then $u_\pm$ is a solution of the transmission problems (6.2) and (6.3) if and only if $v_\pm$ is a solution of the transmission problem with the transmission condition

$$v_+ - v_- = g, \quad (b_+/a_+)\frac{\partial v_+}{\partial n} - (b_-/a_-)\frac{\partial v_-}{\partial n} + (h/a_+)v_+ = f \quad \text{on } \partial\Omega\,.$$

So, we can suppose that $a_+ = a_1 = 1$.

Fix two solutions $u_\pm$ and $v_\pm$ of the problem. Put $w_\pm = u_\pm - v_\pm$. Then $-\Delta w_\pm = 0$ in $\Omega_\pm$, $w_-(x) \to 0$ as $|x| \to \infty$ and $w_+ = w_-$, $b_+ \partial w_+/\partial n - b_- \partial w_-/\partial n + h w_+ = 0$ on $\partial\Omega_+$. Define $w = w_+$ on $\overline{\Omega}_+$, $w = w_-$ on Ω_-, and $c = \sup |w|$. Suppose that $c > 0$. Since w is continuous and $w(x) \to 0$ as $|x| \to \infty$, there exists $z \in R^m$ such that $|w(z)| = c$. For the sake of definiteness, suppose that $w(z) = c$. Denote $\mathcal{M} = \{x \in R^m; w(x) = c\}$. Then $\mathcal{M}$ is a nonempty closed set. We show that $\mathcal{M}$ is open. Let $x \in \mathcal{M}$. Suppose first that $x \notin \partial\Omega_+$. Let ω be a component of $R^m \setminus \partial\Omega_+$ such that $x \in \omega$. Since $v \in H(\omega)$ and v attain their maximum in x, Theorem 2.4.2 forces that $w = c$ in ω. Let now $x \in \partial\Omega_+$. Let ω_+ be the component of Ω_+ and ω_- be the component of Ω_- such that $x \in \partial\omega_+ \cap \partial\omega_-$. Since $w_+ \le w_+(x)$ and n is the outward-pointing normal of Ω_+, we have $\partial w_+(x)/\partial n \ge 0$. Since $w_- \le w_-(x)$ and n is the inward-pointing normal of Ω_-, we obtain $\partial w_-(x)/\partial n \le 0$. Since $b_+ \partial w_+/\partial n - b_- \partial w_-/\partial n + h w_+ = 0$ and $b_+ \partial w_+(x)/\partial n \ge 0$, $-b_- \partial w_-(x)/\partial n \ge 0$, and $h(x) w_+(x) \ge 0$, we infer that $b_+ \partial w_+(x)/\partial n = 0$, $b_- \partial w_-(x)/\partial n = 0$, and $h(x) w_+(x) = 0$. Thus, $h(x) = 0$. So, Ω_+ has boundary of class $\mathcal{C}^2$ in a neighborhood of x. If $w_+ < c = w_+(x)$ in ω_+, then $\partial w_+(x)/\partial n \ne 0$ (see Lemma 1.17.11 and Proposition 3.5.1), what is a contradiction. Therefore, there exists $y \in \omega_+$ such that $w_+(y) = c$. Theorem 2.4.2 gives that $w = w_+ = c$ in ω_+. By the same way, we prove that $w = c$ in ω_-. Continuity of w gives that $w = c$ on a neighborhood of x. Since $\mathcal{M}$ is a nonempty closed open subset of the domain R^m, we infer that $\mathcal{M} = R^m$. But that is a contradiction with the fact that $w(x) \to 0$ as $|x| \to \infty$. Thus, $w \equiv 0$.

6.4 Dirichlet Problem on Bounded Domains

In this section, we study the Dirichlet problem on bounded regular domains. Remark that a bounded domain with Lipschitz boundary is regular (see Theorems 4.5.7 and 4.5.8).

Proposition 6.4.1 *Let Ω be a regular bounded open set with the Green function G_Ω, $f \in L^p(\Omega)$, and $p > m/2$. Then $G_\Omega f$ is a unique solution of the problem $u \in \mathcal{C}^0(\overline{\Omega})$, $u = 0$ on $\partial\Omega$, and $-\Delta u = f$ in Ω in the sense of distributions. If $f \in \mathcal{C}^{0,\alpha}_{loc}(\Omega)$, then the Green potential $G_\Omega f$ is a classical solution of the Dirichlet problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.*

Proof $G_\Omega f \in \mathcal{C}^{0,0}(\Omega)$ by Theorem 3.13.3, $-\Delta G_\Omega f = f$ in the sense of distribution by Theorem 3.11.5. If $z \in \partial\Omega$, then $G_\Omega f(x) \to 0$ as $x \to z$ by Theorem 4.6.3. The uniqueness of a solution of the problem follows from Lemma 6.2.1. If $f \in \mathcal{C}^{0,\alpha}_{loc}(\Omega)$, then $G_\Omega f \in \mathcal{C}^2(\Omega)$ by Proposition 3.18.1.

Theorem 6.4.2 *Let Ω be a bounded regular open set. Let $g \in \mathcal{C}^0(\partial\Omega)$, $f \in \mathcal{C}^{0,\alpha}_{loc}(\Omega) \cap L^p(\Omega)$, $0 < \alpha < 1$, and $m/2 < p \le \infty$. Then there exists a unique classical solution u of the Dirichlet problem for the Poisson equation $-\Delta u = f$ in*

Ω, $u = g$ on $\partial\Omega$. Moreover,

$$\sup_{x\in\Omega} |u(x)| \le \sup_{x\in\partial\Omega} |g(x)| + c\|f\|_{L^p(\Omega)} \tag{6.6}$$

with c depending only on Ω and p. If $p > m$, then

$$\operatorname{dist}(x, \partial\Omega)|\nabla u(x)| \le \tilde{c}\left[\sup_{x\in\partial\Omega} |g(x)| + \|f\|_{L^p(\Omega)}\right], \quad x \in \Omega,$$

with $\tilde{c}$ depending only on Ω and p.

Proof The Green potential $G_\Omega f$ is a classical solution of the Dirichlet problem $-\Delta v = f$, $v = 0$ on $\partial\Omega$ by Proposition 6.4.1. Theorem 3.13.3 gives that there exists a constant c depending only on Ω and p such that $|G_\Omega f| \le c\|f\|_{L^p(\Omega)}$ in Ω. The function g is resolutive by Theorem 4.2.7. So, $H^\Omega g$ is a classical solution of the problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$. Maximum principle (Theorem 2.4.2) gives $|H^\Omega g| \le \max|g|$. Hence, $u = G_\Omega f + H^\Omega_h g$ is a classical solution of the Dirichlet problem for the Poisson equation $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ satisfying (6.6). The uniqueness of a solution follows from Theorem 6.3.1.

Let now $p > m$. Put $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^1(R^m)$ by Proposition 3.15.1. Moreover, $-\Delta Vf = f$ (see Theorem 3.1.5). Thus, $Vf \in \mathcal{C}^2(\Omega)$ by Proposition 3.18.1. According to Proposition 3.15.1, there exists a constant C_1 depending only on Ω and p such that

$$|Vf(x)| + |\nabla Vf(x)| \le C_1\|f\|_{L^p(\Omega)}, \quad x \in \overline{\Omega}.$$

Thus, $u - Vf \in H(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, and $|u - Vf| \le \|g\|_{\mathcal{C}^0(\partial\Omega)} + (c + C_1)\|f\|_{L^p(\Omega)}$ on $\overline{\Omega}$. If $x \in \Omega$, we obtain by Corollary 6.2.4

$$\operatorname{dist}(x, \partial\Omega)|\nabla u(x)| \le \operatorname{dist}(x, \partial\Omega)|\nabla V(x)| + \operatorname{dist}(x, \partial\Omega)|\nabla[u(x) - Vf(x)]|$$

$$\le C_1 \operatorname{diam}\Omega\|f\|_{L^p(\Omega)} + m\left[\|g\|_{\mathcal{C}^0(\partial\Omega)} + (c + C_1)\|f\|_{L^p(\Omega)}\right].$$

6.5 Dirichlet Problem on Unbounded Sets

In this section, we prove the existence of a classical solution of the Dirichlet problem on an unbounded domain $\Omega \subset R^m$ with compact boundary such that $\Omega \cap B(0; r)$ is regular (e.g., a domain with Lipschitz boundary). We also show that a solution of the Dirichlet problem on the complement of a ball is given by the Poisson integral.

Theorem 6.5.1 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact boundary,* $\overline{\Omega} \neq R^m$*. Fix* $r > 0$ *such that* $\partial\Omega \subset B(0; r)$*. Suppose that* $\Omega \cap B(0; r)$ *is a regular set. (It is true if* Ω *has Lipschitz boundary.) Let* $g \in \mathcal{C}^0(\partial\Omega)$ *and* $u_\infty \in R$*.*

- *If* $m > 2$*, then there exists a unique classical solution of the Dirichlet problems* $\Delta u = 0$ *in* Ω *and* $u = g$ *on* $\partial\Omega$ *such that* $u(x) \to u_\infty$ *as* $|x| \to \infty$*. Moreover,*

$$\sup_{x \in \Omega} |u(x)| \leq \max(|u_\infty|, \sup_{x \in \partial\Omega} |g(x)|). \tag{6.7}$$

- *If* $m = 2$*, then there exists a unique bounded classical solution of the Dirichlet problem* $\Delta u = 0$ *in* Ω*,* $u = g$ *on* $\partial\Omega$*. Moreover,*

$$\sup_{x \in \Omega} |u(x)| = \sup_{x \in \partial\Omega} |g(x)|. \tag{6.8}$$

Proof If Ω has Lipschitz boundary, then $\Omega \cap B(0; r)$ is regular by Theorems 4.5.7 and 4.5.8.

Suppose first that $u_\infty = 0$. Fix $z \in R^m$ and $\rho > 0$ such that $\overline{B(z; \rho)} \cap \overline{\Omega} = \emptyset$. We denote

$$y^* = \frac{\rho^2}{|y - z|^2}(y - z) + z, \quad f^*(y) = \left(\frac{\rho}{|y - z|}\right)^{m-2} f(y^*)$$

the Kelvin transformation with respect to $\partial B(z; \rho)$. Denote $\omega = \{y^*; y \in \Omega\} \cup \{z\}$. Then ω is a bounded domain, and $\partial\omega = \{y^*; y \in \partial\Omega\}$. Since $\Omega \cap B(0; r)$ is regular, for each $y \in \partial\Omega$, there exists a neighborhood U of y and a positive $h \in H(U \cap \Omega)$ such that $h(x) \to 0$ as $x \to y$ (see Theorem 4.5.5 and Lemma 4.5.3). So, $h^* \in H(V \cap \Omega)$ is positive, and $h^*(x) \to 0$ as $x \to y^*$ (see Theorem 2.3.1), where V is a neighborhood of y^*. Theorem 4.5.5 gives that ω is regular. According to Theorem 6.4.2, there exists a classical solution of the Dirichlet problem $\Delta v = 0$ in ω, $v = g^*$ on $\partial\omega$. Since $(g^*)^* = g$, the function v^* is a classical solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. (See Theorem 2.3.1.) If $m > 2$, then $v^*(x) \to 0$ as $|x| \to \infty$. If $m = 2$, then $v^*(x) \to v(z)$ as $|x| \to \infty$. The relation (6.8) is a consequence of Corollary 2.4.4 and Proposition 2.4.7.

Let now $m > 2$ and u_∞ be arbitrary. We have proved that there exists a classical solution v of the problem $\Delta v = 0$ in Ω, $v = g - u_\infty$ on $\partial\Omega$, and $v(x) \to 0$ as $|x| \to \infty$. Then $u = v + u_\infty$ is a solution of our problem. The relation (6.7) is a consequence of Corollary 2.4.4.

Let now u and v be two bounded classical solutions of the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If $m > 2$, suppose moreover that $u(x) \to u_\infty$ and $v(x) \to u_\infty$ as $|x| \to \infty$. Denote $w = u - v$. Then w is a bounded classical solution of the problem $\Delta w = 0$ in Ω, $w = 0$ on $\partial\Omega$. If $m > 2$, then $w(x) \to 0$ as $|x| \to \infty$. The relations (6.8) and (6.7) force $w \equiv 0$.

Proposition 6.5.2 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary, $\overline{\Omega} \neq R^m$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Suppose that $\Omega \cap B(0;r)$ is a regular set. (It is true if Ω has Lipschitz boundary.) Let $g \in \mathcal{C}^0(\partial\Omega)$. Let u be a bounded classical solution of the Dirichlet problem $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$. If $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. For $n > r$, denote $\Omega_n = \Omega \cap B(0;n)$. Let u_n be a classical solution of the Dirichlet problem $\Delta u_n = 0$ in Ω_n, $u_n = g$ on $\partial\Omega$, and $u_n = 0$ on $\partial B(0;n)$. If K is a compact subset of $\overline{\Omega}$, then $u_n \to u$ uniformly on K. If $m > 2$, then $u_n \to u$ uniformly on Ω. If $g \geq 0$, then $u_n \uparrow u$.*

Proof If Ω has Lipschitz boundary, then $\Omega \cap B(0;r)$ is regular by Theorems 4.5.7 and 4.5.8.

Suppose first that $g \geq 0$. Then $u, u_n \geq 0$ by Theorem 2.4.2, Corollary 2.4.3, and Proposition 2.4.7. Since $u - u_n \geq 0$ and $u_{n+1} \geq 0$ on $\partial\Omega_n$, Theorem 2.4.2 gives that $u - u_n \geq 0$ and $u_{n+1} - u_n \geq 0$ in Ω_n. Since $\{u_n\}$ is an increasing sequence, there exist $v = \lim u_n$. Since $0 \leq u_n \leq u$, we have $0 \leq v \leq u$. Proposition 2.11.3 gives that $v \in H(\Omega)$. If $y \in \partial\Omega$, then $u(x) \to g(y)$, $u_n(x) \to g(y)$ as $x \to y$. Since $u_n \leq v \leq u$, we deduce that $v(x) \to g|(y)$ as $x \to y$. Thus, v is a bounded classical solution of the problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$. If $m > 2$, then $v(x) \to 0$ as $|x| \to \infty$ because $0 \leq v \leq u$ and $u(x) \to 0$ as $|x| \to \infty$. Theorem 6.5.1 gives that $v = u$. Since $u_n \uparrow u$ and $u_n, u \in \mathcal{C}^0(K)$, Dini's theorem (Theorem 1.2.17) gives that $u_n \to u$ uniformly in K. Suppose now that $m > 2$. Fix $\epsilon > 0$. Then there exists $\rho > 0$ such that $0 \leq u(x) < \epsilon/2$ on $R^m \setminus B(0;\rho)$. We have proved that there exists $n_0 \in N$ such that $|u(x) - u_n(x)| < \epsilon$ for $x \in \overline{\Omega} \cap \overline{B(0;\rho)}$ and $n \geq n_0$. If $x \in \Omega \setminus B(0;\rho)$ and $n \geq n_0$, then $|u(x) - u_n(x)| \leq |u(x)| + |u_n(x)| \leq u(x) + u(x) < \epsilon$, because $0 \leq u_n \leq u$.

Let now g be general. Denote $g_1 = g^+$ and $g_2 = g^-$. The set Ω_n is regular by Theorem 4.5.5. According to Theorem 6.4.2, there exists a classical solution $u_{n,j}$ of the problem $\Delta u_{n,j} = 0$ in Ω_n, $u_{n,j} = g_j$ on $\partial\Omega$, $u_{n,j} = 0$ on $\partial B(0;n)$. There exists a bounded classical solution $u_{0,j}$ of the problem $\Delta u_{0,j} = 0$ in Ω, $u_{0,j} = g_j$. If $m > 2$, we can suppose that $u_{0,j}(x) \to 0$ as $|x| \to \infty$. (See Theorem 6.5.1.) We have proved $u_{n,j} \to u_{o,j}$ uniformly in K. If $m > 2$, then $u_{n,j} \to u_{o,j}$ uniformly in Ω. Clearly, $\Delta(u_{n,1} - u_{n,2}) = 0$ in Ω_n, $u_{n,1} - u_{n,2} = g$ on $\partial\Omega_n$, $u_{n,1} - u_{n,2} = 0$ on $\partial B(0;n)$, $\Delta(u_{0,1} - u_{0,2}) = 0$ in Ω, and $u_{0,1} - u_{0,2} = g$ on $\partial\Omega$. Uniqueness of a solution of the Dirichlet problem (Theorems 6.4.2 and 6.5.1) gives that $u_n = u_{n,1} - u_{n,2}$ and $u = u_{0,1} - u_{0,2}$. Therefore, $u_n \to u$ uniformly on K; $u_n \to u$ uniformly on Ω for $m > 2$.

Theorem 6.5.3 *Let Ω, Ω_n be unbounded open sets in R^m with compact Lipschitz boundary, $m \geq 2$. Suppose that $\Omega_n \subset \Omega$,*

$$\lim_{n\to\infty} \sup_{x \in \partial\Omega} \operatorname{dist}(x, \partial\Omega_n) = 0.$$

Let $g \in \mathcal{C}^0(\overline{\Omega})$, u, u_n be bounded classical solutions of the Dirichlet problems $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, $\Delta u_n = 0$ in Ω_n, and $u_n = g$ on $\partial\Omega_n$. If $m > 2$, suppose *moreover that* $u(x) \to 0$ *and* $u_n(x) \to 0$ *as* $|x| \to \infty$. *Then*

$$\lim_{n\to\infty} \sup_{x\in\Omega_n} |u(x) - u_n(x)| = 0.$$

Proof Fix $\epsilon > 0$. Choose $R > 0$ such that $\partial\Omega \subset B(0; R)$. Then there exists $\delta > 0$ such that $|[g(x)-u(x)]-[g(y)-u(y)]| < \epsilon$ for each $x, y \in \overline{\Omega}\cap\overline{B}(0; R)$ with $|x-y| < \delta$ (see Lemma 1.2.15 and Example 1.2.8). Fix $k \in N$ such that $\operatorname{dist}(x, \partial\Omega) < \delta$ for each $n \geq k$ and $x \in \partial\Omega_n$. Fix $n \geq k$. Since $g - u = 0$ on $\partial\Omega$, we deduce that $|u - g| < \epsilon$ on $\partial\Omega_n$. Hence, $u-u_n \in H(\Omega_n)\cap \mathcal{C}^{0,0}(\overline{\Omega}_n)$, and $|u-u_n| \leq \epsilon$ on $\partial\Omega_n$. If $m > 2$, then, moreover, $u(x) - u_n(x) \to 0$ as $|x| \to \infty$. Corollary 2.4.3 and Proposition 2.4.7 give that $|u - u_n| \leq \epsilon$ on Ω_n.

Theorem 6.5.4 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact boundary,* $\overline{\Omega} \neq R^m$. *Fix* $r > 0$ *such that* $\partial\Omega \subset B(0; r)$. *Suppose that* $\Omega \cap B(0; r)$ *is a regular set. (It is true if* Ω *has Lipschitz boundary.) Let* $g \in \mathcal{C}^0(\partial\Omega)$ *and* f *be a locally Hölder continuous function in* Ω. *Suppose moreover that there exist a bounded open neighborhood* ω *of* $\partial\Omega$ *and* $1 < p < \infty$, $m/2 < q \leq \infty$ *such that* $f \in L^q(\Omega \cap \omega) \cap L^p(\Omega \setminus \omega)$. *Then there exists a classical solution of the problem* $-\Delta u = f$ *in* Ω, $u = g$ *on* $\partial\Omega$.

Proof Put $f = 0$ on $R^m\setminus\Omega$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on a neighborhood of $\partial\Omega$. Put $v_1 = V(f\varphi)$ and $v_2 = V^{m,p}[f(1 - \varphi)]$. Then $v_1, v_2 \in \mathcal{C}^2(\Omega)$ and $-\Delta(v_1 + v_2) = f$ in Ω by Theorem 3.16.1, Propositions 3.18.1 and 3.9.2, and Theorems 3.11.5 and 3.1.5. Moreover, $v_1 + v_2 \in \mathcal{C}^0(R^m)$ by Theorem 3.13.3. According to Theorem 6.5.1, there exists a classical solution v of the problem $\Delta v = 0$ in Ω, $v = h$ on $\partial\Omega$. We have a solution $u = v + v_1 + v_2$ of the problem.

Theorem 6.5.5 *Let* $\Omega = \{x \in R^m; |x| > r\}$. *If* $g \in \mathcal{C}^0(\partial\Omega)$, *then*

$$u(x) = \int_{\partial\Omega} \frac{|x|^2 - r^2}{\mathcal{H}_{m-1}(\partial B(0; 1))|x - y|^m r} g(y)\, d\mathcal{H}_{m-1}(y)$$

is a unique classical solution of the Dirichlet problem for the Laplace equation in Ω *with the boundary condition* g *such that* $u(x) = O(|x|^{2-m})$ *as* $|x| \to \infty$.

Proof Put

$$v(z) = \int_{\partial\Omega} \frac{r^2 - |z|^2}{\mathcal{H}_{m-1}(\partial B(0; 1))r|z - y|^m} g(y)\, d\mathcal{H}_{m-1}(y).$$

Then v is a classical solution of the Dirichlet problem for the Laplace equation in $B(0; r)$ with the boundary condition g by Theorem on the Poisson integral (Theorem 2.5.4). Denote by v^* the Kelvin transform of v with respect to $\partial B(0; r)$. Easy calculation yields that $v^* \in \mathcal{C}^0(\overline{\Omega})$, $v^* = g$ on $\partial\Omega$, and $u(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Since $v^* \in H(\Omega)$ by Theorem 2.3.1, we infer that v^* is a classical solution of the Dirichlet problem for the Laplace equation in Ω with the boundary condition g. By virtue of Lemma 2.3.2

$$v^*(x) = \left(\frac{r}{|x|}\right)^{m-2} \int\limits_{\partial\Omega} \frac{r^2 - r^4|x|^{-2}}{\mathcal{H}_{m-1}(\partial B(0;1)) r |x^* - y|^m} g(y)\, d\mathcal{H}_{m-1}(y)$$

$$= \left(\frac{|x|}{r}\right)^{2} \int\limits_{\partial\Omega} \frac{r^2 - r^4|x|^{-2}}{\mathcal{H}_{m-1}(\partial B(0;1)) |x - y|^m r} g(y)\, d\mathcal{H}_{m-1}(y) = u(x).$$

The uniqueness follows from Theorem 6.3.1.

6.6 Double Layer Potential

We shall look for solutions of boundary value problems in the form of a linear combination of a double layer potential and a single layer potential. For this reason, we need to know further properties of these potentials. In this section, we study a double layer potential with a continuous or Hölder continuous density.

Lemma 6.6.1 *Let Ω be a bounded open set in R^m with boundary of class $C^{1,\alpha}$, $0 < \alpha \leq 1$. Then there is a constant M such that*

$$\int\limits_{B(x;r)\cap\partial\Omega} [|K_\Omega(x,y)| + |K'_\Omega(x,y)|]\, d\mathcal{H}_{m-1}(y) \leq M r^\alpha, \tag{6.9}$$

for each $x \in \partial\Omega$ and $r > 0$.

Proof According to Lemma 1.17.9, there is a positive constant C_1 such that $|K_\Omega(x,y)| + |K'_\Omega(x,y)| \leq C_1 |x - y|^{\alpha+1-m}$ for each $x, y \in \partial\Omega$. According to Corollary 1.17.6, there is a constant C_2 such that $\mathcal{H}_{m-1}(B(x;r) \cap \partial\Omega) \leq C_2 r^{m-1}$ for each $x \in \partial\Omega$ and $r > 0$. Lemma 1.26.1 gives (6.9).

Proposition 6.6.2 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, $0 < \alpha < 1$. Then K_Ω and K'_Ω are bounded linear operators from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\alpha/2}(\partial\Omega)$ and compact bounded linear operators on $\mathcal{C}^0(\partial\Omega)$ and on $L^\infty(\partial\Omega)$.*

Proof There exists a constant C_1 such that $\mathcal{H}_{m-1}(\partial\Omega \cap B(x;r)) \leq C_1 r^{m-1}$ for all $x \in R^m$ and $r > 0$. (See Corollary 1.17.6.) According to Lemma 1.17.9, there exists a constant C_2 such that $|K_\Omega(x,y)| + |K'_\Omega(x,y)| \leq C_2 |x - y|^{\alpha+1-m}$ for each $x, y \in \partial\Omega$. If $f \in L^\infty(\partial\Omega)$ then $K_\Omega f, K'_\Omega f \in \mathcal{C}^0(\partial\Omega)$ by Lemma 1.26.8. Lemma 1.26.1 gives that there exists a constant C_3 such that

$$|K_\Omega f| + |K'_\Omega f| \leq C_3 \|f\|_{L^\infty(\partial\Omega)}.$$

Fix $f \in L^\infty(\partial\Omega)$, $x, z \in \partial\Omega$. Put $M = \|f\|_{L^\infty(\partial\Omega)}$. If $|x-z| \geq 1$, then $|K_\Omega f(x) - K_\Omega f(z)| + |K'_\Omega f(x) - K'_\Omega f(z)| \leq 4C_3 M \leq 4C_3 M|x-z|^{\alpha/2}$. Let now $|x-z| < 1$. Lemma 1.26.1 gives

$$\int\limits_{\partial\Omega\cap B(x;2\sqrt{|x-z|})} |K_\Omega(x,y)f(y)|| \,\mathrm{d}\mathcal{H}_{m-1}(y) \leq \frac{C_1C_2M}{\alpha} 2|x-z|^{\alpha/2},$$

$$\int\limits_{\partial\Omega\cap B(z;3\sqrt{|x-z|})} |K_\Omega(z,y)f(y)|| \,\mathrm{d}\mathcal{H}_{m-1}(y) \leq \frac{C_1C_2M}{\alpha} 3|x-z|^{\alpha/2}.$$

Lemma 2.9.7 gives that there exists a constant C_4 dependent only on m such that $|K_\Omega(x,y) - K_\Omega(z,y)| \leq C_4|x-z||z-y|^{-m}$ for $|z-y| > 2|x-z|$. By virtue of Lemma 1.26.1

$$\int\limits_{\partial\Omega\setminus B(z;2\sqrt{|x-z|})} |K_\Omega(x,y) - K_\Omega(z,y)||f(y)| \,\mathcal{H}_{m-1}(y) \leq \frac{C_1C_4Mm|x-z|}{2\sqrt{|x-z|}}.$$

Thus

$$|K_\Omega f(x) - K_\Omega f(z)| \leq \int\limits_{\partial\Omega\cap B(x;2\sqrt{|x-z|})} |K_\Omega(x,y)f(y)|| \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\cap B(x;2\sqrt{|x-z|})} |K_\Omega(z,y)f(y)|| \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2\sqrt{|x-z|})} |K_\Omega(x,y) - K_\Omega(z,y)||f(y)| \,\mathcal{H}_{m-1}(y) \leq \frac{6C_1C_2M}{\alpha}|x-z|^{\alpha/2}$$

$$+C_1C_4Mm\sqrt{|x-z|} \leq \left[\frac{6C_1C_2M}{\alpha} + C_1C_4Mm\right]|x-z|^{\alpha/2}.$$

So, K_Ω is a bounded linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\alpha/2}(\partial\Omega)$. By the same way, we prove that K'_Ω is a bounded linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\alpha/2}(\partial\Omega)$. Since the identity operator is a compact operator from $\mathcal{C}^{0,\alpha/2}(\partial\Omega)$ to $\mathcal{C}^0(\partial\Omega)$ (see Proposition 1.16.5), K_Ω and K'_Ω are compact operators on $\mathcal{C}^0(\partial\Omega)$ and on $L^\infty(\partial\Omega)$ by Theorem 1.7.4.

Proposition 6.6.3 *Let Ω be an open set in R^m with compact boundary of class $C^{1,\alpha}$, $0 < \alpha \le 1$. Then there exists a constant C such that*

$$|\mathcal{D}^{\Omega} f| \le C\|f\|_{L^\infty(\partial\Omega)} \quad on\ R^m \setminus \partial\Omega$$

for all $f \in L^\infty(\partial\Omega)$. If $f \in \mathcal{C}^0(\partial\Omega)$, then the function $u = \mathcal{D}^{\Omega} f$ in Ω, $u = \frac{1}{2}f + K_\Omega f$ on $\partial\Omega$ is continuous in $\overline{\Omega}$.

Proof According to Proposition 6.6.2, there exists a constant C such that $\|\frac{1}{2}f + K_\Omega f\|_{L^\infty(\partial\Omega)} \le C\|f\|_{L^\infty(\partial\Omega)}$ for all $f \in L^\infty(\partial\Omega)$. Put $g = \frac{1}{2}f + K_\Omega f$. Since $u = \mathcal{D}^{\Omega} f$ is an L^2-solution of the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ (see Theorem 5.3.5) such that $u(x) \to 0$ as $|x| \to \infty$, Corollary 5.16.3 gives $|u(x)| \le \|g\|_{L^\infty(\partial\Omega)} \le C\|f\|_{L^\infty(\partial\Omega)}$.

Let now f be continuous. Then g is continuous by Proposition 6.6.2. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Put $g = u$ in Ω. Then u is an L^2-solution of the problem $\Delta u = 0$ in $B(0; r) \cap \Omega$, $u = g$ on $\partial(B(0; r) \cap \Omega)$. So, u is a PWB-solution of this problem by Theorem 5.16.2. Theorems 4.5.7 and 4.5.8 give that $u \in \mathcal{C}^0(\overline{\Omega})$.

Proposition 6.6.4 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m \ge 2$, $0 < \alpha < 1$. If $0 < \beta < 1$, then K_Ω is a compact linear operator on $\mathcal{C}^{0,\beta}(\partial\Omega)$.*

Proof Let $f \in \mathcal{C}^{0,\beta}(\partial\Omega)$. Put $M = \|f\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$. According Proposition 6.6.2, there exists a constant C_1 depending only on Ω such that $|K_\Omega f| \le C_1 M$ in $\partial\Omega$. According to Lemma 6.6.1, there exists a constant C_2 such that

$$\int\limits_{\Omega\cap B(x;r)} |K_\Omega(x; y)|\, \mathrm{d}\mathcal{H}_{m-1}(y) \le C_2 r^\alpha$$

for each $x \in \partial\Omega$ and $r > 0$.

Fix $x, z \in \partial\Omega$. If $|x - z| \ge 1$, then $|K_\Omega f(x) - K_\Omega f(z)| \le 2M \le 2M|x - z|^c$ for each $c \in (0, 1)$. Suppose that $|x - z| < 1$. Put $g(y) = f(y) - f(x)$. Then $|g(y)| = |f(x) - f(y)| \le M|x - y|^\beta$. Proposition 5.3.2 gives

$$|K_\Omega f(x) - K_\Omega f(z)| = |K_\Omega g(x) - K_\Omega g(z)|.$$

Fix $\gamma \in (\beta/(\beta + \alpha), 1)$. If $y \in B(x; 2|x - z|^\gamma)$, then $|g(y)| \le M|x - y|^\beta \le M[2|x - z|^\gamma]^\beta$. Thus

$$\int\limits_{\Omega\cap B(x;2|x-z|^\gamma)} |K_\Omega(x; y)||g(y)|\, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le M2^\beta|x - z|^{\beta\gamma} \int\limits_{\Omega\cap B(x;2|x-z|^\gamma)} |K_\Omega(x; y)|\, \mathrm{d}\mathcal{H}_{m-1}(y) \le MC_2 2^{\beta+\alpha}|x - z|^{\gamma(\beta+\alpha)},$$

$$\int\limits_{\Omega\cap B(x;2|x-z|^{\gamma})} |K_\Omega(z;y)||g(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq M2^{\beta}|x-z|^{\beta} \int\limits_{\Omega\cap B(z;3|x-z|^{\gamma})} |K_\Omega(z;y)|\,\mathrm{d}\mathcal{H}_{m-1}(y) \leq MC_2 3^{\beta+\alpha}|x-z|^{\gamma(\beta+\alpha)}.$$

According to Lemma 2.9.7, there exists a constant C_3 dependent only on m such that $|K_\Omega(x,y)-K_\Omega(z,y)| \leq C_3|x-z||x-y|^{-m}$ for $|x-y| > 2|x-z|$. Corollary 1.17.6 gives that there exists a constant C_4 dependent only on Ω such that $\mathcal{H}_{m-1}(\partial\Omega\cap B(x;r)) \leq C_4 r^{m-1}$. By virtue of Lemma 1.26.1, we have

$$\int\limits_{\partial\Omega\setminus B(x;2|x-z|^{\gamma})} |K_\Omega(x,y)-K_\Omega(z,y)||g(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq \int\limits_{\partial\Omega\setminus B(x;2|x-z|^{\gamma})} MC_3|x-z||x-y|^{\beta-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq MC_3C_4m(1-\beta)^{-1}|x-z|^{1+\gamma(\beta-1)}.$$

Since $1+\gamma(\beta-1) > \beta$ and $\gamma(\beta+\alpha) > \beta$, there exists $c \in (\beta,1)$ such that $c < 1+(\beta-1)$ and $c < \gamma(\beta+\alpha)$. We have

$$|K_\Omega f(x) - K_\Omega f(y)| \leq \int\limits_{\Omega\cap B(x;2|x-z|^{\gamma})} |K_\Omega(x;y)||g(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\Omega\cap B(x;2|x-z|^{\gamma})} |K_\Omega(z;y)||g(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(x;2|x-z|^{\gamma})} |K_\Omega(x,y)-K_\Omega(z,y)||g(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq M[2C_2 3^{\alpha+\beta} + C_3C_4m(1-\beta)^{-1}]|x-z|^{c}.$$

Thus, K_Ω is a bounded linear operator from $\mathcal{C}^{0,\beta}(\partial\Omega)$ to $\mathcal{C}^{0,c}(\partial\Omega)$. Since the identity operator is a compact operator from $\mathcal{C}^{0,c}(\partial\Omega)$ to $\mathcal{C}^{0,\beta}(\partial\Omega)$ (see Proposition 1.16.5), K_Ω is a compact operator on $\mathcal{C}^{0,\beta}(\partial\Omega)$ (see Theorem 1.7.4).

Lemma 6.6.5 *Let $\Omega \subset R^m$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, and $0 < \beta < 1$. For $f \in \mathcal{C}^{0,\beta}(\partial\Omega)$, put $Tf = \mathcal{D}^{\Omega}f$ on Ω and $Tf = \frac{1}{2}f + \mathcal{D}^{\Omega}f$*

on $\partial\Omega$*. Then* T *is a bounded linear operator from* $\mathcal{C}^{0,\beta}(\partial\Omega)$ *to* $\mathcal{C}^{0,\beta}(\overline{\Omega})$*. If* $f \in \mathcal{C}^{0,\beta}(\partial\Omega)$*, then there exists a constant* C *such that* $|\nabla\mathcal{D}^{\Omega}f(x)| \leq C\,\mathrm{dist}(x,\partial\Omega)^{\beta-1}$*.*

Proof The function Tf is continuous on $\overline{\Omega}$ by Proposition 6.6.3. According to Propositions 6.6.2, 6.6.3 and 6.6.4, there exists a constant C_1 such that $|Tf| \leq C_1\|f\|_{L^\infty(\partial\Omega)}$ and $\|Tf\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \leq C_1\|f\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$. According to Corollary 2.9.6 and Lemma 2.9.7, there exists a constant C_2 such that $|K_\Omega(x,y)| \leq C_2|x-y|^{1-m}$ and $|K_\Omega(x,y) - K_\Omega(z,y)| \leq C_2|x-z||x-y|^{-m}$ for $|x-y| \geq 2|x-z|$. According to Corollary 1.17.6, there exists a constant C_3 such that $\mathcal{H}_{m-1}(B(x;r)\cap\partial\Omega) \leq C_3 r^{m-1}$ for $x \in R^m$ and $r > 0$.

Suppose that $\|f\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \leq 1$. Fix $x, z \in \overline{\Omega}$. If $|x-z| \geq 1$, then $|Tf(x) - Tf(z)| \leq 2C_1 \leq 2C_1|x-z|^\beta$. Suppose that $|x-z| \leq 1$. We can suppose that $x \notin \partial\Omega$. Choose $x_0 \in \partial\Omega$ such that $|x-x_0| = \mathrm{dist}(x,\partial\Omega)$. Define $g(x) = f(x) - f(x_0)$. Then $Tf(x) - Tf(x_0) = Tg(x) - Tg(x_0)$ by Propositions 5.3.1 and 5.3.2. By virtue of Lemma 1.26.1

$$|Tf(x) - Tf(x_0)| \leq \int\limits_{\partial\Omega\cap B(x_0;2|x-x_0|)} |g(y)||K_\Omega(x,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\cap B(x_0;2|x-x_0|)} |g(y)||K_\Omega(x_0,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(x_0;2|x-x_0|)} |g(y)||K_\Omega(x_0,y) - K_\Omega(x,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq \int\limits_{\partial\Omega\cap B(x_0;2|x-x_0|)} |y-x_0|^\beta C_2|x-x_0|^{1-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\cap B(x_0;2|x-x_0|)} |y-x_0|^\beta C_2|y-x_0|^{1-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(x_0;2|x-x_0|)} |y-x_0|^\beta C_2|x-x_0||y-x_0|^{-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq C_2C_3 2^m|x-x_0|^\beta + \frac{2C_2C_3m}{\beta}|x-x_0|^\beta + \frac{C_2C_3m}{1-\beta}|x-x_0|^\beta.$$

So, $|Tf(x) - Tf(x_0)| \leq C_4|x-x_0|^\beta$ with $C_4 = C_2C_3 4^m[1 + \beta^{-1} + (1-\beta)^{-1}]$.

If $z \in \partial\Omega$, then $|z-x| \geq |x-x_0|$ and $|z-x_0| \leq |z-x|+|x-x_0| \leq 2|z-x|$. Thus, $|Tf(x)-Tf(z)| \leq |Tf(x)-Tf(x_0)|+|Tf(x_0)-Tf(x)| \leq C_4|x-x_0|^\beta + C_1|x_0-z|^\beta \leq (C_4+2C_1)|x-z|^\beta$.

Let now $z \notin \partial\Omega$. Choose $z_0 \in \partial\Omega$ such that $|z-z_0| = \operatorname{dist}(z,\partial\Omega)$. If $\operatorname{dist}(x,\partial\Omega) \leq 2|x-z|$, then $|x-x_0| \leq 2|x-z|$, $|z-z_0| \leq |x_0-z| \leq |x_0-x|+|x-z| \leq 3|x-z|$, $|x_0-z_0| \leq |x-x_0|+|x-z|+|z-z_0| \leq 6|x-z|$. Thus, $|Tf(x)-Tf(z)| \leq |Tf(x)-Tf(x_0)|+|Tf(x_0)-Tf(z_0)|+|Tf(z_0)-Tf(z)| \leq C_4|x-x_0|^\beta + C_1|x_0-z_0|^\beta + C_4|z-z_0|^\beta \leq 2C_4|x-z|^\beta + 6C_1|x-z|^\beta + 3C_4|x-z|^\beta$.

Suppose now that $\operatorname{dist}(x,\partial\Omega) > 2|x-z|$. Then $\operatorname{dist}(z,\partial\Omega) > |x-z|$. If $y \in \partial\Omega$, then $|y-x_0| \leq |y-x|+|x-x_0| \leq 2|y-x|$. By virtue of Lemma 1.26.1

$$|Tf(x)-Tf(z)| \leq \int\limits_{\partial\Omega\cap B(x_0;2|x-z|)} |g(y)||K_\Omega(x,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\cap B(x_0;2|x-z|)} |g(y)||K_\Omega(z,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(x_0;2|x-z|)} |g(y)||K_\Omega(z,y)-K_\Omega(x,y)|\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq \int\limits_{\partial\Omega\cap B(x_0;2|x-z|)} |y-x_0|^\beta C_2|x-z|^{1-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\cap B(x_0;2|x-z|)} |y-x_0|^\beta C_2|x-z|^{1-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(x_0;2|x-z|)} |y-x_0|^\beta C_2|x-z||y-x|^{-m}\,\mathrm{d}\mathcal{H}_{m-1}(y) \leq C_2C_3 2^{m+1}|x-z|^\beta$$

$$+ \int\limits_{\partial\Omega\setminus B(x_0;2|x-z|)} |y-x_0|^\beta C_2|x-z|2^m|y-x_0|^{-m}\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq 2C_4|x-z|^\beta + \frac{2^m C_2 C_3 m}{1-\beta}|x-z|^\beta.$$

According to Theorem 2.9.4, there exists a constant C such that

$$|\nabla\mathcal{D}^\Omega f(x)| \leq C\operatorname{dist}(x,\partial\Omega)^{\beta-1}.$$

6.7 Single Layer Potential

In this section, we study single layer potentials with continuous densities on a domain with compact Ljapunov boundary. We show that $\mathcal{S}^\Omega f$ is continuous in R^m and there exists a normal derivative of $\mathcal{S}^\Omega f$ at all points of the boundary.

Proposition 6.7.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $0 \leq \beta < 1$. Then $\mathcal{S}^\Omega$ is a compact bounded linear operator from $L^\infty(\partial\Omega)$ into $\mathcal{C}^{0,\beta}(\partial\Omega)$ (and thus, $\mathcal{S}^\Omega$ is a compact bounded linear operator on $\mathcal{C}^{0,\beta}(\partial\Omega)$). If $m > 2$ or Ω is bounded, put $X = L^\infty(\partial\Omega)$; if $m = 2$ and Ω is unbounded, set $X = \{f \in L^\infty(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1 = 0\}$. Then $\mathcal{S}^\Omega$ is a bounded linear operator from X to $\mathcal{C}^{0,\beta}(\overline{\Omega})$.*

Proof Since there exists a constant C such that $|h_m(x-y)| \leq |x-y|^{3/2-m}$ for $x, y \in \partial\Omega$, the operator $\mathcal{S}^\Omega$ is a compact linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,0}(\partial\Omega)$ by Proposition 1.26.3.

Suppose first that Ω is bounded. If $f \in L^\infty(\partial\Omega)$ and $1 < p < \infty$, then $\mathcal{S}^\Omega f$ is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g = \mathcal{S}^\Omega f \in \mathcal{C}^0(\partial\Omega)$ (see Lemma 5.2.1 and Proposition 5.2.2). For $p = 2$, we obtain that $\mathcal{S}^\Omega f$ is a PWB-solution of this problem (see Theorem 5.16.2). Since $g \in \mathcal{C}^0(\partial\Omega)$, Theorems 4.5.7 and 4.5.8 give that $\mathcal{S}^\Omega f \in \mathcal{C}^0(\overline{\Omega})$. The function $\mathcal{S}^\Omega f \in W^{1,p}(\Omega)$ by Propositions 5.2.2, 5.2.8, and Lemma 1.28.1. For $0 < \beta < 1$ and $p = m/(1-\beta)$, Theorem 1.21.13 gives $\mathcal{S}^\Omega f \in \mathcal{C}^{0,\beta}(\overline{\Omega})$.

Let now Ω be unbounded, $f \in X$, and $0 < \beta < 1$. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $\omega = \Omega \cap B(0;r)$. Put $f = 0$ on $\partial\omega \setminus \partial\Omega$ and $p = m/(1-\beta)$. We have proved that $\mathcal{S}^\Omega f = \mathcal{C}^{0,\beta}(\overline{\omega}) \cap \mathcal{S}^\omega f \in W^{1,p}(\omega)$. According to the behavior of $\mathcal{S}^\Omega f$ and $\nabla \mathcal{S}^\Omega f$ at infinity (see Proposition 3.1.6), we have $\mathcal{S}^\Omega f \in W^{1,p}(\Omega)$. Theorem 1.21.13 and Lemma 1.21.12 yield $\mathcal{S}^\Omega f \in \mathcal{C}^{0,\beta}(\overline{\Omega})$.

If $x \in \Omega$ and $f_k \to f$ in $L^\infty(\partial\Omega)$, then $\mathcal{S}^\Omega f_k(x) \to \mathcal{S}^\Omega f(x)$ by Lebesque's lemma (Lemma 1.13.2). So, $\mathcal{S}^\Omega$ is a closed linear operator from X to $\mathcal{C}^{0,\beta}(\overline{\Omega})$. Closed-graph theorem (Theorem 1.4.15) gives that $\mathcal{S}^\Omega$ is a bounded linear operator from X to $\mathcal{C}^{0,\beta}(\overline{\Omega})$.

Let now $0 \leq \gamma < \beta < 1$. We have proved that $\mathcal{S}^\Omega$ is a bounded linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\beta}(\partial\Omega)$. So, $\mathcal{S}^\Omega$ is a compact linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\gamma}(\partial\Omega)$ by Proposition 1.16.5 and Theorem 1.7.4.

Proposition 6.7.2 *Let $\Omega \subset R^m$, $m \geq 2$, be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, and $f \in \mathcal{C}^0(\partial\Omega)$. If $x \in \partial\Omega$, then*

$$\lim_{t \to 0_-} n^\Omega(x) \cdot \nabla \mathcal{S}^\Omega f(x + t n^\Omega(x)) = \frac{1}{2} f(x) - K'_\Omega f(x), \tag{6.10}$$

$$\lim_{t \to 0_+} n^\Omega(x) \cdot \nabla \mathcal{S}^\Omega f(x + t n^\Omega(x)) = -\frac{1}{2} f(x) - K'_\Omega f(x). \tag{6.11}$$

Proof For $y \in \partial\Omega$ and $z \in R^m \setminus \{y\}$, put $L(z, y) = (n^\Omega(x) - n^\Omega(y)) \cdot \nabla h_m(z - y)$ and

$$Lf(z) = \int_{\partial\Omega} L(z, y) f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Lf is well defined in $(R^m \setminus \partial\Omega) \cup \{x\}$ by Lemma 6.6.1. Fix $\epsilon > 0$. Since $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$, there exist positive constants δ, C_1, and C_2 such that $|n^\Omega(x) - n^\Omega(y)| \le C_1|x-y|^\alpha$ and $|y-(x+tn^\Omega)| \ge C_2|y-x|$ for $y \in B(x;\delta) \cap \partial\Omega$ and $|t| < \delta$ (compare Lemmas 1.17.8 and 1.17.14). For $y \in B(x;\delta) \cap \partial\Omega$ and $|t| < \delta$, we thus have

$$|L(x + tn^\Omega(x), y)| \le \frac{C_1|x-y|^{\alpha+1-m}}{C_2^{m-1}\mathcal{H}_{m-1}(\partial B(0;1))}.$$

Corollary 1.17.6 and Lemma 1.26.1 give that there exists $0 < \rho < \delta$ such that

$$\int_{\partial\Omega \cap B(x;\rho)} |L(x + tn^\Omega(x), y) f(y)|\, \mathrm{d}\mathcal{H}_{m-1}(y) < \epsilon$$

for $|t| < \delta$. According to theorem on continuity of parametrized integrals (Theorem 1.13.6), there exists $s \in (0, \delta)$ such that

$$\int_{\partial\Omega \setminus B(x;\rho)} |L(x + tn^\Omega(x), y) - L(x, y)||f(y)|\, \mathrm{d}\mathcal{H}_{m-1}(y) < \epsilon$$

for $|t| < s$. We have proved that

$$\begin{aligned}\lim_{t\to 0}[n^\Omega(x) \cdot \nabla \mathcal{S}^\Omega f(x + tn^\Omega(x)) - \mathcal{D}^\Omega(x + tn^\Omega(x))] &= \lim_{t\to 0} Lf(x + tn^\Omega(x)) \\ &= Lf(x) = -K'_\Omega f(x) - K_\Omega f(x).\end{aligned}$$

On account of Proposition 6.6.3, we obtain (6.10), (6.11).

6.8 Hölder Continuous Solution of the Dirichlet Problem

In this section, we prove the existence of a Hölder continuous solution of the Dirichlet problem with a Hölder continuous boundary condition and a domain with compact Ljapunov boundary.

Theorem 6.8.1 *Let* $0 < \alpha, \beta, \gamma < 1$, $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$, $p > m/2$, $m \ge 2$, *and* $\beta \le 2 - m/p$. *If* $g \in \mathcal{C}^{0,\beta}(\partial\Omega)$, *and* $f \in$

$\mathcal{C}^{0,\gamma}_{loc}(\Omega) \cap L^p(\Omega)$, then there exists a unique classical solution $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. Moreover,

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C[\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} + \|f\|_{L^p(\Omega)}]$$

where C depends only on Ω, β, and p. If $p > m$, then

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta} |\nabla u(x)| \le M[\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} + \|f\|_{L^p(\Omega)}],$$

where M depends only on Ω, β, and p.

Proof According to Theorem 6.4.2, there exists a unique classical solution of the problem. Define $f = 0$ on $R^m \setminus \Omega$. Then there exists a constant C_1 depending only on Ω, p, and β such that

$$\|Vf\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_1 \|f\|_{L^p(\Omega)}$$

(see Proposition 3.15.2). If $p > m$, then there exists a constant $\tilde{C}_1$ depending only on Ω, p and β such that

$$\|\nabla Vf\|_{\mathcal{C}^{0}(\overline{\Omega})} \le \tilde{C}_1 \|f\|_{L^p(\Omega)}$$

(see Proposition 3.15.1). Denote $v = u + Vf$. Then v is a classical solution of the problem $\Delta v = 0$ in Ω, $v = g + Vf$ on $\partial\Omega$ by Theorems 3.1.5 and 3.14.2.

Suppose first that $m > 2$. Then $v = \mathcal{D}^{\Omega}\psi + \mathcal{S}^{\Omega}\psi$, where $\psi = (\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega})^{-1}(g + Vf)$ in $L^2(\partial\Omega)$ (see Theorem 5.15.2). The operator $\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega}$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\beta}(\overline{\Omega})$ by Propositions 6.7.1 and 6.6.4 and Theorem 1.8.3. So, it is an isomorphism on $\mathcal{C}^{0,\beta}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{0,\beta}(\partial\Omega)$ and

$$\|\psi\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \le C_2 \|g + Vf\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \tag{6.12}$$

where C_2 depends only on Ω and β.

Let now $m = 2$. Then $v = \mathcal{D}^{\Omega}(I - P)\psi + \mathcal{S}^{\Omega}(I - P)\psi + P\psi$, where $P\psi = \oint_{\partial\Omega} \psi \, d\mathcal{H}_{m-1}$ and $\psi = [(\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega})(I - P) + P]^{-1}(g + Vf)$ in $L^2(\partial\Omega)$ (see Theorem 5.15.3). The operator $(\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega})(I - P) + P$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\beta}(\overline{\Omega})$ by Propositions 6.7.1 and 6.6.4 and Theorem 1.8.3. So, it is an isomorphism on $\mathcal{C}^{0,\beta}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{0,\beta}(\partial\Omega)$, and (6.12) holds with C_2 depending only on Ω and β.

Proposition 6.7.1 and Lemma 6.6.5 give that $v \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|v\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_3 \|g + Vf\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$$

where C_3 depends only on Ω and β. We have proved that there exists C_4 such that

$$\|v\|_{\mathcal{C}^{0,\beta}(\Omega)} \leq C_4[\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} + \|f\|_{L^p(\Omega)}].$$

Theorem 2.9.4 gives

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla v(x)| \leq mC_4[\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} + \|f\|_{L^p(\Omega)}].$$

Theorem 6.8.2 *Let* $0 < \alpha, \beta < 1$, *and* $\Omega \subset R^m$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $m > 2$. *If* $g \in \mathcal{C}^{0,\beta}(\partial\Omega)$, *then there exists a unique solution* $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ *of the Dirichlet problem* $\Delta u = 0$ *in* Ω, $u = g$ *on* $\partial\Omega$ *such that* $u(x) \to 0$ *as* $|x| \to \infty$. *Moreover,*

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \leq C\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)},$$

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla u(x)| \leq C\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)},$$

where C *depends only on* Ω *and* β.

Proof According to Theorem 6.5.1, there exists a unique classical solution u of the problem. We have $u = \mathcal{D}^{\Omega}\psi + \mathcal{S}^{\Omega}\psi$, where $\psi = (\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega})^{-1}g$ in $L^2(\partial\Omega)$ (see Theorem 5.15.4). The operator $\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega}$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\beta}(\overline{\Omega})$ by Propositions 6.7.1 and 6.6.4 and Theorem 1.8.3. So, it is an isomorphism on $\mathcal{C}^{0,\beta}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{0,\beta}(\partial\Omega)$ and

$$\|\psi\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \leq C_1\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$$

where C_1 depends only on Ω and β. Proposition 6.7.1 and Lemma 6.6.5 give that $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|u\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \leq C_2\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$$

where C_2 depends only on Ω and β. Theorem 2.9.4 gives

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla u(x)| \leq mC_2\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}.$$

Theorem 6.8.3 *Let* $0 < \alpha, \beta < 1$, *and* $\Omega \subset R^2$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$. *If* $g \in \mathcal{C}^{0,\beta}(\partial\Omega)$, *then there exists a unique solution* $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ *of the Dirichlet problem* $\Delta u = 0$ *in* Ω, $u = g$ *on* $\partial\Omega$.

Moreover,

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)},$$

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla u(x)| \le C\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)},$$

where C depends only on Ω and β.

Proof According to Theorem 6.5.1, there exists a unique bounded classical solution of the problem. We have

$$u = \mathcal{D}^{\Omega}(I-P)\psi + \mathcal{S}^{\Omega}(I-P)\psi + P\psi$$

where

$$P\varphi = \oint_{\partial\Omega} \varphi \, \mathrm{d}\mathcal{H}_1,$$

$$T = \left[\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega}\right](I-P) + P,$$

$\psi = T^{-1}g$ in $L^2(\partial\Omega)$ (see Theorem 5.15.3). The operator T is a Fredholm operator with index 0 on $\mathcal{C}^{0,\beta}(\overline{\Omega})$ by Propositions 6.7.1 and 6.6.4 and Theorem 1.8.3. So, it is an isomorphism on $\mathcal{C}^{0,\beta}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{0,\beta}(\partial\Omega)$ and

$$\|\psi\|_{\mathcal{C}^{0,\beta}(\partial\Omega)} \le C_1\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$$

where C_1 depends only on Ω and β. Proposition 6.7.1 and Lemma 6.6.5 give that $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|u\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_2\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}$$

where C_2 depends only on Ω and β. Theorem 2.9.4 gives

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla u(x)| \le mC_2\|g\|_{\mathcal{C}^{0,\beta}(\partial\Omega)}.$$

6.9 Robin Problem and Neumann Problem

In this section, we study bounded classical solutions of the Neumann problem and the Robin problem for the Laplace equation on a domain with compact Ljapunov boundary. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness. We show that these solutions are in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$.

Theorem 6.9.1 *Let* $\Omega \subset R^m$ *be a domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$, $m > 2$, $0 < \alpha < 1$, $g, h \in \mathcal{C}^0(\partial\Omega)$, *and* $h > 0$. *Then the operator*

$$T\varphi = (1/2)\varphi - K'_\Omega\varphi + h\mathcal{S}^\Omega\varphi \tag{6.13}$$

is an isomorphism on $\mathcal{C}^0(\partial\Omega)$.

- *If* Ω *is bounded, then there exists a unique classical solution* u *of the Robin problem for the Laplace equation*

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \quad \text{on } \partial\Omega. \tag{6.14}$$

- *If* Ω *is unbounded, then there exists a unique classical solution* u *of the Robin problem for the Laplace equation (6.14) such that* $u(x) \to 0$ *as* $|x| \to \infty$.

Moreover, $u = \mathcal{S}^\Omega\varphi$ *where* $\varphi = T^{-1}g$ *and*

$$\inf_{x\in\partial\Omega} \frac{g(x)}{h(x)} \le \inf_{x\in\Omega} u(x) \le \sup_{x\in\Omega} u(x) \le \sup_{x\in\partial\Omega} \frac{g(x)}{h(x)}, \tag{6.15}$$

$$\operatorname{dist}(x, \partial\Omega)|\nabla u(x)| \le m \sup_{y\in\partial\Omega} \frac{|g(y)|}{h(y)}, \qquad x \in \Omega. \tag{6.16}$$

Proof The uniqueness of a solution follows from Theorem 6.3.1.

T is an isomorphism on $L^2(\partial\Omega)$ by Theorem 5.10.4. The operator $T - (1/2)I$ is compact on $\mathcal{C}^0(\partial\Omega)$ (see Theorem 1.7.4 and Propositions 6.6.2 and 6.7.1). Theorem 1.7.7 gives that T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. If $\varphi = T^{-1}g$, then $\mathcal{S}^\Omega\varphi$ is a solution of the problem by Propositions 6.7.2 and 6.7.1.

If u is a solution of the problem (6.14), then $h^{-1}(\partial u/\partial n) + u = g/h$ on $\partial\Omega$. Corollary 6.2.4 gives (6.15) and (6.16).

Theorem 6.9.2 *Let* $\Omega \subset R^2$ *be a domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, *and* $h \in \mathcal{C}^0(\partial\Omega)$, $h > 0$. *Denote*

$$P\varphi = \oint_{\partial\Omega} \varphi \, d\mathcal{H}_1, \tag{6.17}$$

$$T\varphi = \frac{1}{2}(I - P)\varphi - K'_\Omega(I - P)\varphi + h[\mathcal{S}^\Omega(I - P)\varphi + P\varphi]. \tag{6.18}$$

Then the operator T *is an isomorphism on* $\mathcal{C}^0(\partial\Omega)$. *If* $g \in \mathcal{C}^0(\partial\Omega)$, *then there exists a unique bounded classical solution* u *of the Robin problem for the Laplace* *equation (6.14). This solution is given by the formula*

$$u = \mathcal{S}^{\Omega}(I - P)\varphi + P\varphi, \tag{6.19}$$

where $\varphi = T^{-1}g$, *and the estimates (6.15) and (6.16) hold.*

Proof The uniqueness of a solution follows from Theorem 6.3.1.

T is an isomorphism on $L^2(\partial\Omega)$ by Theorem 5.10.4. The operator $T - (1/2)I$ is compact on $\mathcal{C}^0(\partial\Omega)$ (see Theorem 1.7.4 and Propositions 6.6.2 and 6.7.1). Theorem 1.7.7 gives that T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. If $\varphi = T^{-1}g$, then u given by (6.19) is a solution of the problem by Propositions 6.7.2 and 6.7.1.

If u is a solution of the problem (6.14), then $h^{-1}(\partial u/\partial n) + u = g/h$ on $\partial\Omega$. Corollary 6.2.4 gives (6.15) and (6.16).

Corollary 6.9.3 *Let* $\Omega \subset R^m$ *be a domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$, $m \geq 2$, $0 < \alpha < 1$, *and* $g, h \in \mathcal{C}^0(\partial\Omega)$. *If* $1 < p < \infty$ *and u is bounded classical solution of the Robin problem (6.14), then u is an* L^p*-solution of the problem (6.14).*

Proof Define $\tilde{h} = h + |h| + 1$ and $\tilde{g} = g + (|h| + 1)u$ on $\partial\Omega$. Then $\tilde{h} > 0$ and u is classical solution of the problem $\Delta u = 0$ in Ω and $\partial u/\partial n + \tilde{h}u = \tilde{g}$ on $\partial\Omega$. If Ω is bounded or $m > 2$, then there exists a constant c and $\varphi \in \mathcal{C}^0(\partial\Omega)$ such that $u = \mathcal{S}^{\Omega}\varphi + c$ (see Theorems 6.9.1 and 6.9.2). Let now $m > 2$ and Ω is unbounded. According to Proposition 2.17.1, there exists a constant c such that $u(x) \to c$ as $|x| \to \infty$. Since $v = u - c$ is a solution of the problem $\Delta v = 0$ in Ω, and $\partial v/\partial n + \tilde{h}v = \tilde{g} - c\tilde{h}$ on $\partial\Omega$, $v(x) \to 0$ as $|x| \to \infty$, there exists $\varphi \in \mathcal{C}^0(\partial\Omega)$ such that $v = \mathcal{S}^{\Omega}\varphi$ (see Theorem 6.9.1). Thus, $u = \mathcal{S}^{\Omega}\varphi + c$. Propositions 5.2.2 and 5.2.8, Theorems 5.4.7 and 5.4.8, and Proposition 6.7.2 give that u is an L^p solution of the problem (6.14).

Theorem 6.9.4 *Let* $\Omega \subset R^m$ *be a domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$, $m > 2$, $0 < \alpha < 1$, $0 \leq \beta < 1$, $g, h \in \mathcal{C}^0(\partial\Omega)$, *and* $h \geq 0$. *Suppose that* Ω *is unbounded or* $h \not\equiv 0$. *Then the operator T given by (6.13) is an isomorphism on* $\mathcal{C}^0(\partial\Omega)$.

- *If* Ω *is bounded, then there exists a unique classical solution u of the Robin problem for the Laplace equation (6.14).*
- *If* Ω *is unbounded, then there exists a unique classical solution u of the Robin problem for the Laplace equation (6.14) such that* $u(x) \to 0$ *as* $|x| \to \infty$.

Moreover, $u = \mathcal{S}^{\Omega}\varphi$ *where* $\varphi = T^{-1}g$, $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$, *and*

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \leq C\|g\|_{\mathcal{C}^0(\partial\Omega)}, \tag{6.20}$$

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta}|\nabla u(x)| \leq C\|g\|_{\mathcal{C}^0(\partial\Omega)}, \tag{6.21}$$

with a constant C dependent on Ω, *h, and* β *only.*

Proof T is an isomorphism on $L^2(\partial\Omega)$ by Theorems 5.10.4 and 5.9.11. The operator $T-(1/2)I$ is compact on $\mathcal{C}^0(\partial\Omega)$ (see Theorem 1.7.4 and Propositions 6.6.2 and 6.7.1). Theorem 1.7.7 gives that T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. If $\varphi = T^{-1}g$, then $\mathcal{S}^\Omega\varphi$ is a solution of the problem by Propositions 6.7.2 and 6.7.1.

Let u be a solution of the problem with $g = 0$. If Ω is unbounded, suppose that $u(x) \to 0$ as $|x| \to \infty$. Then u is an L^2-solution of the problem by Corollary 6.9.3. So, $u \equiv 0$ by Proposition 5.10.1 and Theorem 5.9.11.

There exists a constant C_1 such that $\|\mathcal{S}^\Omega\psi\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_1\|\psi\|_{\mathcal{C}^0(\partial\Omega)}$ (see Proposition 6.7.1). Put $\varphi = T^{-1}g$ and $u = \mathcal{S}^\Omega\varphi$. Then

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C_1\|T^{-1}\|\,\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Theorems 2.9.2 and 2.9.4 give

$$\sup_{x\in\Omega} \operatorname{dist}(x,\partial\Omega)^{1-\beta}|\nabla u(x)| \le mC_1\|T^{-1}\|\,\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Theorem 6.9.5 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0<\alpha<1$, $0\le\beta<1$, $h\in\mathcal{C}^0(\partial\Omega)$, $h\ge 0$, and $h\not\equiv 0$. Then the operator T given by (6.18) is an isomorphism in $\mathcal{C}^0(\partial\Omega)$. If $g\in\mathcal{C}^0(\partial\Omega)$, then there exists a unique bounded classical solution u of the Robin problem for the Laplace equation (6.14). This solution is given by the formula (6.19), where $\varphi = T^{-1}g$. Moreover, $u\in\mathcal{C}^{0,\beta}(\overline{\Omega})$, and the estimates (6.20) and (6.21) hold with a constant C dependent on Ω, h, and β only.*

Proof Let u be a bounded solution of the problem with $g = 0$. Then u is an L^2-solution of the problem by Corollary 6.9.3. So, $u\equiv 0$ by Proposition 5.10.1.

T is an isomorphism on $L^2(\partial\Omega)$ by Theorem 5.10.4. The operator $T-(1/2)I$ is compact on $\mathcal{C}^0(\partial\Omega)$ (see Theorem 1.7.4 and Propositions 6.6.2 and 6.7.1). Theorem 1.7.7 gives that T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. If $\varphi = T^{-1}g$, then $\mathcal{S}^\Omega\varphi$ is a solution of the problem by Propositions 6.7.2 and 6.7.1.

Denote $X = \{\psi\in\mathcal{C}^0(\partial\Omega); \int_{\partial\Omega}\psi\,\mathrm{d}\mathcal{H}_1 = 0\}$. Then there exists a constant C_1 such that $\|\mathcal{S}^\Omega\psi\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_1\|\psi\|_{\mathcal{C}^0(\partial\Omega)}$ for $\psi\in X$ (see Proposition 6.7.1). Let $\varphi = T^{-1}g$ and u be given by (6.19). Then

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le (C_1\|I-P\| + \|P\|)\|T^{-1}\|\|g\|_{\mathcal{C}^0(\partial\Omega)},$$

where P is given by (6.17). Theorems 2.9.2 and 2.9.4 give (6.21) with $C = m(C_1\|I-P\| + \|P\|)\|T^{-1}\|$.

Theorem 6.9.6 *Let Ω be a bounded domain in R^m with boundary of class $\mathcal{C}^{1,\alpha}$, $0<\alpha<1$, $0\le\beta<1$, and $m\ge 2$. Denote by X the set of all $g\in\mathcal{C}^0(\partial\Omega)$ satisfying*

$$\int_{\partial\Omega} g\,\mathrm{d}\mathcal{H}_{m-1} = 0 \tag{6.22}$$

and by T the restriction of $(1/2)I - K'_\Omega$ onto X. Then T is an isomorphism. If $g \in \mathcal{C}^0(\partial\Omega)$, then there exists a classical solution of the Neumann problem for the Laplace equation

$$\Delta u = 0 \quad in\ \Omega, \qquad \frac{\partial u}{\partial n} = g \quad on\ \partial\Omega \tag{6.23}$$

if and only if g satisfies (6.22). The general form of this solution is $u = \mathcal{S}^\Omega(T^{-1}g) + c$, where c is an arbitrary constant. If u is a solution of the problem (6.23), then $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$, and the estimate (6.21) holds with a constant C dependent only on Ω and β.

Proof The operator K'_Ω is a compact operator on X by Lemma 5.9.6 and Proposition 6.6.2. The operator T is one-to-one by Proposition 5.9.4. So, the operator T is an isomorphism on X by Theorem 1.7.7.

Let now u be a classical solution of the Neumann problem (6.23). Then u is an L^2-solution of the problem by Corollary 6.9.3. Lemma 5.9.1 forces $g \in X$. If $g = 0$, then u is constant by Lemma 5.9.2.

If $g \in X$, put $v = \mathcal{S}^\Omega(T^{-1}g)$. If c is a constant, then $u = v + c$ is a classical solution of the Neumann problem (6.23) by Propositions 6.7.1 and 6.7.2. There exists a constant C_1 such that $\|\mathcal{S}^\Omega\psi\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_1\|\psi\|_{\mathcal{C}^0(\partial\Omega)}$ (see Proposition 6.7.1). Thus, $v \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|v\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C_1\|T^{-1}\|\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Since $\nabla u = \nabla v$, Theorems 2.9.2 and 2.9.4 give

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta}|\nabla u(x)| \le mC_1\|T^{-1}\|\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Theorem 6.9.7 *Let $\Omega \subset R^2$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \alpha < 1$. Denote by X the set of all $g \in \mathcal{C}^0(\partial\Omega)$ satisfying (6.22). Then $(1/2)I - K'_\Omega$ is an isomorphism on $\mathcal{C}^0(\partial\Omega)$ and on X. If $g \in \mathcal{C}^0(\partial\Omega)$, then there exists a classical solution u of the Neumann problem for the Laplace equation (6.23) such that $u(x) = o(|x|)$ as $|x| \to \infty$. The general form of this solution is $\mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + c$, where c is an arbitrary constant. This solution u is bounded if and only if $g \in X$. Suppose that $g \in X$. Fix a constant c. Then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + c$ is a unique classical solution of the problem (6.23) such that $u(x) \to c$ as $|x| \to \infty$. Let $c = 0$ and $0 \le \beta < 1$. Then $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$, and the estimates (6.20) and (6.21) hold with a constant C dependent only on Ω and β.*

Proof The operator K'_Ω is compact by Proposition 6.6.2. Lemma 5.9.6 gives that $\varphi/2 - K'_\Omega\varphi \in X$ if and only if $\varphi \in X$. If $\varphi/2 - K'_\Omega\varphi = 0$, then $\varphi \in X$, and Proposition 5.9.4 gives that $\varphi = 0$. So, $(1/2)I - K'_\Omega$ is an isomorphism on $\mathcal{C}^0(\partial\Omega)$ and on X by Theorem 1.7.7.

If g in $\mathcal{C}^0(\partial\Omega)$ and c is a constant, then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + c$ is a classical solution of the Neumann problem (6.23) by Propositions 6.7.1 and 6.7.2. Moreover, $u(x) = o(|x|)$ as $|x| \to \infty$. If $g \in X$, then u is bounded by Proposition 3.1.6.

If u is a bounded classical solution of the problem (6.23), then u is an L^2-solution of the problem by Corollary 6.9.3. Theorem 5.9.11 gives that $g \in X$. Moreover, $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g + c$ is a unique classical solution of the problem (6.23) such that $u(x) \to c$ as $|x| \to \infty$.

Let now $g \equiv 0$. Suppose that u is a classical solution of the Neumann problem (6.23) such that $u(x) = o(|x|)$ as $|x| \to \infty$. Fix $z \in R^2 \setminus \overline{\Omega}$. Then there exists a constant c_1 such that $v(x) = u(x) - c_1 h_2(x-z)$ is bounded (see Proposition 2.17.6). Put $\psi = c_1/\mathcal{H}_1(\partial\Omega)$ on $\partial\Omega$. Then $\mathcal{S}^\Omega\psi(x) - c_1 h_2(x-z)$ is bounded in a neighborhood of the infinity. So, $v = u - \mathcal{S}^\Omega\psi(x)$ is bounded in a neighborhood of the infinity. There exists $f \in X$ such that v is a bounded classical solution of the problem $\Delta v = 0$ in Ω and $\partial v/\partial n = f$ (see Propositions 6.7.1 and 6.7.2). The function v is a bounded classical solution of the problem $\Delta v = 0$ in Ω and $\partial v/\partial n + v = \tilde f$ with $\tilde f = f + v$. According Theorem 6.9.2, there exists $\varphi \in X$ and a constant c_2 such that $v = \mathcal{S}^\Omega\varphi + c_2$. So, $u = \mathcal{S}^\Omega(\varphi + \psi) + c_2$. So, $[(1/2)I - K'_\Omega](\varphi + \psi) = 0$ (compare Proposition 6.7.2). Since $[(1/2)I - K'_\Omega]$ is an isomorphism, we infer that $\varphi + \psi = 0$. Thus, $u = \mathcal{S}^\Omega(\varphi + \psi) + c_2 = c_2$.

Let $g \in X$ and u be a classical solution of the problem (6.23) such that $u(x) \to 0$ as $|x| \to \infty$. Then $u = \mathcal{S}^\Omega[(1/2)I - K'_\Omega]^{-1}g$. There exists a constant C_1 such that $\|\mathcal{S}^\Omega\psi\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_1\|\psi\|_{\mathcal{C}^0(\partial\Omega)}$ for $\psi \in X$ (see Proposition 6.7.1). Since $[(1/2)I - K'_\Omega]^{-1}g \in X$, we have $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C_1\|[(1/2)I - K'_\Omega]^{-1}\|\;\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Theorems 2.9.2 and 2.9.4 give

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta}|\nabla u(x)| \le mC_1\|[(1/2)I - K'_\Omega]^{-1}\|\|g\|_{\mathcal{C}^0(\partial\Omega)}.$$

Theorem 6.9.8 *Let Ω be a bounded domain in R^m with boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \alpha < 1$. Suppose that $g, h \in \mathcal{C}^0(\partial\Omega)$, $h \ge 0$, $h \not\equiv 0$, and $f \in \mathcal{C}^{0,\alpha}_{loc}(\Omega) \cap L^p(\Omega)$ with $p > m$. Then there exists a unique classical solution u of the Robin problem for the Poisson equation*

$$-\Delta u = f \quad in\ \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \quad on\ \partial\Omega. \tag{6.24}$$

If $0 \le \beta < 1$, then $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ and

$$\|u\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C[\|g\|_{\mathcal{C}^0(\partial\Omega)} + \|f\|_{L^p(\Omega)}], \tag{6.25}$$

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta}|\nabla u(x)| \le C[\|g\|_{\mathcal{C}^0(\partial\Omega)} + \|f\|_{L^p(\Omega)}], \tag{6.26}$$

where a constant C depends on Ω, h, p, and β only.

Proof The uniqueness of a solution follows from Theorems 6.9.4 and 6.9.5.

Put $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(R^m)$ by Theorem 3.14.2 and Proposition 3.15.1. Moreover, $-\Delta Vf = f$ (see Theorem 3.1.5). Put $b = g - \partial Vf/\partial n - hVf$. According to Theorems 6.9.4 and 6.9.5, there exists a classical solution v of the problem $-\Delta v = 0$ in Ω, $\partial v/\partial n + hv = b$ on $\partial\Omega$. Clearly, $u = v + Vf$ is a classical solution of the problem (6.24).

According to Proposition 3.15.1, there exists a constant C_1 depending only on Ω and p such that

$$|Vf(x)| + |\nabla Vf(x)| \le C_1 \|f\|_{L^p(\Omega)}, \quad x \in \overline{\Omega}.$$

So, there exists a constant C_2 dependent only on Ω, p, and β such that

$$\|Vf\|_{\mathcal{C}^{0,\beta}(\overline{\Omega})} \le C_2 \|f\|_{L^p(\Omega)}, \quad \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta} |\nabla Vf(x)| \le C_2 \|f\|_{L^p(\Omega)}.$$

Theorems 6.9.4 and 6.9.5 give that there exists a constant C_3 depending only on Ω, h, and β such that

$$\|v\|_{\mathcal{C}^{0,\beta}(\Omega)} \le C_3 \|b\|_{\mathcal{C}^0(\partial\Omega)},$$

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta} |\nabla v(x)| \le C_3 \|b\|_{\mathcal{C}^0(\partial\Omega)}.$$

Since

$$\|b\|_{\mathcal{C}^0(\partial\Omega)} \le \|g\|_{\mathcal{C}^0(\partial\Omega)} + C_1 \|f\|_{L^p(\Omega)} + \|h\|_{\mathcal{C}^0(\partial\Omega)} C_1 \|f\|_{L^p(\Omega)}$$

we obtain the estimates (6.25) and (6.26).

Theorem 6.9.9 *Let Ω be a bounded domain in R^m with boundary of class $\mathcal{C}^{1,\alpha}$, $f \in \mathcal{C}^{0,\alpha}_{loc}(\Omega) \cap L^p(\Omega)$, $g \in \mathcal{C}^0(\partial\Omega)$, $2 \le m < p$, and $0 < \alpha < 1$. Then there exists a classical solution u of the Neumann problem for the Poisson equation*

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \tag{6.27}$$

if and only if

$$\int_{\partial\Omega} g \, d\mathcal{H}_{m-1} + \int_{\Omega} f \, d\mathcal{H}_m = 0. \tag{6.28}$$

If v is another solution, then $u - v$ is constant. If $0 \le \beta < 1$, then $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$, and (6.26) holds with a constant C dependent on Ω, p, and β only.

Proof Put $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(R^m)$ by Theorem 3.14.2 and Proposition 3.15.1. Moreover, $-\Delta Vf = f$ (see Theorem 3.1.5). Let Ω_j be a sequence of domains from Lemma 1.17.13. Then the Green theorem (Theorem 1.19.1) and Lebesgue lemma (Lemma 1.13.2) give

$$\int\limits_{\partial\Omega} \frac{\partial Vf}{\partial n}\, d\mathcal{H}_{m-1} = \lim_{j\to\infty} \int\limits_{\partial\Omega_j} \frac{\partial Vf}{\partial n}\, d\mathcal{H}_{m-1} = \lim_{j\to\infty} \int\limits_{\Omega_j} \Delta Vf \, d\mathcal{H}_m = -\int\limits_{\Omega} f \, d\mathcal{H}_m.$$

Put $h = g - \partial Vf/\partial n$ and $w = u - Vf$. Then u is a classical solution of the problem (6.27) if and only if w is a classical solution of the problem $\Delta w = 0$ in Ω, $\partial w/\partial n = h$ on $\partial\Omega$. According to Theorem 6.9.6, there exists a classical solution w of this problem if and only if $\int_{\partial\Omega} h\, \mathcal{H}_{m-1} = 0$. Thus, there is a solution of the problem (6.27) if and only if

$$0 = \int\limits_{\partial\Omega} h\, d\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} g\, d\mathcal{H}_{m-1} - \int\limits_{\partial\Omega} \frac{\partial Vf}{\partial n}\, d\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} g\, d\mathcal{H}_{m-1} + \int\limits_{\Omega} f\, d\mathcal{H}_m.$$

If v is another solution of the problem (6.27), then $u - v$ is constant by Theorem 6.9.6.

According to Proposition 3.15.1, there exists a constant C_1 depending only on Ω and p such that

$$|\nabla Vf(x)| \le C_1 \|f\|_{L^p(\Omega)}, \quad x \in \overline{\Omega}.$$

So, there exists a constant C_2 dependent only on Ω, p, and β such that

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta} |\nabla Vf(x)| \le C_2 \|f\|_{L^p(\Omega)}.$$

According to Theorem 6.9.6, there exists a constant C_3 dependent only on Ω and β, such that

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\beta} |\nabla w(x)| \le C_3 \|h\|_{\mathcal{C}^0(\partial\Omega)}.$$

Since

$$\|h\|_{\mathcal{C}^0(\partial\Omega)} \le \|g\|_{\mathcal{C}^0(\partial\Omega)} + C_1 \|f\|_{L^p(\Omega)}$$

we obtain (6.26).

6.10 The Third Problem for Hölder Continuous Boundary Data

In this section, we study the Neumann problem and the Robin problem for the Laplace equation with a boundary condition from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ for a domain Ω with compact Ljapunov boundary. We find a necessary and sufficient condition for the existence of a solution from $\mathcal{C}^{1,\lambda}(\overline{\Omega})$.

Proposition 6.10.1 *Let Ω be a domain in R^m with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, and $m \geq 2$. Then $K'_\Omega + K_\Omega$ is a bounded linear operator from $L^\infty(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$.*

Proof According to Lemma 6.6.1, there exists a constant M_1 such that

$$\int\limits_{B(x;r)\cap\partial\Omega} [|K_\Omega(x,y)| + |K'_\Omega(x,y)|] \, d\mathcal{H}_{m-1}(y) \leq M_1 r^\alpha$$

for each $x \in \partial\Omega$ and $r > 0$. In particular, $K_\Omega + K'_\Omega$ is a bounded linear operator on $L^\infty(\partial\Omega)$.

There exists a constant M_2 such that $\mathcal{H}_{m-1}(B(x;r) \cap \partial\Omega) \leq M_2 r^{m-1}$ for each $x \in \partial\Omega$ and $r > 0$ (see Corollary 1.17.6). There exists a constant M_3 such that $|n^\Omega(x) - n^\Omega(y)| \leq M_3|x-y|^\alpha$ (see Lemma 1.17.8). According to Lemma 2.9.7 and Corollary 2.9.6, there exists a constant M_4 such that if $x, y, z \in R^m$, and $|z-y| > 2|x-z|$, then $|\nabla h_m(x-y) - \nabla h_m(z-y)| \leq M_4|x-z||z-y|^{-m}$, $|\nabla h_m(x-y)| \leq M_4|x-y|^{1-m}$. Let $\varphi \in L^\infty(\partial\Omega)$, $|\varphi| \leq 1$. Fix $x, z \in \partial\Omega$. If $|x-z| \geq 1$, then $|K_\Omega\varphi + K'_\Omega\varphi| \leq M_1(\operatorname{diam}\partial\Omega)^\alpha \leq M_1(\operatorname{diam}\partial\Omega)^\alpha|x-z|^\alpha$. Suppose now that $|x-z| < 1$. Lemma 1.26.1 gives

$$|[K_\Omega + K'_\Omega]\varphi(x) - [K_\Omega + K'_\Omega]\varphi(z)| \leq \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [|K_\Omega(z,y)|$$

$$+|K'_\Omega(z,y)|] \, d\mathcal{H}_{m-1}(y) + \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [|K_\Omega(x,y)| + |K'_\Omega(x,y)|] \, d\mathcal{H}_{m-1}(y)$$

$$\int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |[K_\Omega(x,y)| + K'_\Omega(x,y)] - [K_\Omega(z,y) + K'_\Omega(z,y)]| \, d\mathcal{H}_{m-1}(y)$$

$$\leq M_1(2|x-z|)^\alpha + \int\limits_{B(x;3|x-z|)\cap\partial\Omega} [|K_\Omega(x,y)| + |K'_\Omega(x,y)|] \, d\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega\setminus B(z;2|x-z|)} |(n^\Omega(x) - n^\Omega(y)) \cdot \nabla h_m(x-y) - (n^\Omega(z) - n^\Omega(y)) \cdot \nabla h_m(z-y)|$$

$$d\mathcal{H}_{m-1}(y) \leq 2M_1|x-z|^\alpha + M_1(3|x-z|)^\alpha + \int_{\partial\Omega\setminus B(z;2|x-z|)} [|n^\Omega(x)$$

$$-n^\Omega(z)||\nabla h_m(x-y)| + |n^\Omega(z) - n^\Omega(y)||\nabla h_m(x-y) - \nabla h_m(z-y)| \, d\mathcal{H}_{m-1}(y)$$

$$\leq 5M_1|x-z|^\alpha + M_3|x-z|^\alpha \int_{\partial\Omega\setminus B(x;|x-z|)} M_4|x-y|^{1-m} \, d\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega\setminus B(z;2|x-z|)} M_3M_4 \frac{|x-z|}{|z-y|^{m-\alpha}} \, d\mathcal{H}_{m-1}(y) \leq 5M_1|x-z|^\lambda$$

$$+M_3M_4|x-z|^\lambda \int_{\partial\Omega\setminus B(x;|x-z|)} |x-y|^{1-m+\alpha-\lambda} \, d\mathcal{H}_{m-1}(y) + \frac{M_2M_3M_4m}{1-\alpha}|x-z|^\alpha$$

$$\leq |x-z|^\lambda \left[5M_1 + \frac{M_2M_3M_4m}{\alpha-\lambda}(\operatorname{diam}\Omega)^{\alpha-\lambda} + \frac{M_2M_3M_4m}{1-\alpha}\right].$$

Corollary 6.10.2 *Let Ω be an open set in R^m with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, and $m \geq 2$. Then the operator K'_Ω is a compact linear operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$.*

Proof $K'_\Omega + K_\Omega$ is a bounded linear operator from $\mathcal{C}^{0,0}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.1. The identity operator I is a compact linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,0}(\partial\Omega)$ (see Proposition 1.16.5). Thus, $K'_\Omega + K_\Omega = (K'_\Omega + K_\Omega)I$ is a compact linear operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (compare Theorem 1.7.4). The operator K_Ω is a compact on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.6.4. This yields that $K'_\Omega = (K'_\Omega + K_\Omega) - K_\Omega$ is a compact linear operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (compare Theorem 1.7.4).

Proposition 6.10.3 *Let Ω be a domain in R^m with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, $m \geq 2$, $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, and $h \geq 0$.*

- *Suppose $m > 2$. If Ω is unbounded or $h \not\equiv 0$, denote $T = (1/2)I - K'_\Omega + h\mathcal{S}^\Omega$ and $X = \mathcal{C}^0(\partial\Omega)$.*
- *If $m = 2$ and $h \not\equiv 0$, denote $Pf = \oint_{\partial\Omega} f \, d\mathcal{H}_{m-1}$, $T = (1/2)(I - P) - K'_\Omega(I - P) + h[\mathcal{S}^\Omega(I - P) + P]$ and $X = \mathcal{C}^0(\partial\Omega)$.*
- *Suppose $h \equiv 0$. If Ω is bounded or $m = 2$, denote $T = (1/2)I - K'_\Omega$ and $X = \{f \in \mathcal{C}^0(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$.*

Let $Y = X \cap \mathcal{C}^{0,\lambda}(\partial\Omega)$ equipped with the norm induced form $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Then T is an isomorphism on Y.

Proof Corollary 6.10.2 and Proposition 6.7.1 give that K'_Ω and $\mathcal{S}^\Omega$ are compact operators on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Theorems 6.9.4, 6.9.5, 6.9.6, and 6.9.7 give that T is an isomorphism X. Thus, T is an injective operator on Y. Since $T-(1/2)I$ is compact (see Lemma 1.16.8 and Theorem 1.7.4), T is an isomorphism on Y by Theorem 1.7.7.

Proposition 6.10.4 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1.\alpha}$, $m \geq 2$, and $0 < \lambda < \alpha < 1$. Then $\mathcal{S}^\Omega$ is a bounded linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$.*

Proof For $j,k \in \{1,\dots,m\}$, denote $\tau_{jk} = [\delta_{1j}n_k - \delta_{1k}n_j, \dots, \delta_{mj}n_k - \delta_{mk}n_j]$. For $z \in \partial\Omega$ and $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, define

$$T_{jk}f(z) = \lim_{\epsilon\downarrow 0} \int\limits_{\partial\Omega\setminus B(z;\epsilon)} f(y)\tau_{jk}(z)\cdot\nabla h_m(z-y)\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

We show that $T_{jk}f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Fix a constant M_1 such that $|f| \leq M_1$ and $|f(x)-f(y)| \leq M_1|x-y|^\lambda$. According to Corollary 1.17.6 and Lemma 1.17.8, there exist constants C_1 and C_2 such that $\mathcal{H}_{m-1}(B(x;r)\cap\partial\Omega) \leq C_1 r^{m-1}$ for all $x \in R^m$ and $r > 0$ and $|\tau_{jk}(z)-\tau_{jk}(y)| \leq C_2|z-y|^\alpha$ for $z, y \in \partial\Omega$. According to Lemma 2.9.7 and Corollary 2.9.6, there exist constants C_3 and C_4 such that if $x, y, z \in R^m$, $|z-y| > 2|x-z|$, then $|\nabla h_m(x-y) - \nabla h_m(z-y)| \leq C_3|x-z||z-y|^{-m}$ and $|\nabla h_m(x-y)| \leq C_4|x-y|^{1-m}$. According to Lemma 5.4.1

$$\int\limits_{\partial\Omega\setminus B(x;\epsilon)} \tau_{jk}(y)\cdot\nabla h_m(y-z)\,\mathrm{d}\mathcal{H}_{m-1}(y) = 0 \tag{6.29}$$

and thus

$$T_{jk}f(z) = \int\limits_{\partial\Omega} [f(y)-f(z)]\tau_{jk}(z)\cdot\nabla h_m(z-y)\,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+f(z)\int\limits_{\partial\Omega} [\tau_{jk}(z)-\tau_{jk}(y)]\cdot\nabla h_m(z-y)\,\mathrm{d}\mathcal{H}_{m-1}(y).$$

By virtue of Lemma 1.26.1

$$|T_{jk}f(z)| \leq \int\limits_{\partial\Omega} 2M_1C_4|z-y|^{\lambda+1-m} + \int\limits_{\partial\Omega} M_1C_2C_4|z-y|^{\alpha+1-m}$$

$$\leq \frac{2M_1C_4m(\mathrm{diam}\,\partial\Omega)^\lambda}{\lambda} + \frac{M_1C_2C_4m(\mathrm{diam}\,\partial\Omega)^\alpha}{\alpha} < \infty.$$

Let $x, z \in \partial\Omega$. By virtue of (6.29)

$$T_{jk}f(z) - T_{jk}f(x) = \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [f(y) - f(z)]\tau_{jk}(z) \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y)$$

$$+f(z) \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [\tau_{jk}(z) - \tau_{jk}(y)] \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y)$$

$$- \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [f(y) - f(x)]\tau_{jk}(x) \cdot \nabla h_m(x-y)\, d\mathcal{H}_{m-1}(y)$$

$$-f(x) \int\limits_{B(z;2|x-z|)\cap\partial\Omega} [\tau_{jk}(x) - \tau_{jk}(y)] \cdot \nabla h_m(x-y)\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} [f(y) - f(x)]\tau_{jk}(x) \cdot [\nabla h_m(z-y) - \nabla h_m(x-y)]\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} f(x)[\tau_{jk}(x) - \tau_{jk}(y)] \cdot [\nabla h_m(z-y) - \nabla h_m(x-y)]\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} [f(y) - f(x)][\tau_{jk}(z) - \tau_{jk}(x)] \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} [f(z) - f(x)][\tau_{jk}(x) - \tau_{jk}(y)] \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} [f(x) - f(z)][\tau_{jk}(z) - \tau_{jk}(y)] \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} f(z)[\tau_{jk}(z) - \tau_{jk}(x)] \cdot \nabla h_m(z-y)\, d\mathcal{H}_{m-1}(y).$$

Denote $C_5 = \max(C_2, C_3, C_4)$. If $|z-y| \geq |x-z|$, then $|x-y| \leq |x-z| + |z-y| \leq 2|z-y|$. By virtue of Lemma 1.26.1

$$|T_{jk}f(z) - T_{jk}f(x)| \leq \int\limits_{B(z;2|x-z|)\cap\partial\Omega} M_1 C_5 |y-z|^{1-m+\lambda}\, d\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{B(z;2|x-z|)\cap\partial\Omega} M_1 C_5^2 |\, y-z|^{1-m+\alpha} + \int\limits_{B(x;3|x-z|)\cap\partial\Omega} M_1 C_5 |\, y-x|^{1-m+\lambda}$$

$$+ \int\limits_{B(x;3|x-z|)\cap\partial\Omega} M_1 C_5^2 |\, y-x|^{1-m+\alpha} + \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} M_1 2|z-y|^{\lambda} C_5 |x-z||z-y|^{-m}$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} M_1 2|z-y|^{\alpha} C_5^2 |x-z||z-y|^{-m} \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} M_1 C_5^2 2|z-y|^{\lambda} |x-z|^{\alpha} |z-y|^{1-m} \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+3M_1 C_5^2 |x-z|^{\lambda} \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |\, y-z|^{\alpha+1-m} \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+M_1 \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} C_5^2 |z-x|^{\alpha} |\, y-z|^{1-m} \le M_1 C_5 \frac{2mC_1|x-z|^{\lambda}}{\lambda}$$

$$+M_1 C_5^2 \frac{2mC_1|x-z|^{\alpha}}{\alpha} + M_1 C_5 \frac{3mC_1|x-z|^{\lambda}}{\lambda} + M_1 C_5^2 \frac{3mC_1|x-z|^{\alpha}}{\alpha}$$

$$+2M_1 C_1 C_5^2 |x-z| \frac{m|x-z|^{\lambda-1}}{1-\lambda} + +2M_1 C_1 C_5^2 |x-z| \frac{m|x-z|^{\alpha-1}}{1-\alpha}$$

$$+2M_1 C_1 C_5^2 |z-x|^{\alpha} \frac{m(\operatorname{diam}\partial\Omega)^{\lambda}}{\lambda} + 3M_1 C_1 C_5^2 |z-x|^{\lambda} \frac{m(\operatorname{diam}\partial\Omega)^{\alpha}}{\alpha}$$

$$+M_1 C_5^2 |x-z|^{\lambda} \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |\, y-z|^{1-m+\alpha-\lambda} \,\mathrm{d}\mathcal{H}_{m-1}(y) \le C_6 |x-z|^{\lambda},$$

where $C_6 = 3mC_1C_5M_1[2\lambda^{-1} + 2C_5\alpha^{-1}(\operatorname{diam}\partial\Omega)^{\alpha-\lambda} + C_5(1-\lambda)^{-1} + C_5(1-\alpha)^{-1} + C_5(\operatorname{diam}\partial\Omega)^{\alpha}(\lambda^{-1}+\alpha^{-1}) + C_5(\operatorname{diam}\partial\Omega)^{\alpha-\lambda}(\alpha-\lambda)^{-1}]$. Hence, $T_{jk}f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$.

For $\theta \in R^m$, denote

$$T_{\theta}f(z) = \lim_{\epsilon\downarrow 0} \int\limits_{\partial\Omega\setminus B(z;\epsilon)} f(y)\theta\cdot\nabla h_m(z-y) \,\mathrm{d}\mathcal{H}_{m-1}.$$

We show that $T_\theta f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Fix $z \in \partial\Omega$. Fix $k \in \{1, \dots, m\}$ such that $n_k^\Omega(z) \neq 0$. Clearly, $\tau_{1,k}(z), \dots, \tau_{k-1,k}(z), \tau_{k+1,k}(z), \dots, \tau_{m,k}(z)$ form a basis of the tangential hyperplane of $\partial\Omega$ at z. Thus, $\tau_{1,k}(z), \dots, \tau_{k-1,k}(z), \tau_{k+1,k}(z), \dots, \tau_{m,k}(z), n^\Omega(z)$ is a basis of R^m. Denote by $A(x)$ the matrix with rows $\tau_{1,k}(x), \dots, \tau_{k-1,k}(x), \tau_{k+1,k}(x), \dots, \tau_{m,k}(x), n^\Omega(x)$. Then there exists $r > 0$ such that $A(x)$ is invertible for $x \in B(z; 2r) \cap \partial\Omega$. According to the rules for the calculation of an inverse matrix and Lemma 1.16.8, all entries of $A^{-1}(x)$ are from $\mathcal{C}^{0,\alpha}(B(z, r) \cap \partial\Omega) \subset \mathcal{C}^{0,\lambda}(B(z; r) \cap \partial\Omega)$. So, there exist $a_j \in \mathcal{C}^{0,\lambda}(B(0; r) \cap \partial\Omega)$ such that

$$\theta = \sum_{j \neq k} a_j(x)\tau_{jk}(x) + a_0(x)n^\Omega(x)$$

for $x \in B(z; r) \cap \partial\Omega$. Therefore

$$T_\theta f(x) = \sum_{j \neq k} a_j(x)T_{jk}f(x) + a_0(x)K'_\Omega f(x)$$

for $x \in B(z; r) \cap \partial\Omega$. We have proved that $T_{jk}f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Corollary 6.10.2 gives that $K'_\Omega f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Lemma 1.16.8 forces that $T_\theta f \in \mathcal{C}^{0,\lambda}(B(z; r) \cap \partial\Omega)$. Since $\partial\Omega$ is compact, we infer that $T_\theta f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$.

Fix $j \in \{1, \dots, m\}$. Denote $T_j f = T_\theta f$ with $\theta = (\delta_{1j}, \dots, \delta_{mj})$. If $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $g = f(z)n_j(z) + T_j f(z) \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Lemma 1.16.8). The function $\partial_j \mathcal{S}^\Omega f$ is an L^2-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ (see Proposition 5.2.8 and Theorem 5.4.7). This solution is unique by Proposition 5.15.1. Since $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, Theorem 6.8.1 gives that $\partial_j \mathcal{S}^\Omega f \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$. Since $\mathcal{S}^\Omega$ is a bounded linear operator from $\mathcal{C}^0(\partial\Omega)$ to $\mathcal{C}^0(\overline{\Omega})$ by Proposition 6.7.1, $\mathcal{S}^\Omega$ is a closed linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. Hence, $\mathcal{S}^\Omega$ is a bounded linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ by the closed-graph theorem (Theorem 1.4.15).

Proposition 6.10.5 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, and $0 < \lambda < \alpha < 1$. Then $\partial_j \mathcal{S}^\Omega$ is a bounded linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$. If $m > 2$, denote $X = \mathcal{C}^{0,\lambda}(\partial\Omega)$. If $m = 2$, denote $X = \{f \in \mathcal{C}^{0,\lambda}(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_1 = 0\}$. Then $\mathcal{S}^\Omega$ is a bounded linear operator from X to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$.*

Proof Fix $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Define $f = 0$ on $R^m \setminus \partial\Omega$. Fix $r > 1$ such that $\partial\Omega \subset B(0; r)$. Then $\mathcal{S}^\Omega f = \mathcal{S}^{\Omega \cap B(0;3r)} f \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap B(0; 3r)})$ (see Proposition 6.10.4). Moreover, $\mathcal{S}^\Omega f \in \mathcal{C}^\infty(\Omega)$ and $\nabla \mathcal{S}^\Omega f(x) \to 0$ as $|x| \to \infty$ (see Proposition 3.1.6). Thus, $\partial_j \mathcal{S}^\Omega f \in \mathcal{C}^0(\overline{\Omega})$, and there exists a constant C_1 such that $|\nabla \mathcal{S}^\Omega f(x)| \leq C_1$ for $x \in \overline{\Omega}$. According to Proposition 3.1.6, there exists a constant C_2 such that $|\nabla^2 \mathcal{S}^\Omega f(x)| \leq C_2$ for $|x| \geq r$. Let now $x \in \overline{\Omega}$ and $y \in \Omega \setminus B(0; 3r)$. If $|x - y| \geq 1$, then $|\partial_j \mathcal{S}^\Omega f(x) - \partial_j \mathcal{S}^\Omega f(y)| \leq 2C_1 \leq 2C_1 |x - y|^\lambda$. Let now $|x - y| < 1$. Then $|y| > 2r$. There exists $\theta \in (0, 1)$ such that $\partial_j \mathcal{S}^\Omega f(x) -$

$\partial_j \mathcal{S}^\Omega f(y) = (x-y)\cdot\nabla\partial_j\mathcal{S}^\Omega f(z)$ with $z = x+\theta(y-x)$. Since $|z| > 2r$, we have $|\partial_j\mathcal{S}^\Omega f(x) - \partial_j\mathcal{S}^\Omega f(y)| \le C_2|x-y| \le C_2|x-y|^\lambda$. Thus, $\partial_j\mathcal{S}^\Omega f \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$. Since $\mathcal{S}^\Omega$ is a closed linear operator from $\mathcal{C}^{0,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$, it is bounded by closed-graph theorem (Theorem 1.4.15).

$\mathcal{S}^\Omega$ is a bounded linear operator from X to $\mathcal{C}^{0,0}(\overline{\Omega})$ by Proposition 6.7.1.

Theorem 6.10.6 *Let $\Omega \subset R^m$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m > 2$, $0 < \lambda < \alpha < 1$, $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, and $h \ge 0$. Suppose that Ω is unbounded or $h \not\equiv 0$. Then the operator $T = (1/2)I - K'_\Omega + h\mathcal{S}^\Omega$ is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Let $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Put $\varphi = T^{-1}g$ and $u = \mathcal{S}^\Omega\varphi$. Then $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$, and*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} \tag{6.30}$$

with a constant C dependent on Ω, h, and λ only.

- *If Ω is bounded, then u is a unique classical solution of the Robin problem for the Laplace equation (6.14).*
- *If Ω is unbounded, then u is a unique classical solution of the Robin problem (6.14) such that $u(x) \to 0$ as $|x| \to \infty$.*

Proof T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.3. Propositions 6.10.4 and 6.10.5 give that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and the estimate (6.30) holds. If Ω is bounded, then u is a unique classical solution of the problem (6.14). If Ω is unbounded, then u is a unique classical solution of the problem (6.14) such that $u(x) \to 0$ as $|x| \to \infty$. (See Theorem 6.9.4.)

Theorem 6.10.7 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $h \ge 0$, and $h \not\equiv 0$. Denote $P\varphi = \oint_{\partial\Omega}\varphi\,\mathrm{d}\mathcal{H}_1$ and $T = \frac{1}{2}(I-P) - K'_\Omega(I-P) + h[\mathcal{S}^\Omega(I-P)+P]$. Then the operator T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Fix $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Put $\varphi = T^{-1}g$ and $u = \mathcal{S}^\Omega(I-P)\varphi + P\varphi$. Then u is a unique bounded classical solution of the Robin problem for the Laplace equation $\Delta u = 0$ in Ω and $\partial u/\partial n + hu = g$ on $\partial\Omega$. Moreover, $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$, and the estimate (6.30) holds with a constant C dependent on Ω, h, and λ only.*

Proof T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.3. The function u is a unique bounded classical solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$ by Theorem 6.9.5. Propositions 6.10.4 and 6.10.5 give that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and the estimate (6.30) holds.

Theorem 6.10.8 *Let Ω be a bounded domain in R^m with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, and $m \ge 2$. Denote by X the set of all $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ satisfying $\int_{\partial\Omega} f\,\mathrm{d}\mathcal{H}_{m-1} = 0$ and by T the restriction of $(1/2)I - K'_\Omega$ onto X. Then T is an isomorphism. Fix $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Then there exists a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ such that $\Delta u = 0$ in Ω and $\partial u/\partial n = g$ on $\partial\Omega$ if and only if $\int_{\partial\Omega} g\,\mathrm{d}\mathcal{H}_{m-1} = 0$. The general form of this solution is $u = \mathcal{S}^\Omega(T^{-1}g) + c$, where* *c is an arbitrary constant. There exists a unique solution of the problem such that*

$$\int_{\Omega} u \, \mathrm{d}\mathcal{H}_m = 0.$$

This solution satisfies the estimate (6.30) with a constant C dependent on Ω and λ only.

Proof T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.3. According to Proposition 6.10.4, we have $\mathcal{S}^{\Omega}(T^{-1}g) \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and

$$\|\mathcal{S}^{\Omega}(T^{-1}g)\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C_1 \|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}$$

with a constant C_1 dependent only on Ω and λ. Theorem 6.9.6 gives that the general form of a solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ is $u = \mathcal{S}^{\Omega}(T^{-1}g) + c$, where c is an arbitrary constant. The function

$$u = \mathcal{S}^{\Omega}(T^{-1}g) - \oint_{\Omega} \mathcal{S}^{\Omega}(T^{-1}g) \, \mathrm{d}\mathcal{H}_m$$

is a unique solution of the problem such that $\int_{\Omega} u \, \mathrm{d}\mathcal{H}_m = 0$. Clearly, the estimate (6.30) holds with a constant C dependent only on Ω and λ.

Theorem 6.10.9 *Let $\Omega \subset R^2$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, and $0 < \lambda < \alpha < 1$. Denote $X = \{f \in \mathcal{C}^{0,\lambda}(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1 = 0\}$ and by T the restriction of $(1/2)I - K'_{\Omega}$ onto X. Then T is an isomorphism. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then there exists $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $\Delta u = 0$ in Ω and $\partial u/\partial n = g$ on $\partial\Omega$ if and only if $g \in X$. The general form of this solution is $u = \mathcal{S}^{\Omega}T^{-1}g + c$, where c is an arbitrary constant. This is a unique solution of the problem such that $u(x) \to c$ as $|x| \to \infty$. Moreover,*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C\left[\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + |c|\right] \tag{6.31}$$

with a constant C dependent only on Ω and λ.

Proof T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.3. If there exists a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the problem, then $g \in X$ and $u = \mathcal{S}^{\Omega}T^{-1}g + c$ (see Theorem 6.9.7). Proposition 6.10.5 gives that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and the estimate (6.31) holds. The function $u = \mathcal{S}^{\Omega}T^{-1}g + c$ is a unique solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$, and $u(x) \to c$ as $|x| \to \infty$ (see Theorem 6.9.7).

Theorem 6.10.10 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, $m \geq 2$, and $0 < \beta < 1$. Let $h, g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $h \geq 0$, $h \not\equiv 0$, and $f \in \mathcal{C}^{0,\beta}_{loc}(\Omega) \cap L^{\infty}(\Omega)$. Then there exists a unique $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $-\Delta u = f$ in Ω and $\partial u/\partial n + hu = g$ on $\partial\Omega$. Moreover,*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C\left[\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + \|f\|_{L^{\infty}(\Omega)}\right] \tag{6.32}$$

with a constant C dependent only on Ω, h, and λ.

Proof Define $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and

$$\|Vf\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C_1 \|f\|_{L^\infty(\Omega)}$$

with a constant C_1 dependent only on Ω and λ (see Proposition 3.15.1). Theorem 3.1.5 gives that $-\Delta Vf = f$ in the sense of distributions. According to Proposition 3.18.1, we have $Vf \in \mathcal{C}^2(\Omega)$. So, u is a solution of the problem if and only if $v = u - Vf$ is a solution of the problem $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$, $-\Delta v = 0$ in Ω, $\partial v/\partial n + hv = G := g - \partial Vf/\partial n - hVf$ on $\partial\Omega$. Lemma 1.16.8 forces $G \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$ and

$$\|G\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} \leq \|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + C_1 \|f\|_{L^\infty(\Omega)} + 2\|h\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} C_1 \|f\|_{L^\infty(\Omega)}.$$

According to Theorems 6.10.6 and 6.10.7, there exists a unique solution v and

$$\|v\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C_2 \|G\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}$$

with a constant C_2 dependent only on Ω, h, and λ.

Theorem 6.10.11 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, $m \geq 2$, and $0 < \beta < 1$. Let $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ and $f \in \mathcal{C}^{0,\beta}_{loc}(\Omega) \cap L^\infty(\Omega)$. Then there exists $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $-\Delta u = f$ in Ω and $\partial u/\partial n = g$ on $\partial\Omega$ if and only if*

$$\int_\Omega f \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{6.33}$$

Suppose that (6.33) holds true. If u and v are two solutions of the problem, then $u - v$ is constant. If u is the unique solution of the problem such that $\int_\Omega u \, \mathrm{d}\mathcal{H}_m = 0$, then

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C \left[\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)} \right] \tag{6.34}$$

with a constant C dependent only on Ω and λ.

Proof Let u be a solution of the problem. Then (6.33) holds true. If v is another solution of the problem, then $u - v$ is constant. (See Theorem 6.9.9.)

Let now (6.33) holds true. Define $f = 0$ on $R^m \setminus \Omega$. Then $Vf \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and

$$\|Vf\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C_1 \|f\|_{L^\infty(\Omega)}$$

with a constant C_1 dependent only on Ω and λ (see Proposition 3.15.1). Theorem 3.1.5 gives that $-\Delta Vf = f$ in the sense of distributions. According to

Proposition 3.18.1, we have $Vf \in \mathcal{C}^2(\Omega)$. We have proved that

$$\int_{\Omega} f \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \frac{\partial Vf}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

The function u is a solution of the problem if and only if $v = u - Vf$ is a solution of the problem $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$, $-\Delta v = 0$ in Ω, $\partial v/\partial n = G := g - \partial Vf/\partial n$ on $\partial\Omega$. Clearly, $G \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$ and

$$\|G\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} \leq \|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + C_1 \|f\|_{L^\infty(\Omega)}.$$

Since

$$\int_{\partial\Omega} G \, \mathrm{d}\mathcal{H}_{m-1} = \left[\int_{\partial\Omega} g + \int_{\Omega} f \right] - \left[\int_{\Omega} f + \int_{\partial\Omega} \frac{\partial Vf}{\partial n} \right] = 0,$$

Theorem 6.10.8 gives that there exists a unique $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $-\Delta v = 0$ in Ω, $\partial v/\partial n = G$, and $\int_\Omega v \, \mathrm{d}\mathcal{H}_m = 0$. Moreover,

$$\|v\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C_2 \|G\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}$$

with a constant C_2 dependent only on Ω, h, and λ. Define

$$u = v + Vf - \fint_{\Omega} (v + Vf) \, \mathrm{d}\mathcal{H}_m.$$

Clearly, $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega})$, $-\Delta u = f$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$, $\int_\Omega u \, \mathrm{d}\mathcal{H}_m = 0$, and the estimate (6.34) holds.

6.11 Hölder Continuous Derivatives of Solutions of the Dirichlet Problem

In this section, we prove the existence of a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the Dirichlet problem with the boundary condition $f \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ provided Ω is a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$.

Proposition 6.11.1 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \lambda < \alpha < 1$. Let $f \in \mathcal{C}^{1,\lambda}(\partial\Omega)$. Define $u_f = \mathcal{D}^\Omega f$ in Ω and $u_f = \frac{1}{2} f + K_\Omega f$ on $\partial\Omega$. Then $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and*

$$\|u_f\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C \|f\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} \tag{6.35}$$

with a constant C dependent only on Ω and λ. The hypersingular operator $H^\Omega f = \partial\mathcal{D}^\Omega f/\partial n^\Omega = \partial u_f/\partial n^\Omega$ is a bounded linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Proof $u_f \in \mathcal{C}^0(\overline{\Omega})$ by Proposition 6.6.3 and

$$\|u_f\|_{\mathcal{C}(\overline{\Omega})} \le c_1\|f\|_{\mathcal{C}(\partial\Omega)}$$

where c_1 depends only on Ω. Lemma 5.5.1 gives

$$\partial_j\mathcal{D}^\Omega f = \sum_{k=1}^m \partial_k\mathcal{S}^\Omega(\partial_{\tau_{jk}} f),$$

where $\partial_{\tau_{jk}} = n_j^\Omega\partial_k - n_k^\Omega\partial_j$. Since $\partial_{\tau_{jk}} f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, Proposition 6.10.5 gives that $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and (6.35) holds. Since

$$\|H^\Omega f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} \le 2\|n^\Omega\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}\|\nabla u_f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}$$

by Lemma 1.16.8, the hypersingular operator H^Ω is a bounded linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Proposition 6.11.2 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \lambda < \alpha < 1$. Then K_Ω is a compact linear operator on $\mathcal{C}^{1,\lambda}(\partial\Omega)$.*

Proof The operator K_Ω is a bounded linear operator on $\mathcal{C}^{1,\alpha}(\partial\Omega)$ by Proposition 6.11.1. Proposition 6.6.4 gives that K_Ω is a compact operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Denote $\partial_{\tau_{jk}} = n_j^\Omega\partial_k - n_k^\Omega\partial_j$

$$T_{jk}f(z) = \int_{\partial\Omega} \frac{(y_j - z_j)(n_k^\Omega(y) - n_k^\Omega(z))}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}.$$

According to Lemma 5.14.5

$$\partial_{\tau_{jk}} Kf(z) = \sum_{i=1}^m [T_{ik}\partial_{\tau_{ji}} f(z) - T_{ij}\partial_{\tau_{ki}} f(z)] + K_\Omega\partial_{\tau_{jk}} f(z).$$

$f \mapsto \partial_{\tau_{ki}} f$ is a bounded linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$. The operator K_Ω is a compact linear operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. So, it is enough to prove that T_{jk} is a compact linear operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Theorem 1.7.4).

There exists a constant C_1 such that $|n^\Omega(x) - n^\Omega(y)| \le C_1|x-y|^\alpha$ for all $x, y \in \partial\Omega$ (see Lemma 1.17.8). So, T_{jk} is a bounded linear operator on $\mathcal{C}^0(\partial\Omega)$ by Proposition 1.26.3. According to Corollary 1.17.6, there exists a constant C_2 such that $\mathcal{H}_{m-1}(B(x;r) \cap \partial\Omega) \le C_2 r^{m-1}$ for all $x \in R^m$ and $r > 0$. Clearly, there exists a constant C_3 such that $(x_j - y_j)|x-y|^{-m} = C_3\partial_j h_m(x-y)$. According to Lemma 2.9.7, there exists a constant C_4 such that $|(x_j - y_j)|x-y|^{-m} - (z_j - y_j)|z-y|^{-m}| \leq C_4|x-z||z-y|^{-m}$ for $|z-y| > 2|x-z|$.

Suppose that $|f| \leq 1$. Fix $x, z \in \partial\Omega$. According to Lemma 1.26.1

$$|T_{jk}f(x) - T_{jk}f(y)| \leq \int_{B(z;2|x-z|)\cap\partial\Omega} \left| \frac{(y_j - z_j)(n_k^\Omega(y) - n_k^\Omega(z))}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m} f(y) \right| \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int_{B(z;2|x-z|)\cap\partial\Omega} \left| \frac{(y_j - x_j)(n_k^\Omega(y) - n_k^\Omega(x))}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} f(y) \right| \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega\setminus B(z;2|x-z|)} \left| \frac{(y_j - x_j)(n_k^\Omega(z) - n_k^\Omega(x))}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} f(y) \right| \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega\setminus B(z;2|x-z|)} \left| \frac{(y_j - x_j)}{|x-y|^m} - \frac{(y_j - z_j)}{|z-y|^m} \right| \left| \frac{f(y)(n_k^\Omega(y) - n_k^\Omega(z))}{\mathcal{H}_{m-1}(\partial B(0;1))} \right| \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq \frac{C_1 C_2 (m-1)}{\alpha \mathcal{H}_{m-1}(\partial B(0;1))}(2|x-z|)^\alpha + \int_{B(x;3|x-z|)\cap\partial\Omega} \frac{C_1|y-x|^{1+\alpha-m}}{\mathcal{H}_{m-1}(\partial B(0;1))} \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ C_1|x-z|^\alpha \int_{\partial\Omega\setminus B(x;|x-z|)} \frac{|y-x|^{1-m}}{\mathcal{H}_{m-1}(\partial B(0;1))} \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$+ \int_{\partial\Omega\setminus B(z;2|x-z|)} \frac{C_4|z-x|C_1|z-y|^\alpha}{|z-y|^m \mathcal{H}_{m-1}(\partial B(0;1))} \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\leq 5\frac{C_1 C_2 (m-1)}{\alpha \mathcal{H}_{m-1}(\partial B(0;1))}|x-z|^\alpha + C_1|x-z|^\lambda \int_{\partial\Omega\setminus B(x;|x-z|)} \frac{|y-x|^{1-m+\alpha-\lambda}}{\mathcal{H}_{m-1}(\partial B(0;1))}$$

$$+ \frac{C_1 C_2 C_4 |z-x| m}{\mathcal{H}_{m-1}(\partial B(0;1))(1-\alpha)}(2|x-z|)^{\alpha-1} \leq \frac{5 C_1 C_2 m (\operatorname{diam}\partial\Omega)^{\alpha-\lambda}}{\alpha \mathcal{H}_{m-1}(\partial B(0;1))}|x-z|^\lambda$$

$$+ \frac{C_1 C_2 m (\operatorname{diam}\partial\Omega)^{\alpha-\lambda}}{(\alpha-\lambda)\mathcal{H}_{m-1}(\partial B(0;1))}|x-z|^\lambda + \frac{C_1 C_2 C_4 m (\operatorname{diam}\partial\Omega)^{\alpha-\lambda}}{\mathcal{H}_{m-1}(\partial B(0;1))(1-\alpha)}|x-z|^\lambda.$$

Hence, T_{jk} is a bounded linear operator from $\mathcal{C}^0(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$. So, T_{jk} is a compact operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 1.16.5 and Theorem 1.7.4.

Theorem 6.11.3 *Let* $0 < \lambda < \alpha$, $0 < \gamma < 1$, *and* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$. *If* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, *and* $f \in \mathcal{C}^{0,\gamma}_{loc}(\Omega) \cap L^\infty(\Omega)$, *then there exists a unique classical solution* $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ *of the Dirichlet problem* $-\Delta u = f$ *in* Ω, $u = g$ *on* $\partial\Omega$. *Moreover,*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\Omega)} \le C[\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}]$$

where C *depends only on* Ω, λ, *and* p.

Proof According to Theorem 6.4.2, there exists a unique classical solution of the problem. Define $f = 0$ on $R^m \setminus \Omega$. Then there exists a constant C_1 depending only on Ω and λ such that

$$\|Vf\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C_1\|f\|_{L^\infty(\Omega)}$$

(see Proposition 3.15.1). Denote $v = u + Vf$. Then v is a classical solution of the problem $\Delta v = 0$ in Ω, $v = g + Vf$ on $\partial\Omega$ by Theorems 3.1.5 and 3.14.2.

Suppose first that $m > 2$. Then $v = \mathcal{D}^\Omega\psi + \mathcal{S}^\Omega\psi$, where $\psi = (\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)^{-1}(g + Vf)$ in $L^2(\partial\Omega)$ (see Theorem 5.15.2). The operator $\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega$ is a Fredholm operator with index 0 on $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Propositions 6.11.2, 6.10.4, and 1.16.5 and Theorems 1.7.4 and 1.8.3. So, it is an isomorphism on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, and

$$\|\psi\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} \le C_2\|g + Vf\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} \tag{6.36}$$

where C_2 depends only on Ω and λ.

Let now $m = 2$. Then $v = \mathcal{D}^\Omega(I - P)\psi + \mathcal{S}^\Omega(I - P)\psi + P\psi$, where $P\psi = \oint_{\partial\Omega}\psi \, d\mathcal{H}_{m-1}$ and $\psi = [(\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)(I - P) + P]^{-1}(g + Vf)$ in $L^2(\partial\Omega)$ (see Theorem 5.15.3). The operator $(\frac{1}{2}I + K_\Omega + \mathcal{S}^\Omega)(I - P) + P$ is a Fredholm operator with index 0 on $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Propositions 6.11.2, 6.10.4, and 1.16.5 and Theorems 1.7.4 and 1.8.3. So, it is an isomorphism on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, and (6.36) holds with C_2 depending only on Ω and λ.

Propositions 6.11.1 and 6.10.4 give that $v \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and

$$\|v\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C_3\|g + Vf\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)}$$

where C_3 depends only on Ω and λ. Thus

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C_3\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} + (1 + C_3)C_1\|f\|_{L^\infty(\Omega)}.$$

Theorem 6.11.4 *Let* $0 < \lambda < \alpha < 1$ *and* $\Omega \subset R^m$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $m > 2$. *If* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, *then there exists a unique solution* $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ *of the Dirichlet problem* $\Delta u = 0$ *in* Ω, $u = g$

on $\partial\Omega$ *such that* $u(x) \to 0$ *as* $|x| \to \infty$*. Moreover,*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\Omega)} \leq C\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} \tag{6.37}$$

where C *depends only on* Ω *and* λ*.*

Proof According to Theorem 6.5.1, there exists a unique classical solution u of the problem. We have $u = \mathcal{D}^{\Omega}\psi + \mathcal{S}^{\Omega}\psi$, where $\psi = (\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega})^{-1}g$ in $L^2(\partial\Omega)$ (see Theorem 5.15.4). The operator $\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega}$ is a Fredholm operator with index 0 on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Propositions 6.11.2, 6.10.4, and 1.16.5 and Theorems 1.7.4 and 1.8.3. So, it is an isomorphism on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ and

$$\|\psi\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} \leq C_1\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)}$$

where C_1 depends only on Ω and λ. Propositions 6.11.1 and 6.10.5 give (6.37).

Theorem 6.11.5 *Let* $0 < \lambda < \alpha < 1$ *and* $\Omega \subset R^2$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$*. If* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$*, then there exists a unique solution* $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ *of the Dirichlet problem* $\Delta u = 0$ *in* Ω*,* $u = g$ *on* $\partial\Omega$*. Moreover, (6.37) holds with* C *depending only on* Ω *and* λ*.*

Proof According to Theorem 6.5.1, there exists a unique bounded classical solution of the problem. We have $u = \mathcal{D}^{\Omega}(I-P)\psi + \mathcal{S}^{\Omega}(I-P)\psi + P\psi$ where $P\varphi = \fint_{\partial\Omega}\varphi \, \mathrm{d}\mathcal{H}_1$, $T = \left[\frac{1}{2}I + K_{\Omega} + \mathcal{S}^{\Omega}\right](I-P) + P$, and $\psi = T^{-1}g$ in $L^2(\partial\Omega)$ (see Theorem 5.15.3). The operator T is a Fredholm operator with index 0 on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Propositions 6.11.2, 6.10.4, and 1.16.5 and Theorems 1.7.4 and 1.8.3. So, it is an isomorphism on $\mathcal{C}^{1,\lambda}(\partial\Omega)$ by Lemma 1.8.4. Thus, $\psi \in \mathcal{C}^{1,\alpha}(\partial\Omega)$ and

$$\|\psi\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)} \leq C_1\|g\|_{\mathcal{C}^{1,\partial}(\partial\Omega)}$$

where C_1 depends only on Ω and β. Propositions 6.11.1 and 6.10.5 give (6.37).

6.12 Regularity of the Green Function

Proposition 6.12.1 *Let* $\Omega \subset R^m$ *be an open set with nonempty compact Lipschitz boundary. Let* $G_{\Omega}(x,y) = h_m(x-y) - g_{\Omega}(x,y)$ *be the Green function of* Ω*. For* $y \in \Omega$ *and* $x \in \partial\Omega$*, put* $g_{\Omega}(x,y) = h_m(x-y)$*. Then* $g_{\Omega} \in \mathcal{C}^0(\overline{\Omega} \times \Omega)$*. If* Ω *has boundary of class* $\mathcal{C}^{1,\alpha}$*,* $0 < \lambda < \alpha < 1$*, and* $y \in \Omega$*, then* $g_{\Omega}(\cdot, y) \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$*.*

Proof Since $g_{\Omega} \in H(\Omega \times \Omega)$ by Proposition 3.9.12, we have $g_{\Omega} \in \mathcal{C}^2(\Omega \times \Omega)$.

Fix $y \in \Omega$. Denote $f_y(x) = h_m(x-y)$. Suppose first that Ω is bounded. According to Theorem 6.4.2, there exists a unique classical solution u_y of the Dirichlet problem $\Delta u_y = 0$ in Ω, $u_y = f_y$ on $\partial\Omega$. Lemma 3.9.9 gives that $g_\Omega(x, y) = u_y(x)$. Let now Ω be unbounded and $m = 2$. According to Theorem 6.5.1, there exists a unique bounded classical solution u_y of the Dirichlet problem $\Delta u_y = 0$ in Ω, $u_y = f_y$ on $\partial\Omega$. Lemma 3.9.9 gives that $g_\Omega(x, y) = u_y(x)$. Let now Ω be unbounded and $m > 2$. According to Theorem 6.5.1, there exists a unique classical solution u_y of the Dirichlet problem $\Delta u_y = 0$ in Ω, $u_y = f_y$ on $\partial\Omega$ such that $u_y(x) \to 0$ as $|x| \to \infty$. One has $u_y(x) = O(|x|^{2-m})$ as $|x| \to \infty$ by Proposition 2.17.3. Lemma 3.9.9 gives that $g_\Omega(x, y) = u_y(x)$.

According to Theorems 6.4.2 and 6.5.1

$$|g_\Omega(x, y) - g_\Omega(x, z)| = |u_y(x) - u_z(x)| \le \|f_y - f_z\|_{L^\infty(\partial\Omega)} \tag{6.38}$$

for $y, z \in \Omega$ and $x \in \overline{\Omega}$.

Let now $x \in \partial\Omega$, $x_n \in \overline{\Omega}$, $y_n \in \Omega$, and $[x_n, y_n] \to [x, y]$. We show that $g_\Omega(x_n, y_n) \to g_\Omega(x, y)$. According to (6.38)

$$|g_\Omega(x, y) - g_\Omega(x_n, y_n)| \le |g_\Omega(x, y) - g_\Omega(x_n, y)| + |g_\Omega(x_n, y) - g_\Omega(x_n, y_n)|$$

$$\le |g_\Omega(x, y) - g_\Omega(x_n, y)| + \|h_m(\cdot - y) - h_m(\cdot - y_n)\|_{L^\infty(\partial\Omega)} \to 0.$$

If Ω has boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \lambda < \alpha < 1$, then $g_\Omega(\cdot, y) = u_y \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Theorems 6.11.3, 6.11.4, and 6.11.5.

6.13 Representation of a Solution of the First Problem

Proposition 6.13.1 *Suppose that $\Omega \subset R^m$ is a bounded domain with boundary of class $C^{1,\alpha}$, $m \ge 2$ and $0 < \alpha < 1$. Let G_Ω be the Green function of Ω. Let u be a classical solution of the Dirichlet problem*

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

If $f \in L^p(\Omega)$ with $p > m/2$, then

$$u(x) = G_\Omega f(x) - \int_{\partial\Omega} g \frac{\partial G_\Omega(x, \cdot)}{\partial n} \, d\mathcal{H}_{m-1}, \qquad x \in \Omega.$$

Proof $G_\Omega f \in \mathcal{C}^0(\overline{\Omega})$ with $G_\Omega f = 0$ on $\partial\Omega$ (see Theorems 3.13.3, 4.6.3, 4.5.7, and 4.5.8). Since $-\Delta G_\Omega f = f$ in the sense of distribution (see Theorem 3.11.5), the function $v = u - G_\Omega f$ is harmonic in Ω by Corollary 2.18.3. Since v is a classical solution of the Dirichlet problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$, Theorem 5.16.2 gives

$$v(x) = -\int_{\partial\Omega} g \frac{\partial G_\Omega(x,\cdot)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}, \qquad x \in \Omega .$$

6.14 Green Function for the Third Problem

In this section, we define the Green function for the third problem. The Green function for the third problem has the same properties like the Green function (i.e., the Green function for the Dirichlet problem). Using the Green function for the third problem, we get a similar integral representation formula for solutions of the third problem for the Poisson equation like for solutions of the Dirichlet problem using the Green function.

Let $\Omega \subset R^m$ be a bounded domain with boundary of class $C^{1,\alpha}$, $m \geq 2$, $0 < \alpha < 1$, $c(x) \in \mathcal{C}^0(\partial\Omega)$, $c \geq 0$, and $c \not\equiv 0$. For a fixed $x \in \Omega$, there exists a unique classical solution $g^c_\Omega(\cdot, x)$ of the problem

$$\Delta g^c_\Omega(\cdot, x) = 0 \text{ in } \Omega , \qquad \frac{\partial g^c_\Omega(\cdot, x)}{\partial n} + c g^c_\Omega = \frac{\partial h_m(\cdot - x)}{\partial n} + c h_m(\cdot - x) \quad \text{on } \partial\Omega$$

(see Theorems 6.9.4 and 6.9.5). If $c \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ with $0 < \lambda < \alpha$, then $g^c_\Omega \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Theorems 6.10.6 and 6.10.7. Denote

$$R^c_\Omega(y, x) = h_m(x - y) - g^c_\Omega(y, x)$$

the Green function for the third problem

$$-\Delta u = f \quad \text{in } \Omega , \qquad \frac{\partial u}{\partial n} + cu = h \quad \text{on } \partial\Omega .$$

According to Theorem 3.1.5

$$-\Delta R^c_\Omega(\cdot, x) = \delta_x \quad \text{in } \Omega , \qquad \frac{\partial R^c_\Omega(\cdot, x)}{\partial n} + c R^c_\Omega = 0 \quad \text{on } \partial\Omega .$$

Proposition 6.14.1 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $C^{1,\alpha}$, $c(x) \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $c \geq 0$, $c \not\equiv 0$, $0 < \lambda < 1$ and $0 < \alpha < 1$. Then $R^c_\Omega(x, y) = R^c_\Omega(y, x)$ for each $x, y \in \Omega$.*

Proof Let $x, y \in \Omega$, $x \neq y$. Put $u(z) = R^c_\Omega(z, x)$ and $v(z) = R^c_\Omega(z, y)$. For $r > 0$, denote $\Omega(r) = \Omega \setminus \overline{[B(x; r) \cup B(y; r)]}$. Since $u \in H(\Omega \setminus \{x\})$ and $v \in H(\Omega \setminus \{y\})$, we obtain by Corollary 2.2.2 and theorem on three potentials

(Theorem 2.2.4)

$$0 = \lim_{r \to 0_+} \int\limits_{\partial\Omega(r)} \left[u \frac{\partial v}{\partial n^{\Omega(r)}} - v \frac{\partial u}{\partial n^{\Omega(r)}} \right] \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} u \left[\frac{\partial v}{\partial n} + cv \right] \mathrm{d}\mathcal{H}_{m-1}$$

$$- \int\limits_{\partial\Omega} v \left[\frac{\partial u}{\partial n} + cu \right] \mathrm{d}\mathcal{H}_{m-1} + \lim_{r \to 0_+} \left\{ \int\limits_{\partial B(x;r)} \left[v \frac{\partial u}{\partial n^{B(x;r)}} - u \frac{\partial v}{\partial n^{B(x;r)}} \right] \mathrm{d}\mathcal{H}_{m-1} \right.$$

$$\left. + \int\limits_{\partial B(y;r)} \left[v \frac{\partial u}{\partial n^{B(y;r)}} - u \frac{\partial v}{\partial n^{B(y;r)}} \right] \right\} = \lim_{r \to 0_+} \left[-\mathcal{D}^{B(x;r)} v(x) - \mathcal{S}^{B(x;r)} \frac{\partial v}{\partial n}(x) \right.$$

$$\left. + \mathcal{D}^{B(y;r)} v(y) + \mathcal{S}^{B(y;r)} \frac{\partial v}{\partial n}(y) \right] = -v(x) + u(y) = R^c_\Omega(y, x) - R^c_\Omega(x, y).$$

Proposition 6.14.2 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $C^{1,\alpha}$, $c(x) \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $c \geq 0$, $c \not\equiv 0$, $0 < \lambda < 1$ and $0 < \alpha < 1$. Then $g^c_\Omega \in H(\Omega \times \Omega)$.*

Proof The function g^c_Ω is locally bounded in $\Omega \times \Omega$ by Theorem 6.9.8. Since $g^c_\Omega(\cdot, x) = g^c_\Omega(x, \cdot) \in H(\Omega)$ (see Proposition 6.14.1), Theorem 2.7.1 gives that $g^c_\Omega \in H(\Omega \times \Omega)$.

Proposition 6.14.3 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $C^{1,\alpha}$, $c(x) \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $c \geq 0$, $c \not\equiv 0$, $0 < \lambda < 1$ and $0 < \alpha < 1$. Then $R^c_\Omega \geq 0$.*

Proof Fix $x \in \Omega$. Denote $v(y) = R^c_\Omega(y, x)$ and $v^-(y) = \min(v, 0)$. Since $R^c_\Omega(y, x) \to \infty$ as $y \to x$, there exists $0 < r < \operatorname{dist}(x, \partial\Omega)$ such that $v^- = 0$ on $\overline{B(x; r)}$. Denote $\omega = \Omega \setminus \overline{B(x; r)}$. We have $v^- \in W^{1,2}(\omega)$ by Theorem 1.21.5, and $\nabla v^-(x) = \nabla v(x)$ for $v(x) \leq 0$, and $\nabla v^-(x) = 0$ elsewhere. According to Lemma 1.21.3, there exists a sequence $\varphi_k \in \mathcal{C}^\infty_c(R^m)$ such that $\varphi_k \to v^-$ in $W^{1,2}(\omega)$. Theorem 1.24.1 gives $\varphi_k \to v^-$ in $L^2(\partial\omega)$. By virtue of Lemma 5.8.1

$$\int\limits_{\partial\omega} v^- \frac{\partial v}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{k \to \infty} \int\limits_{\partial\omega} \varphi_k \frac{\partial v}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \lim_{k \to \infty} \int\limits_{\Omega} \nabla\varphi_k \cdot \nabla v \, \mathrm{d}\mathcal{H}_m$$

$$= \int\limits_{\Omega} \nabla v^- \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} |\nabla v^-|^2 \, \mathrm{d}\mathcal{H}_m.$$

Define $h = 0$ on $\partial\omega \setminus \partial\Omega$. Since $\partial v / \partial n + hv = 0$ on $\partial\Omega$, and $v^- = 0$ on $\partial\omega \setminus \partial\Omega$, we have

$$0 = \int\limits_{\partial\omega} v^- \left[\frac{\partial v}{\partial n} + hv \right] \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} |\nabla v^-|^2 \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} h |v^-|^2 \, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus, $\nabla v^- = 0$ in ω. So, v^- is constant (compare Lemma 1.20.1). Since $v^- = 0$ on $\partial B(x; r)$, we infer that $v^- \equiv 0$. Thus, $R^c_\Omega(y, x) \geq 0$.

Theorem 6.14.4 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, $c(x) \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $c \geq 0$, $c \not\equiv 0$, $0 < \lambda < 1$ and $0 < \alpha < 1$. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be a classical solution of the problems $-\Delta u = f$ in Ω and $\partial u/\partial n + cu = g$ on $\partial\Omega$. If $f \in L^1(\Omega)$, then*

$$u(x) = \int_\Omega R^c_\Omega(x, y) f(y)\, \mathcal{H}_m(y) + \int_{\partial\Omega} R^c_\Omega(x, y) g(y)\, \mathcal{H}_{m-1}(y).$$

Proof Let $\Omega(k)$ be a sequence of domains from Lemma 1.17.13. Since $g^c_\Omega(\cdot, x) \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega(k)})$, we have by second Green formula (Theorem 2.2.1)

$$0 = \int_{\partial\Omega(k)} \left[u \frac{\partial g^c_\Omega(\cdot, x)}{\partial n} - g^c_\Omega(\cdot, x) \frac{\partial u}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1} - \int_{\Omega(k)} g^c_\Omega(\cdot, x) f \, \mathrm{d}\mathcal{H}_m.$$

If $x \in \Omega(k)$, then theorem on three potentials (Theorem 2.2.4) gives

$$u(x) = \mathcal{S}^{\Omega(k)} \frac{\partial u}{\partial n}(x) + \mathcal{D}^{\Omega(k)} u(x) + V(f \chi_{\Omega(k)}).$$

Adding we get

$$u(x) = \int_{\partial\Omega(k)} R^c_\Omega(y, x) \frac{\partial u(y)}{\partial n} - \int_{\partial\Omega(k)} u \frac{\partial R^c_\Omega(\cdot, x)}{\partial n} + \int_{\Omega(k)} R^c_\Omega(y, x) f.$$

Letting $k \to \infty$, we get by virtue of Lebesgue lemma (Lemma 1.13.2)

$$u(x) = \int_{\partial\Omega} R^c_\Omega(y, x) \frac{\partial u(y)}{\partial n} - \int_{\partial\Omega} u \frac{\partial R^c_\Omega(\cdot, x)}{\partial n} + \int_\Omega R^c_\Omega(y, x) f(y).$$

Since $\partial R^c_\Omega(\cdot, x)/\partial n + c R^c_\Omega(\cdot, x) = 0$ on $\partial\Omega$, we obtain

$$u(x) = \int_{\partial\Omega} [R^c_\Omega(y, x) \frac{\partial u(y)}{\partial n} + c(y) u(y) R^c_\Omega(y, x)] + \int_\Omega R^c_\Omega(y, x) f(y)$$

$$= \int_{\partial\Omega} R_\Omega(y, x) g(y)\, \mathrm{d}\mathcal{H}_{m-1}(y) + \int_\Omega R^c_\Omega(y, x) f(y)\, \mathrm{d}\mathcal{H}_m(y).$$

The symmetry of R^c_Ω (Proposition 6.14.1) gives the proposition.

6.15 Neumann Function

In this section, we define the Neumann function, i.e., the Green function for the Neumann problem. The Neumann function has the same properties like the Green function. Using the Neumann function, we get a similar integral representation formula for solutions of the Neumann problem for the Poisson equation like for solutions of the Dirichlet problem using the Green function.

Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, and $0 < \alpha < 1$. Fix $x \in \Omega$. Choose $0 < r < \operatorname{dist}(x, \partial\Omega)$. According to Corollary 2.2.3

$$\int\limits_{\partial\Omega} \frac{\partial h_m(\cdot - x)}{\partial n} \, \mathrm{d}\mathcal{H}_m = \int\limits_{\partial(\Omega \setminus B(x;r))} \frac{\partial h_m(\cdot - x)}{\partial n} \, \mathrm{d}\mathcal{H}_m$$

$$+ \int\limits_{\partial B(x;r)} \frac{\partial h_m(\cdot - x)}{\partial n} \, \mathrm{d}\mathcal{H}_m = 0 + \int\limits_{\partial B(x;r)} \frac{-r^{1-m}}{\mathcal{H}_{m-1}(\partial B(0;1))} = -1.$$

Therefore, there exists a unique classical solution $g_\Omega^N(\cdot, x)$ of the Neumann problem

$$\Delta g_\Omega^N(\cdot, x) = 0 \quad \text{in } \Omega, \qquad \frac{\partial g_\Omega^N(\cdot, x)}{\partial n} = \frac{\partial h_m(\cdot - x)}{\partial n} + \frac{1}{\mathcal{H}_{m-1}(\partial\Omega)} \quad \text{on } \partial\Omega,$$

such that

$$\int\limits_{\partial\Omega} g_\Omega^N(y, x) \, \mathcal{H}_{m-1}(y) = \int\limits_{\partial\Omega} h_m(y - x) \, \mathcal{H}_{m-1}(y).$$

Moreover, $g_\Omega^N(\cdot, x) \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ for all $\lambda \in \langle 0, \alpha)$. (See Theorem 6.10.11.) Denote

$$N_\Omega(y, x) = h_m(x - y) - g_\Omega^N(y, x)$$

so-called Neumann function of Ω. According to Theorem 3.1.5

$$-\Delta N_\Omega(\cdot, x) = \delta_x \quad \text{in } \Omega, \tag{6.39}$$

$$\frac{\partial N_\Omega(\cdot, x)}{\partial n} = \frac{-1}{\mathcal{H}_{m-1}(\partial\Omega)} \quad \text{on } \partial\Omega, \tag{6.40}$$

$$\int\limits_{\partial\Omega} N_\Omega(\cdot, x) \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{6.41}$$

Proposition 6.15.1 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, and $0 < \alpha < 1$. Then $N_\Omega(x, y) = N_\Omega(y, x)$ for each $x, y \in \Omega$.*

Proof Let $x, y \in \Omega$ and $x \neq y$. Put $u(z) = N_\Omega(z,x)$ and $v(z) = N_\Omega(z,y)$. For $r > 0$, denote $\Omega(r) = \Omega \setminus \overline{[B(x;r) \cup B(y;r)]}$. Since $u \in H(\Omega \setminus \{x\})$, and $v \in H(\Omega \setminus \{y\})$, we obtain by Corollary 2.2.2, (6.40), (6.41), and theorem on three potentials (Theorem 2.2.4)

$$0 = \lim_{r\to 0+} \int\limits_{\partial\Omega(r)} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] \mathrm{d}\mathcal{H}_{m-1} = \oint\limits_{\partial\Omega} v \,\mathrm{d}\mathcal{H}_{m-1} - \oint\limits_{\partial\Omega} u \,\mathrm{d}\mathcal{H}_{m-1}$$

$$+ \lim_{r\to 0+} \left\{ \int\limits_{\partial B(x;r)} \left[v\frac{\partial u}{\partial n^{B(x;r)}} - u\frac{\partial v}{\partial n^{B(x;r)}}\right] \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\partial B(y;r)} \left[v\frac{\partial u}{\partial n^{B(y;r)}}\right.\right.$$

$$\left.\left. -u\frac{\partial v}{\partial n^{B(y;r)}}\right] \mathrm{d}\mathcal{H}_{m-1}\right\} = \lim_{r\to 0+} [-\mathcal{D}^{B(x;r)} v(x) - \mathcal{S}^{B(x;r)} \frac{\partial v}{\partial n}(x)$$

$$\left. +\mathcal{D}^{B(y;r)} v(y) + \mathcal{S}^{B(y;r)} \frac{\partial v}{\partial n}(y)\right] = -v(x) + u(y) = N_\Omega(y,x) - N_\Omega(x,y).$$

Proposition 6.15.2 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $C^{1,\alpha}$ *and* $0 < \alpha < 1$. *Then* $g_\Omega^N \in H(\Omega \times \Omega)$.

Proof Fix $\lambda \in (0,\alpha)$. Define a continuous mapping $T = [T_1, T_2] : H(\Omega) \cap \mathcal{C}^{1,\lambda}(\overline{\Omega}) \to \mathcal{C}^{0,\lambda}(\partial\Omega) \times R^1$ by

$$T_1 u = \frac{\partial u}{\partial n}, \quad T_2 u = \int\limits_{\partial\Omega} u \,\mathrm{d}\mathcal{H}_{m-1}.$$

T is one-to-one and onto by Theorem 6.10.8. So, T is an isomorphism by Theorem 1.6.3. This forces that the function g_Ω^N is locally bounded in $\Omega \times \Omega$. Since $g_\Omega^N(\cdot, x) = g_\Omega^N(x, \cdot) \in H(\Omega)$ (see Proposition 6.15.1), Theorem 2.7.1 gives that $g_\Omega^N \in H(\Omega \times \Omega)$.

Theorem 6.15.3 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$, $m \geq 2$, *and* $0 < \alpha < 1$. *Let* $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ *be a classical solution of the problem* $-\Delta u = f$ *in* Ω, $\partial u/\partial n = g$ *on* $\partial\Omega$. *If* $f \in L^1(\Omega)$, *then*

$$u(x) = \int\limits_{\Omega} N_\Omega(x,y) f(y)\, \mathcal{H}_m(y) + \int\limits_{\partial\Omega} N_\Omega(x,y) g(y)\, \mathcal{H}_{m-1}(y) + \oint\limits_{\partial\Omega} u \,\mathrm{d}\mathcal{H}_{m-1}.$$

Proof Let $\Omega(k)$ be a sequence of domains from Lemma 1.17.13. Since $g_\Omega^N(\cdot, x) \in H(\Omega) \cap \mathcal{C}^1(\overline{\Omega}(k))$, we have by second Green formula (Theorem 2.2.1)

$$0 = \int\limits_{\partial\Omega(k)} \left[u\frac{\partial g_\Omega^N(\cdot,x)}{\partial n} - g_\Omega^N(\cdot,x)\frac{\partial u}{\partial n}\right] \mathrm{d}\mathcal{H}_{m-1} - \int\limits_{\Omega(k)} g_\Omega^N(\cdot,x) f \,\mathrm{d}\mathcal{H}_m.$$

If $x \in \Omega(k)$, then theorem on three potentials (Theorem 2.2.4) gives

$$u(x) = V(f\chi_{\Omega(k)}) + \mathcal{S}^{\Omega(k)}\frac{\partial u}{\partial n}(x) + \mathcal{D}^{\Omega(k)}u(x).$$

Adding we get

$$u(x) = \int\limits_{\Omega(k)} N_\Omega(\cdot, x)f + \int\limits_{\partial\Omega(k)} \left[N_\Omega(\cdot, x)\frac{\partial u}{\partial n} - u\frac{\partial N_\Omega(\cdot, x)}{\partial n}\right].$$

Letting $k \to \infty$, we get by virtue of Lebesgue lemma (Lemma 1.13.2)

$$u(x) = \int\limits_{\Omega} N_\Omega(y, x)f + \int\limits_{\partial\Omega} N_\Omega(y, x)g(y) + \int\limits_{\partial\Omega} \frac{u(y)}{\mathcal{H}_{m-1}(\partial\Omega)}.$$

The symmetry of N_Ω gives the proposition (see Proposition 6.15.1).

6.16 Classical Solutions of the Transmission Problem

In this section, we prove the existence of a solution $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_\pm)$ of the transmission problem for transmission conditions from $\mathcal{C}^{1,\lambda}(\partial\Omega_+) \times \mathcal{C}^{0,\lambda}(\partial\Omega_+)$ and an interface of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$.

Let $\Omega_+ \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$, $m \geq 2$, and $\Omega_- = R^m \setminus \overline{\Omega}_+$. Let $h \in \mathcal{C}^{0,\lambda}(\partial\Omega_+)$ and $a_\pm$, $b_\pm$ be positive constants. We shall study classical solutions of the transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{6.42}$$

$$a_+u_+ - a_-u_- = g \quad \text{on } \partial\Omega, \qquad b_+\frac{\partial u_+}{\partial n} - b_-\frac{\partial u_-}{\partial n} + hu_+ = f \quad \text{on } \partial\Omega \tag{6.43}$$

We suppose that $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ and $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$.

Lemma 6.16.1 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \alpha < 1$. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$. Let a_+, a_-, b_+, and b_- be positive constants and $h \in \mathcal{C}^0(\partial\Omega_+)$. If $u_\pm$ is a bounded classical solution of the transmission problems (6.42)–(6.43), then $u_\pm$ is an L^p-solution of the transmission problems (6.42)–(6.43) for every $p \in (1, \infty)$.*

Proof The function $u_\pm$ is a classical solution of some Robin problem on $\Omega_\pm$. So, it is an L^q-solution of some Robin problem by Corollary 6.9.3. Thus, $u_\pm$ is an L^q-solution of the transmission problems (6.42) and (6.43).

Theorem 6.16.2 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$, $m > 2$, *and* $0 < \lambda < \alpha < 1$. *Denote* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Let* a_+, a_-, b_+, *and* b_- *be positive constants,* $h \in \mathcal{C}^{0,\lambda}(\partial\Omega_+)$, *and* $h \geq 0$. *Then the operator* $T = [(b_+ + b_-)/2]I - (b_+ - b_-)K'_\Omega + h\mathcal{S}^\Omega$ *is an isomorphism on* $\mathcal{C}^{0,\lambda}(\partial\Omega)$. *Fix* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ *and* $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. *If* $u_\pm$ *is a bounded classical solution of the transmission problem (6.42)–(6.43), then there exists a constant* u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *On the other hand, if* $u_\infty \in R^1$ *is given, then there exists a unique classical solution of the transmission problem (6.42)–(6.43) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,* $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$ *and*

$$\|u_\pm\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})} \leq C(\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} + \|f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + |u_\infty|) \tag{6.44}$$

where C *depends only on* Ω, $a_\pm$, $b_\pm$, h, *and* λ. *If* $a_+ = a_- = 1$, *then* $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega T^{-1}F + u_\infty$ *where* $F = f - (b_+ - b_-)H^\Omega g - h(\frac{1}{2}g + K_\Omega g + u_\infty)$.

Proof The operators K'_Ω, $h\mathcal{S}^\Omega$ are compact linear operators on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Corollary 6.10.2, Proposition 6.7.1, and Theorem 1.7.4. Thus, T is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Theorem 1.8.3). The operator T is an isomorphism on $L^2(\partial\Omega)$ by Propositions 5.12.2 and 5.12.4. Thus, T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Lemma 1.8.4.

Let now $u_\pm$ be a bounded classical solution of the transmission problem (6.42)–(6.43). According to Proposition 2.17.1, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Lemma 6.16.1 gives that $u_\pm$ is an L^2-solution of the transmission problem (6.42)–(6.43). According to Propositions 5.12.2 and 5.12.4, there exists at most one classical solution $u_\pm$ of the transmission problem (6.42)–(6.43) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$.

Let now $a_+ = a_- = 1$. If $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega T^{-1}F + u_\infty$, then $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$ is a solution of the transmission problem (6.42)–(6.43) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$, and the estimate (6.44) holds. (See Propositions 6.11.1, 6.10.4, 6.10.5, and 6.7.2.)

Let now a_+, a_- be arbitrary. Then $u_\pm$ is a solution of the transmission problem (6.42)–(6.43) if and only if $v_\pm = a_\pm u_\pm$ is a solution of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$v_+ - v_- = g \quad \text{on } \partial\Omega, \quad \frac{b_+}{a_+}\frac{\partial v_+}{\partial n} - \frac{b_-}{a_-}\frac{\partial v_-}{\partial n} + \frac{h}{a_+}v_+ = f \quad \text{on } \partial\Omega.$$

Theorem 6.16.3 *Let* $\Omega = \Omega_+ \subset R^2$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $0 < \lambda < \alpha < 1$. *Denote* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Let* a_+, a_-, b_+, *and* b_- *be positive constants and* $h \equiv 0$. *Denote* $X = \{v \in \mathcal{C}^{0,\lambda}(\partial\Omega); \int_{\partial\Omega} v \, d\mathcal{H}_1 = 0\}$. *Then the operator* $T = [(b_+ + b_-)/2]I - (b_+ - b_-)K'_\Omega$ *is an isomorphism on* X. *Fix* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ *and* $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. *Then there exists a bounded classical solution of the transmission problem (6.42)–(6.43) if and only if* $f \in X$. *If* $u_\pm$ *is* *a bounded classical solution of the transmission problem (6.42)–(6.43), then there exists a constant* u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *On the other hand, if* $u_\infty \in R^1$ *is given and* $f \in X$, *then there exists a unique classical solution of the transmission problem (6.42)–(6.43) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,* $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$, *and the estimate (6.44) holds. If* $a_+ = a_- = 1$, *then* $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega T^{-1} F + u_\infty$ *where* $F = f - (b_+ - b_-) H^\Omega g$.

Proof The operators K'_Ω, $h\mathcal{S}^\Omega$ are compact linear operators on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Corollary 6.10.2, Proposition 6.7.1, and Theorem 1.7.4. Thus, T is a Fredholm operator with index 0 on X (see Theorem 1.8.3 and Lemmas 5.9.6 and 1.8.7). The operator T is an isomorphism on $\{v \in L^2(\partial\Omega); \int_{\partial\Omega} v = 0\}$ by Proposition 5.12.3. Thus, T is an isomorphism on X by Lemma 1.8.4.

Let now $u_\pm$ be a bounded classical solution of the transmission problem (6.42)–(6.43). According to Proposition 2.17.1, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Lemma 6.16.1 gives that $u_\pm$ is an L^2-solution of the transmission problem (6.42)–(6.43). According to Proposition 5.12.3, we have $g \in X$, and there exists at most one classical solution $u_\pm$ of the transmission problem (6.42)–(6.43) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$.

Let now $a_+ = a_- = 1$. If $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega T^{-1} F + u_\infty$, then $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$ is a solution of the transmission problem (6.42)–(6.43) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$, and the estimate (6.44) holds. (See Propositions 6.11.1, 6.10.4, 6.10.5, and 6.7.2.)

Let now a_+ and a_- be arbitrary. Then $u_\pm$ is a solution of the transmission problem (6.42)–(6.43) if and only if $v_\pm = a_\pm u_\pm$ is a solution of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$v_+ - v_- = g \quad \text{on } \partial\Omega, \qquad \frac{b_+}{a_+}\frac{\partial v_+}{\partial n} - \frac{b_-}{a_-}\frac{\partial v_-}{\partial n} = f \quad \text{on } \partial\Omega.$$

Theorem 6.16.4 *Let* $\Omega = \Omega_+ \subset R^2$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $0 < \lambda < \alpha < 1$. *Denote* $\Omega_- = R^2 \setminus \overline{\Omega_+}$. *Let* a_+, a_-, b_+, *and* b_- *be positive constants,* $h \in \mathcal{C}^{0,\lambda}(\partial\Omega_+)$, $h \geq 0$, *and* $h \not\equiv 0$. *Denote* $T\varphi = (b_+ + b_-)(\varphi - \oint \varphi)/2 - (b_+ - b_-)K'_\Omega(\varphi - \oint \varphi) + h[\mathcal{S}^\Omega(\varphi - \oint \varphi) + \oint \varphi]$. *Then* T *is an isomorphism on* $\mathcal{C}^{0,\lambda}(\partial\Omega)$. *Fix* $g \in \mathcal{C}^{1,\lambda}(\partial\Omega)$ *and* $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. *Then there exists a unique bounded classical solution* $u_\pm$ *of the transmission problem (6.42)–(6.43). Moreover,* $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$ *and*

$$\|u_\pm\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})} \leq C(\|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega)} + \|f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}) \tag{6.45}$$

where C *depends only on* Ω, $a_\pm$, $b_\pm$, h, *and* λ. *If* $a_+ = a_- = 1$, *then* $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega(T^{-1}F - \oint_{\partial\Omega} T^{-1}F) + \oint_{\partial\Omega} T^{-1}F$ *where* $F = f - (b_+ - b_-)H^\Omega g - h(\frac{1}{2}g + K_\Omega g)$.

Proof The operators K'_Ω and $h\mathcal{S}^\Omega$ are compact linear operators on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Corollary 6.10.2, Proposition 6.7.1, and Theorem 1.7.4. Thus, T is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Theorem 1.8.3). The operator T is an isomorphism on $L^2(\partial\Omega)$ by Proposition 5.12.5. Thus, T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Lemma 1.8.4.

Let now $u_\pm$ be a bounded classical solution of the transmission problem (6.42)–(6.43). Lemma 6.16.1 gives that $u_\pm$ is an L^2-solution of the transmission problem (6.42)–(6.43). According to Proposition 5.12.5, there exists at most one classical solution $u_\pm$ of the transmission problem (6.42)–(6.43).

Let now $a_+ = a_- = 1$. If $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega(T^{-1}F - \oint_{\partial\Omega} T^{-1}F) + \oint_{\partial\Omega} T^{-1}F$, then $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega_\pm})$ is a solution of the transmission problem (6.42)–(6.43), and the estimate (6.45) holds. (See Propositions 6.11.1, 6.10.4, 6.10.5, and 6.7.2.)

Let now a_+, a_- be arbitrary. Then $u_\pm$ is a solution of the transmission problem (6.42)–(6.43) if and only if $v_\pm = a_\pm u_\pm$ is a solution of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$v_+ - v_- = g \quad \text{on } \partial\Omega, \qquad \frac{b_+}{a_+}\frac{\partial v_+}{\partial n} - \frac{b_-}{a_-}\frac{\partial v_-}{\partial n} + \frac{h}{a_+}v_+ = f \quad \text{on } \partial\Omega.$$

6.17 Classical Solutions of the Transmission Problem for Several Domains

In this section, we find a necessary and sufficient condition for the existence of a solution $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega_j})$ of the transmission problem, where the Euclidean space is divided into open set $\Omega_1, \dots, \Omega_k$ with compact Ljapunov boundary.

Let $\Omega_1, \dots, \Omega_k$ be disjoint open sets with compact boundary such that $R^m = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \dots \cup \overline{\Omega}_k$, $k \geq 2$. Suppose that for each component B of $\partial\Omega_i$, there exists $j \in \{1, \dots, k\}$ and $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \not\in \{i, j\}$. We suppose that $0 < \lambda < \alpha < 1$ and Ω_j have boundary of class $\mathcal{C}^{1,\alpha}$. We denote by n^j the unit exterior normal of Ω_j. Let $b_1, \dots, b_k$ be positive numbers. For $i < j$, let $f_{ij} \in \mathcal{C}^{0,\lambda}(\partial\Omega_i \cap \partial\Omega_j)$ and $g_{ij} \in \mathcal{C}^{1,\lambda}(\partial\Omega_i \cap \partial\Omega_j)$. We shall study classical solutions of the transmission problem

$$\Delta u_j = 0 \quad \text{in } \Omega_j, \tag{6.46}$$

$$u_i - u_j = g_{ij}, \qquad b_i\frac{\partial u_i}{\partial n^i} - b_j\frac{\partial u_j}{\partial n^i} = f_{ij} \quad \text{on } \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j, i < j. \tag{6.47}$$

Theorem 6.17.1 *Let $m > 2$ and Ω_1 be unbounded. If $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega_j})$ for $j = 1, \dots, k$ and $(u_1, \dots, u_k)$ is a solution of the transmission problem (6.46)–(6.47), then there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. Let $u_\infty \in R^1$ be given.*

Then there exist unique $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, *and* $j = 1, \dots, k$, *such that* $(u_1, \dots, u_k)$ *is a solution of the transmission problem (6.46)–(6.47) such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\|u_l\|_{\mathcal{C}^{1,\lambda}(\partial\Omega_l)} \leq C \sum_{1\leq i<j\leq k} \left(\|g_{ij}\|_{\mathcal{C}^{1,\lambda}(\Gamma_{ij})} + \|f_{ij}\|_{\mathcal{C}^{0,\lambda}(\Gamma_{ij})}\right) + |u_\infty| \tag{6.48}$$

where C *does not depend on* $[g_{ij}, f_{ij}]$ *and* u_∞.

Proof Let $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$ for $j = 1, \dots, k$ and $(u_1, \dots, u_k)$ be a solution of (6.46) and (6.47). According to Theorem 5.26.6, there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

According to Theorem 5.26.6, there is a unique L^2-solution $(u_1, \dots, u_k)$ of the transmission problem (6.46)–(6.47) such that $u_1(x) \to 0$ as $|x| \to \infty$. Let B be a component of $\partial\Omega_i$. Then there exists $j \in \{1, \dots, k\}$, and $j \neq i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i, j\}$. We choose disjoint open sets ω_i and ω_j such that $R^m = \overline{\omega}_i \cup \overline{\omega}_j$, $B = \partial\omega_i = \partial\omega_j$, $\Omega_i \subset \omega_i$, and $\Omega_j \subset \omega_j$. Corollary 5.6.2 gives that

$$u_i = \mathcal{D}^{\Omega_i} u_i + \mathcal{S}^{\Omega_i} \frac{\partial u_i}{\partial n^i} \text{ in } \Omega_i, \quad u_j = \mathcal{D}^{\Omega_j} u_j + \mathcal{S}^{\Omega_j} \frac{\partial u_j}{\partial n^j} \text{ in } \Omega_j. \tag{6.49}$$

Define

$$v_i = \mathcal{D}^{\omega_i} u_i + \mathcal{S}^{\omega_i} \frac{\partial u_i}{\partial n^i} \text{ in } \omega_i, \quad v_j = \mathcal{D}^{\omega_j} u_j + \mathcal{S}^{\omega_j} \frac{\partial u_j}{\partial n^j} \text{ in } \omega_j,$$

$g = v_i - v_j$, $f = b_i(\partial v_i/\partial n^i) - b_j(\partial v_j/\partial n^i)$. Choose an open set ω such that $B \subset \omega$ and $\overline{\omega} \cap \partial\Omega_l \subset B$ for $l = 1, \dots, k$. Since $u_i - v_i \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i \cap \overline{\omega})$, and $u_j - v_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j \cap \overline{\omega})$, we infer that $g \in \mathcal{C}^{1,\lambda}(B)$ and $f \in \mathcal{C}^{0,\lambda}(B)$. Since (v_i, v_j) is an L^2-solution of the transmission problem

$$\Delta v_i = 0 \text{ in } \omega_i, \quad \Delta v_j = 0 \text{ in } \omega_j,$$

$$v_i - v_j = g, \quad b_i \frac{\partial v_i}{\partial n_i} - b_j \frac{\partial v_j}{\partial n_i} = f \quad \text{on } B,$$

Theorems 5.26.6 and 6.16.2 give that $v_i \in \mathcal{C}^{1,\lambda}(\overline{\omega}_i)$, $v_j \in \mathcal{C}^{1,\lambda}(\overline{\omega}_j)$. Thus $u_i \in \mathcal{C}^{1,\lambda}(\overline{\omega} \cap \overline{\omega}_i)$. Hence, $u_i \in \mathcal{C}^{1,\lambda}(\partial\Omega_i)$, and $\partial u_i/\partial n^i \in \mathcal{C}^{0,\lambda}(\partial\Omega_i)$. The representation (6.49) and Propositions 6.10.4, 6.10.5, and 6.11.1 give that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$. If $u_\infty \in R^1$, then $w_j = u_j + u_\infty \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$ is a unique solution of the transmission problem (6.46)–(6.47) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

For a given u_∞, $[g_{ij},f_{ij}]$ we denote by $T(u_\infty, g_{ij},f_{ij})$ the solution of the transmission problem (6.46)–(6.47) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. We have proved that

$$T : R^1 \times \prod_{1\le i<j\le k} \mathcal{C}^{1,\lambda}(\Gamma_{ij}) \times \mathcal{C}^{0,\lambda}(\Gamma_{ij}) \to \prod_{1\le i\le k} \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$$

is a linear operator. The operator T is closed by Theorem 5.26.6. So, T is bounded by Theorem 1.4.15.

Theorem 6.17.2 *Let $m = 2$, Ω_1 be unbounded. If $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$ for $j = 1,\dots,k$ and $(u_1,\dots,u_k)$ is a solution of the transmission problem (6.46)–(6.47), then*

$$\sum_{1\le i<j\le k} \int_{\Gamma_{ij}} f_{ij}\, \mathrm{d}\mathcal{H}_1 = 0 \tag{6.50}$$

and there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. If (6.50) is satisfied and $u_\infty \in R^1$ is given, then there exist unique $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1,\dots,k$, such that $(u_1,\dots,u_k)$ is a solution of the transmission problem (6.46), (6.47) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. Moreover, (6.48) holds where C does not depend on $[g_{ij},f_{ij}]$ and u_∞.

Proof Let $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$ for $j = 1,\dots,k$ and $(u_1,\dots,u_k)$ be a solution of (6.46) and (6.47). According to Theorem 5.26.7, the relation (6.50) holds, and there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

Suppose that (6.50) holds. According to Theorem 5.26.7, there exists a unique L^2-solution $(u_1,\dots,u_k)$ of the transmission problem (6.46)–(6.47) such that $u_1(x) \to 0$ as $|x| \to \infty$. Let B be a component of $\partial\Omega_i$. Then there exists $j \in \{1,\dots,k\}$, and $j \ne i$, such that $B \subset \partial\Omega_i \cap \partial\Omega_j$ and $B \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i,j\}$. We choose disjoint open sets ω_i and ω_j such that $R^2 = \overline{\omega}_i \cup \overline{\omega}_j$, $B = \partial\omega_i = \partial\omega_j$, $\Omega_i \subset \omega_i$, and $\Omega_j \subset \omega_j$. Corollary 5.6.2 gives that

$$u_i = \mathcal{D}^{\Omega_i} u_i + \mathcal{S}^{\Omega_i} \frac{\partial u_i}{\partial n^i} \text{ in } \Omega_i, \quad u_j = \mathcal{D}^{\Omega_j} u_j + \mathcal{S}^{\Omega_j} \frac{\partial u_j}{\partial n^j} \text{ in } \Omega_j. \tag{6.51}$$

Choose constants c_i and c_j such that

$$\int_B \left(\frac{\partial u_i}{\partial n^i} + c_i\right) \mathrm{d}\mathcal{H}_1 = 0, \quad \int_B \left(\frac{\partial u_j}{\partial n^j} + c_j\right) \mathrm{d}\mathcal{H}_1 = 0.$$

Define

$$v_i = \mathcal{D}^{\omega_i} u_i + \mathcal{S}^{\omega_i} \left(\frac{\partial u_i}{\partial n^i} + c_i\right) \text{ in } \omega_i, \quad v_j = \mathcal{D}^{\omega_j} u_j + \mathcal{S}^{\omega_j} \left(\frac{\partial u_j}{\partial n^j} + c_j\right) \text{ in } \omega_j,$$

$g = v_i - v_j, f = b_i(\partial v_i/\partial n^i) - b_j(\partial v_j/\partial n^i)$. Choose a bounded open set ω such that $B \subset \omega$ and $\overline{\omega} \cap \partial\Omega_l \subset B$ for $l = 1, \dots, k$. Then $\mathcal{S}^{\omega_i} c_i \in \mathcal{C}^{1,\lambda}(\overline{\omega})$, and $\mathcal{S}^{\omega_j} c_j \in \mathcal{C}^{1,\lambda}(\overline{\omega})$ by Proposition 6.10.4. Since $u_i - v_i \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i \cap \overline{\omega})$, and $u_j - v_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j \cap \overline{\omega})$, we infer that $g \in \mathcal{C}^{1,\lambda}(B)$ and $f \in \mathcal{C}^{0,\lambda}(B)$. Since (v_i, v_j) is an L^2-solution of the transmission problem

$$\Delta v_i = 0 \text{ in } \omega_i, \quad \Delta v_j = 0 \text{ in } \omega_j,$$

$$v_i - v_j = g, \quad b_i \frac{\partial v_i}{\partial n_i} - b_j \frac{\partial v_j}{\partial n_i} = f \quad \text{on } B,$$

Theorems 5.26.7 and 6.16.3 give that $v_i \in \mathcal{C}^{1,\lambda}(\overline{\omega}_i)$ and $v_j \in \mathcal{C}^{1,\lambda}(\overline{\omega}_j)$. Thus $u_i \in \mathcal{C}^{1,\lambda}(\overline{\omega} \cap \overline{\omega}_i)$. Hence, $u_i \in \mathcal{C}^{1,\lambda}(\partial\Omega_i)$ and $\partial u_i/\partial n^i \in \mathcal{C}^{0,\lambda}(\partial\Omega_i)$. The representation (6.51), Proposition 6.10.4, Theorem 6.9.7, and Propositions 6.10.5 and 6.11.1 give that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$. If $u_\infty \in R^1$, then $w_j = u_j + u_\infty \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$ is a unique solution of the transmission problem (6.46)–(6.47) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

Denote by Y the space of all $[g_{ij}, f_{ij}] \in \mathcal{C}^{1,\lambda}(\Gamma_{ij}) \times \mathcal{C}^{0,\lambda}(\Gamma_{ij})$, and $1 \le i < j \le k$, satisfying (6.50). For given $u_\infty \in R^1$, $[g_{ij}, f_{ij}] \in Y$, we denote by $T(u_\infty, g_{ij}, f_{ij})$ the solution of the transmission problem (6.46)–(6.47) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. We have proved that

$$T : R^1 \times Y \to \prod_{1 \le i \le k} \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$$

is a linear operator. The operator T is closed by Theorem 5.26.7. So, T is bounded by Theorem 1.4.15.

6.18 Classical Solution of the Jump Problem on Cracks

In this section, we shall study regularity of L^p-solutions of the generalized jump problem. The necessary and sufficient conditions for the existence of an L^p-solution of the generalized jump problem are stated in Theorem 5.13.2.

Theorem 6.18.1 *Let $\Omega_+ \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$ and $\Omega_- = R^m \setminus \overline{\Omega}_+$. Denote by n the exterior unit normal of Ω_+. Let Γ be a closed subset of $\partial\Omega_+$. Let $h, f \in \mathcal{C}^{0,\lambda}(\partial\Omega_+)$, $g \in \mathcal{C}^{1,\lambda}(\partial\Omega_+)$, and $h \ge 0$. Suppose that $h = 0, f = 0$, and $g = 0$ on $\partial\Omega_+ \setminus \Gamma$. Let $1 < p < \infty$. Suppose that u is an L^p-solution of the generalized jump problem*

$$\Delta u = 0 \quad \text{in } R^m \setminus \Gamma,$$

$$[u]_+ - [u]_- = g, \quad \left[\frac{\partial u}{\partial n}\right]_+ - \left[\frac{\partial u}{\partial n}\right]_- + h[u]_+ = f \quad \text{on } \Gamma.$$

Then there exist $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_\pm)$ *and a constant* u_∞ *such that* $u = u_\pm$ *on* $\Omega_\pm$ *and* $u(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\|u_\pm\|_{\mathcal{C}^{1,\lambda}(\Omega_\pm)} \le C_1 \left[\|f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega_+)} + \|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega_+)} + |u_\infty| \right] \tag{6.52}$$

with a constant C_1 *that does not depend on* f, g, *and* u_∞. *If* $m = 2$ *and* $h \not\equiv 0$, *then*

$$\|u_\pm\|_{\mathcal{C}^{1,\lambda}(\Omega_\pm)} \le C_2 \left[\|f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega_+)} + \|g\|_{\mathcal{C}^{1,\lambda}(\partial\Omega_+)} | \right] \tag{6.53}$$

with a constant C_2 *that does not depend on* f *and* g.

Proof Since u is bounded at infinity, Proposition 2.17.1 gives that there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$.

Denote $u_\pm = u$ on $\Omega_\pm$. Then $u_\pm$ is an L^p-solution of the transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$u_+ - u_- = g, \quad \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} + h u_+ = f \quad \text{on } \partial\Omega_+.$$

Theorems 6.16.2, 6.16.3, and 6.16.4 $u_\pm \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_\pm)$ and (6.52). If $m = 2$ and $h \not\equiv 0$, then (6.53) holds by Theorem 6.16.4.

6.19 Classical Solution of the Oblique Derivative Problem in the Plane

In this section, we find a necessary and sufficient condition for the existence of a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the derivative oblique problem for a planar domain Ω with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$.

Let $\Omega \subset R^2$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \alpha < 1$. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector function on $\partial\Omega$. For $a \in \mathcal{C}^{1,\lambda}(R^2)$, we shall study classical solutions of the derivative oblique problem

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + a \frac{\partial u}{\partial \tau} + hu = 0 \text{ on } \partial\Omega. \tag{6.54}$$

Proposition 6.19.1 *Let* $\Omega \subset R^2$ *be a domain with connected compact boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $0 < \lambda < \alpha < 1$. *Let* $a \in \mathcal{C}^{1,\lambda}(R^2)$ *and* $c, h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. *Define*

$\tau = (-n_2^\Omega, n_1^\Omega)$ and $\Theta = n^\Omega + a\tau$. For $f \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, define

$$Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1,$$

$$u_f = \mathcal{S}^\Omega[\partial f/\partial\tau + c(f - Pf)] - \mathcal{D}^\Omega[a(f - Pf)] + Pf.$$

Then $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. Define

$$Tf = \Theta \cdot \nabla u_f + hu_f.$$

Then $T : \mathcal{C}^{1,\lambda}(\partial\Omega) \to \mathcal{C}^\lambda(\partial\Omega)$ is a bounded linear operator. Moreover,

$$f \mapsto Tf - (1 + a^2)\left(\frac{1}{2}I - K'_\Omega\right)\frac{\partial f}{\partial\tau}$$

is a compact operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Proof $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$, and $T : \mathcal{C}^{1,\lambda}(\partial\Omega) \to \mathcal{C}^{0,\lambda}(\partial\Omega)$ is a bounded linear operator by Propositions 6.10.4 and 6.11.1.

Denote $g = f - Pf$. According to Corollary 5.5.3, Propositions 5.2.2 and 5.2.3, and Theorems 5.3.5 and 5.4.8

$$Tf = \frac{1}{2}(\partial g/\partial\tau + cg) - K'_\Omega(\partial g/\partial\tau + cg) - \frac{\partial}{\partial\tau}\mathcal{S}^\Omega\left[\frac{\partial}{\partial\tau}(ag)\right]$$

$$+a\frac{\partial}{\partial\tau}\mathcal{S}^\Omega(\partial g/\partial\tau + cg) + \frac{a}{2}\left[\frac{\partial}{\partial\tau}(ag)\right] - aK'_\Omega\left[\frac{\partial}{\partial\tau}(ag)\right]$$

$$+h[\mathcal{S}^\Omega(\partial g/\partial\tau + cg) - \frac{1}{2}ag - K_\Omega ag + Pf].$$

By virtue of Theorem 5.4.7

$$Tf = (1 + a^2)\left(\frac{1}{2}I - K'_\Omega\right)\frac{\partial f}{\partial\tau} + Lf,$$

where $L = L_1 + L_2 + L_3 + L_4 + hP$ with

$$L_1 f = \frac{c}{2}(I - P)f - K'_\Omega c(I - P)f - \frac{\partial}{\partial\tau}\mathcal{S}^\Omega[(\partial a/\partial\tau)(I - P)f] + a\frac{\partial}{\partial\tau}\mathcal{S}^\Omega[c(I - P)f]$$

$$+\frac{a}{2}(\partial a/\partial\tau)(I - P)f - aK'_\Omega[(\partial a/\partial\tau)(I - P)f] - \frac{1}{2}ah(I - P)f - hK_\Omega a(I - P)f,$$

$$L_2 f = h\mathcal{S}^{\Omega}[\partial f/\partial\tau + c(I-P)f],$$

$$L_3 f(x) = \int\limits_{\partial\Omega} \frac{\partial h_2(x-y)}{\partial\tau}[a(x)-a(y)]\frac{\partial f(y)}{\partial\tau}\, \mathrm{d}\mathcal{H}_1(y),$$

$$L_4 f(x) = a(x)\int\limits_{\partial\Omega} \frac{\partial h_2(x-y)}{\partial n}[a(x)-a(y)]\frac{\partial f(y)}{\partial\tau}\, \mathrm{d}\mathcal{H}_1(y).$$

It is enough to prove that L_1, L_2, L_3, and L_4 are compact operators from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Theorem 1.7.4).

The identity is a compact linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 1.16.5. Since the operators K'_{Ω}, K_{Ω}, $\mathcal{S}^{\Omega}$, and $\frac{\partial}{\partial\tau}\mathcal{S}^{\Omega}$ are bounded linear operators on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Propositions 6.6.4 and 6.7.1, Corollary 6.10.2, Proposition 6.10.5), the operator L_1 is a compact linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Theorem 1.7.4 and Lemma 1.16.8.

The mapping $f \mapsto [\partial f/\partial\tau + c(I-P)f]$ is a bounded linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (compare Lemma 1.16.8). Since $\mathcal{S}^{\Omega}$ is a compact linear operator on $\mathcal{S}^{0,\lambda}(\partial\Omega)$ by Proposition 6.7.1, the operator L_2 is a compact linear operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Theorem 1.7.4.

Let $v = (v_1, v_2)$ be a continuous vector function on $\partial\Omega$ such that $|v| \le 1$. Define

$$Z\varphi(x) = \int\limits_{\partial\Omega} v(y)\cdot\nabla h_2(x-y)[a(x)-a(y)]\varphi(y)\, \mathrm{d}\mathcal{H}_1(y).$$

Choose a constant C_1 such that $|a(y)-a(x)| \le C_1|x-y|$ for $x, y \in \partial\Omega$. According to Corollary 2.9.6 and Lemma 2.9.7, there exists a constant C_2 such that $|\nabla h_2(x-y)| \le C_2|x-y|^{-1}$ and $|\nabla h_2(x-y) - \nabla h_2(z-y)| \le C_2|x-z||x-y|^{-2}$ for $x, y, z \in \partial\Omega$ with $|x-y| \ge 2|z-x|$. Since $|v(y)\cdot\nabla h_2(x-y)[a(x)-a(y)]| \le C_1C_2$, Proposition 1.26.3 gives that Z is a bounded linear operator on $\mathcal{C}^0(\partial\Omega)$. According to Corollary 1.17.6, there exists a constant C_3 such that $\mathcal{H}_1(B(x;r)\cap\partial\Omega) \le C_3 r$ for all $x \in R^2$ and $r > 0$. Fix $x, z \in \partial\Omega$, $\varphi \in \mathcal{C}^0(\partial\Omega)$ with $|\varphi| \le 1$. By virtue of Lemma 1.26.1

$$\begin{aligned}
|Z\varphi(x) - Z\varphi(z)| \le &\int\limits_{B(z;2|x-z|)\cap\partial\Omega} |v(y)\cdot\nabla h_2(x-y)[a(x)-a(y)]\varphi(y)|\, \mathrm{d}\mathcal{H}_1(y) \\
&+ \int\limits_{B(z;2|x-z|)\cap\partial\Omega} |v(y)\cdot\nabla h_2(z-y)[a(z)-a(y)]\varphi(y)|\, \mathrm{d}\mathcal{H}_1(y) \\
&+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |\nabla h_2(x-y)[a(x)-a(y)] - \nabla h_2(z-y)[a(z)-a(y)]|\, \mathrm{d}\mathcal{H}_1(y)
\end{aligned}$$

$$\le C_1C_2C_3 2|x-z| + C_1C_2C_3 2|x-z| + \int\limits_{\partial\Omega\setminus B(x;|x-z|)} |\nabla h_2(x-y)[a(x)-a(z)]|$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |\nabla h_2(x-y) - \nabla h_2(z-y)||a(z)-a(y)| \, \mathrm{d}\mathcal{H}_1(y)$$

$$\le 4C_1C_2C_3|x-z| + C_1|z-x|^\lambda \int\limits_{\partial\Omega\setminus B(x;|x-z|)} C_2|x-y|^{-\lambda} \, \mathrm{d}\mathcal{H}_1(y)$$

$$+ \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} C_1C_2\frac{|x-z|}{|z-y|} \, \mathrm{d}\mathcal{H}_1(y) \le 4C_1C_2C_3|x-z|$$

$$+C_1C_2C_3|z-x|^\lambda \frac{(\operatorname{diam}\partial\Omega)^{1-\lambda}}{1-\lambda} + C_1C_2|x-z|^\lambda \int\limits_{\partial\Omega\setminus B(z;2|x-z|)} |z-y|^{-\lambda} \, \mathrm{d}\mathcal{H}_1(y)$$

$$\le |x-z|^\lambda C_1C_2C_3(\operatorname{diam}\partial\Omega)^{1-\lambda}\left(4+\frac{2}{1-\lambda}\right).$$

Thus, Z is a bounded linear operator from $\mathcal{C}^0(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Since $\mathcal{C}^{0,\lambda}(\partial\Omega) \hookrightarrow \mathcal{C}^0(\partial\Omega)$ compactly by Proposition 1.16.5, the operator Z is compact on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Theorem 1.7.4. For $v = \tau$, we obtain with help of Theorem 1.7.4 that L_3 is a compact operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$. For $v = n^\Omega$, we deduce by virtue of Theorem 1.7.4 and Lemma 1.16.8 that L_4 is a compact operator from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Proposition 6.19.2 *Let* $\Omega \subset R^2$ *be a domain with connected compact boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $0 < \lambda < 1$, $a \in \mathcal{C}^{1,\lambda}(R^2)$, *and* $\tilde h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ *be such that* $\tilde h > 0$ *and* $|a| > 0$ *on* $\partial\Omega$. *Denote* $\tau = (-n_2^\Omega, n_1^\Omega)$*:* $\Theta = n^\Omega + a\tau$,

$$h = \tilde h + \frac{1}{2}\frac{\partial a}{\partial \tau},$$

$$c = \frac{3}{a}\frac{\partial a}{\partial \tau} - a.$$

For $f \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, *define*

$$Pf = \oint\limits_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1,$$

$$u_f = \mathcal{S}^\Omega[\partial f/\partial\tau + c(f-Pf)] - \mathcal{D}^\Omega[a(f-Pf)] + Pf.$$

Then $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. *Define*

$$Tf = \Theta \cdot \nabla u_f + hu_f.$$

Then $T : \mathcal{C}^{1,\lambda}(\partial\Omega) \to \mathcal{C}^{0,\lambda}(\partial\Omega)$ *is an isomorphism. Let* $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. *Put* $f = T^{-1}g$. *Then* u_f *is a unique solution of the derivative oblique problem (6.54) in* $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. *Moreover,*

$$\|u_f\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq C\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} \tag{6.55}$$

where a constant C *does not depend on* g.

Proof If $f \in \mathcal{C}^{1,\lambda}(\partial\Omega)$, then $u_f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Proposition 6.19.1.

$T_1 : f \to \partial f/\partial\tau$ is a Fredholm operator with index 0 from $\mathcal{C}^{1,\lambda}(\partial\Omega)$ to $\mathcal{C}^{0,\lambda}(\Omega)$. The operator $(1 + a^2)[(1/2)I - K'_\Omega]$ is Fredholm with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Corollary 6.10.2, Theorem 1.8.3, Lemma 1.16.8, and Theorem 1.8.6. So, $(1 + a^2)[(1/2)I - K'_\Omega] \circ T_1$ is a Fredholm operator with index 0 by Theorem 1.8.6. The operator $T - (1 + a^2)[(1/2)I - K'_\Omega] \circ T_1$ is compact by Proposition 6.19.1. Thus, $T : \mathcal{C}^{1,\lambda}(\partial\Omega) \to \mathcal{C}^{0,\lambda}(\partial\Omega)$ is a Fredholm operator with index 0 by Theorem 1.8.3. The operator $T : W^{1,2}(\partial\Omega) \to L^2(\partial\Omega)$ is an isomorphism by Proposition 5.24.3. Therefore, $T : \mathcal{C}^{1,\lambda}(\partial\Omega) \to \mathcal{C}^{0,\lambda}(\partial\Omega)$ is an isomorphism by Lemma 1.8.4.

Let now $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Put $f = T^{-1}g$. Then u_f is a solution of the derivative oblique problem (6.54) by Proposition 6.19.1. The uniqueness of a solution of the problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ follows from Proposition 5.24.3. The estimate (6.55) is a consequence of Propositions 6.10.4 and 6.11.1.

Proposition 6.19.3 *Let* $\Omega \subset R^2$ *be a domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$ *and* $0 < \alpha < 1$. *Let* $0 < \lambda < 1$, $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, *and* $a \in \mathcal{C}^{1,\lambda}(R^2)$. *For each component* Λ *of* $\partial\Omega$, *suppose that* $|a| > 0$ *on the whole* Λ *or* $a = 0$ *on the whole* Λ. *Define* $\tau = (-n_2^\Omega, n_1^\Omega)$ *and* $\Theta = n^\Omega + a\tau$. *For* $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, *define* $Pf = \oint_{\partial\Omega} f \, d\mathcal{H}_1$. *If* $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, *then* $\mathcal{S}^\Omega f \in \mathcal{C}^{1,\lambda}(\overline{\Omega} \cap B(0; r))$ *for every* $r > 0$. *For* $x \in \partial\Omega$, *define*

$$T_\Theta f(x) = \Theta(x) \cdot \nabla \mathcal{S}^\Omega f(y).$$

Then $T_\Theta(I-P) + h[\mathcal{S}^\Omega(I-P) + P]$ *is a Fredholm operator with index* 0 *on* $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Proof If $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $\mathcal{S}^\Omega f \in \mathcal{C}^{1,\lambda}(\overline{\Omega} \cap B(0; r))$ by Proposition 6.10.4.

Suppose first that $\partial\Omega$ is connected. Define

$$\tilde{h} = 1 + \frac{1}{2}\frac{\partial a}{\partial \tau} \quad \text{on } \partial\Omega.$$

If $a = 0$ on $\partial\Omega$, then $T_\Theta(I-P) + \tilde{h}[\mathcal{S}^\Omega(I-P) + P]$ is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.10.3. Let now $|a| > 0$ on $\partial\Omega$. If $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $u = \mathcal{S}^\Omega(I-P)f + Pf$ is a solution of the derivative oblique problem $\Delta u = 0$ in Ω,and $\Theta \cdot \nabla u + \tilde{h}u = g$ on $\partial\Omega$ if $T_\Theta(I-P)f + \tilde{h}[\mathcal{S}^\Omega(I-P)f + Pf] = g$. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then there exists a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of this derivative oblique problem by Proposition 6.19.2. Denote $\tilde{g} = \partial u/\partial n + u$. Then u is a solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + u = \tilde{g}$ on $\partial\Omega$. According to Theorem 6.10.7, there exists $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ such that $u = \mathcal{S}^{\Omega}(I-P)f + Pf$. So, $T_{\Theta}(I-P)f + \tilde{h}[\mathcal{S}^{\Omega}(I-P)f + Pf] = \Theta \cdot \nabla u + \tilde{h}u = g$ and $\{T_{\Theta}(I-P) + \tilde{h}[\mathcal{S}^{\Omega}(I-P) + P]\}(\mathcal{C}^{0,\lambda}(\partial\Omega)) = \mathcal{C}^{0,\lambda}(\partial\Omega))$. The operator $T_{\Theta}(I-P) + \tilde{h}[\mathcal{S}^{\Omega}(I-P) + P]$ is one-to-one by Theorem 5.24.7. Thus, $T_{\Theta}(I-P) + \tilde{h}[\mathcal{S}^{\Omega}(I-P) + P]$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Since $\{T_{\Theta}(I-P)+\tilde{h}[\mathcal{S}^{\Omega}(I-P)+P]\}-\{T_{\Theta}(I-P)+h[\mathcal{S}^{\Omega}(I-P)+P]\}$ is a compact operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ (see Proposition 6.7.1, Lemma 1.16.8, and Theorems 1.7.4), 1.8.3 gives that $T_{\Theta}(I-P) + h[\mathcal{S}^{\Omega}(I-P) + P]$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Suppose now that $\partial\Omega$ is not connected. Let $C_1, \dots, C_k$ be components of $R^m \setminus \Omega$. Denote $\Omega(j) = R^m \setminus C_j$ for $j = 1, \dots, k$. For $f \in \mathcal{C}^{0,\lambda}(\partial\Omega(j))$, denote

$$T_j f = \Theta \cdot [\nabla \mathcal{S}^{\Omega(j)} f]_{\Omega(j)}.$$

We have proved that $f \mapsto T_j(f - \oint_{\partial\Omega(j)} f)$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega(j))$. The operator T_j is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega(j))$ by Theorem 1.8.3. Define $Tf = T_j(f|\partial\Omega(j))$ on $\partial\Omega(j)$. Then T is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Since $T_{\Theta}(I-P) + h[\mathcal{S}^{\Omega}(I-P) + P] - T$ is a compact operator on $\mathcal{C}^{0,\lambda}(\partial\Omega)$, Theorem 1.8.3 gives that $T_{\Theta}(I-P) + h[\mathcal{S}^{\Omega}(I-P) + P]$ is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$.

Theorem 6.19.4 *Let $\Omega \subset R^2$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, and $h \equiv 0$ and a be a function that is constant on each component of $\partial\Omega$. Denote $\tau = (-n_2^{\Omega}, n_1^{\Omega})$ the unit tangential vector. Define $\Theta = n^{\Omega} + a\tau$. Denote $Y = \{g \in \mathcal{C}^{0,\lambda}(\partial\Omega); \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$. Then $T_{\Theta}(\mathcal{C}^{0,\lambda}(\partial\Omega)) = Y$. Denote by T the restriction of T_{Θ} onto Y. Then T is an isomorphism. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then there exists a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the derivative oblique problem (6.54) if and only if $g \in Y$. Let now $g \in Y$. Then the general form of a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the derivative oblique problem (6.54) is $\mathcal{S}^{\Omega}(T^{-1}g) + c$, where c is an arbitrary constant. If $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a solution of the derivative oblique problem (6.54), then*

$$\|\nabla u\|_{\mathcal{C}^{0,\lambda}(\overline{\Omega})} \le C\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)};$$

if moreover

$$\int_{\Omega} u \, \mathrm{d}\mathcal{H}_2 = 0, \tag{6.56}$$

then

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C\|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}. \tag{6.57}$$

Here a constant C does not depend on g.

Proof If $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a solution of the derivative oblique problem (6.54), then $g \in Y$ by Proposition 5.23.9. If $g \equiv 0$, then there exists a constant c such that $u = c$ in Ω (see Proposition 5.23.7).

Let $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Then $u = \mathcal{S}^{\Omega} f \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is solution of the derivative oblique problem (6.54) with $g = T_{\theta} f$. Thus $T_{\theta} f = g \in Y$. So, $T_{\theta}(\mathcal{C}^{0,\lambda}(\partial\Omega)) \subset Y$. The operator T_{Θ} is a Fredholm operator with index 0 on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.19.3. Since $T_{\Theta}(Y) \subset Y$, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. The operator T_{Θ} is an isomorphism on $\{f \in L^2(\partial\Omega); \int_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1 = 0\}$ (see Theorem 5.24.5). Lemma 1.8.4 forces that T is an isomorphism on Y.

Let $g \in Y$. If c is a constant, then $\mathcal{S}^{\Omega}(T^{-1} g) + c \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a solution of the derivative oblique problem (6.54). Moreover, $u = \mathcal{S}^{\Omega}(T^{-1} g) - \oint_{\Omega} \mathcal{S}^{\Omega}(T^{-1} g) \, \mathrm{d}\mathcal{H}_2$ is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that (6.56) holds. Proposition 6.10.4 gives that there exists a constant C such that (6.57) holds. If $v \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is another solution of the problem (6.54), then $u - v$ is constant. Thus, $\|\nabla v\|_{\mathcal{C}^{0,\lambda}(\overline{\Omega})} \le C \|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)}$.

Theorem 6.19.5 *Let $\Omega \subset R^2$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \lambda < \alpha < 1$, and $h \equiv 0$ and a be a function that is constant on each component of $\partial\Omega$. Denote $\tau = (-n_2^{\Omega}, n_1^{\Omega})$ the unit tangential vector. Define $\Theta = n^{\Omega} + a\tau$. Denote $Y = \{g \in \mathcal{C}^{0,\lambda}(\partial\Omega); \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$. Then $T_{\Theta}(\mathcal{C}^{0,\lambda}(\partial\Omega)) = Y$. Denote by T the restriction of T_{Θ} onto Y. Then T is an isomorphism. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then there exists a solution $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ of the derivative oblique problem (6.54) if and only if $g \in Y$. If $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a solution of the derivative oblique problem (6.54), then there exists a constant u_{∞} such that $u(x) \to u_{\infty}$ as $|x| \to \infty$. On the other hand, if $g \in Y$, and $u_{\infty} \in R$, then $u = \mathcal{S}^{\Omega} T^{-1} g + u_{\infty}$ is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $u(x) \to u_{\infty}$ as $|x| \to \infty$. Moreover,*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \le C \|g\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + |u_{\infty}|, \tag{6.58}$$

where a constant C does not depend on g and u_{∞}.

Proof Let $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ be a solution of the derivative oblique problem (6.54). Then $g \in Y$ by Proposition 5.23.9. Since u is bounded in a neighborhood of infinity, there exists a constant u_{∞} such that $u(x) \to u_{\infty}$ as $|x| \to \infty$. (See Proposition 2.17.1.)

Define $Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1$. The operator $T_{\Theta}(I - P)$ is a Fredholm operator with index 0 in $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.19.3. Since $T_{\Theta}(I - P)(Y) \subset Y$ by Theorem 5.24.6 and $T_{\Theta}(I - P) = T_{\Theta}$ on Y, the operator T is a Fredholm operator with index 0 by Lemma 1.8.8. The kernel of T is trivial by Theorem 5.24.6. Since T is a Fredholm operator with index 0, we have $T(Y) = Y$. Thus, T is an isomorphism by Theorem 1.6.3.

Let $g \in Y$ and $u_{\infty} \in R$. Then $u = \mathcal{S}^{\Omega}(T^{-1} g) + u_{\infty} \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Proposition 6.10.5. According to Theorem 5.24.6, u is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Proposition 6.10.5 gives that there exists a constant C such that (6.58) holds.

Theorem 6.19.6 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $a \in \mathcal{C}^{1,\lambda}(R^2)$ with $0 < \lambda < \alpha < 1$. Suppose that for each component Λ of $\partial\Omega$, we have $|a| > 0$ on the whole Λ or $a = 0$ on the whole Λ. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector. Define $\Theta = n^\Omega + a\tau$. Let $\tilde{h} \in \mathcal{C}^{0,\lambda}(\partial\Omega)$ and $\tilde{h} \geq 0$. Put $h = \tilde{h} + \frac{1}{2}\frac{\partial a}{\partial \tau}$. Suppose*

$$\int_{\partial\Omega} (|h| + |\tilde{h}|)\, \mathrm{d}\mathcal{H}_1 > 0,$$

i.e., $\tilde{h} \not\equiv 0$ or a is not locally constant. Define $Pf = \oint_{\partial\Omega} f\, \mathrm{d}\mathcal{H}_1$ and $Tf = T_\Theta(I - P)f + h[\mathcal{S}^\Omega(I-P)f + Pf]$. Then T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. Moreover, the estimate (6.57) holds with a constant C that does not depend on g.

Proof If $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a solution of the derivative oblique problem (6.54) with $g \equiv 0$, then $u \equiv 0$ by Theorem 5.24.7.

T is a Fredholm operator with index 0 in $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Proposition 6.19.3. Theorem 5.24.7 gives that T is an isomorphism on $L^2(\partial\Omega)$. Hence, T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Lemma 1.8.4.

Let $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Then $u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Propositions 6.10.4 and 6.10.5. Clearly, u is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. The estimate (6.57) is consequence of Propositions 6.10.4 and 6.10.5.

Theorem 6.19.7 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $a \in \mathcal{C}^{1,\lambda}(R^2)$ with $0 < \lambda < \alpha < 1$. Suppose that for each component Λ of $\partial\Omega$, we have $|a| > 0$ on the whole Λ or $a = 0$ on the whole Λ. Denote $\tau = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector. Define $\Theta = n^\Omega + a\tau$. Let $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $h \geq 0$, and $h \not\equiv 0$. Suppose that for each $x \in \partial\Omega$, we have $h(x) > 0$ or Ω has boundary of class $\mathcal{C}^2$ in a neighborhood of x. Define $Pf = \oint_{\partial\Omega} f\, \mathrm{d}\mathcal{H}_1$ and $Tf = T_\Theta(I-P)f + h[\mathcal{S}^\Omega(I - P)f + Pf]$. Then T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$. If $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. Moreover, the estimate (6.57) holds with a constant C that does not depend on g.*

Proof The uniqueness follows from Theorem 6.3.1.

If $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, then $\mathcal{S}^\Omega(I - P)f + Pf \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Propositions 6.10.4 and 6.10.5. Proposition 6.19.3 gives that T is a Fredholm operator with index 0 in $\mathcal{C}^{0,\lambda}(\partial\Omega)$. Let $f \in \operatorname{Ker} T$. Then $u = \mathcal{S}^\Omega(I-P)f + Pf$ is a solution of the problem (6.54) with $g \equiv 0$. Thus, $u = 0$ in Ω. So, u is a solution of the problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \Theta \cdot \nabla u + \left[h + \frac{1}{2}\frac{\partial a}{\partial \tau}\right] u = 0 \quad \text{on } \partial\Omega.$$

Theorem 6.19.6 gives that $f \equiv 0$. Since T is a Fredholm operator with index 0, it is onto. Hence, T is an isomorphism by Theorem 1.6.3.

Let $g \in \mathcal{C}^{0,\lambda}(\partial\Omega)$. Then $u = \mathcal{S}^{\Omega}(I-P)T^{-1}g + PT^{-1}g \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ is a unique solution of the derivative oblique problem (6.54) in $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. The estimate (6.57) is consequence of Propositions 6.10.4 and 6.10.5.

Corollary 6.19.8 *Let $\Omega \subset R^2$ be a domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ and $a \in \mathcal{C}^{1,\lambda}(R^2)$ with $0 < \lambda < \alpha < 1$. Suppose that for each component Λ of $\partial\Omega$, we have $|a| > 0$ on the whole Λ or $a = 0$ on the whole Λ. Denote $\tau = (-n_2^{\Omega}, n_1^{\Omega})$ the unit tangential vector. Define $\Theta = n^{\Omega} + a\tau$. Let $h \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $h \geq 0$, and $h \not\equiv 0$. Suppose that for each $x \in \partial\Omega$, we have $h(x) > 0$ or Ω has boundary of class $\mathcal{C}^2$ in a neighborhood of x. Let $1 < p < \infty$. For $f \in L^p(\partial\Omega)$, define $Pf = \oint_{\partial\Omega} f \, \mathrm{d}\mathcal{H}_1$ and $Tf = \Theta \cdot [\mathcal{S}^{\Omega}(I-P)f]_{\Omega} + h[\mathcal{S}^{\Omega}(I-P)f + Pf]$. Then T is an isomorphism on $L^p(\partial\Omega)$. If $g \in L^p(\partial\Omega)$, then $u = \mathcal{S}^{\Omega}(I-P)T^{-1}g + PT^{-1}g$ is a unique L^p-solution of the derivative oblique problem (6.54). Moreover,*

$$\|M_a(u)\|_{L^p(\partial\Omega)} + \|M_a(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)} \tag{6.59}$$

where a constant C does not depend on g.

Proof Fix $\lambda \in (0, \alpha)$. Then T is an isomorphism on $\mathcal{C}^{0,\lambda}(\partial\Omega)$ by Theorem 6.19.7. Since T is a Fredholm operator with index 0 in $L^p(\partial\Omega)$ by Proposition 5.24.4, Lemma 1.8.4 gives that T is an isomorphism in $L^p(\partial\Omega)$. If $g \in L^p(\partial\Omega)$, then $u = \mathcal{S}^{\Omega}(I-P)T^{-1}g + PT^{-1}g$ is an L^p-solution of the derivative oblique problem (6.54) by Lemma 5.2.1; Propositions 5.2.2, 5.2.3, and 5.2.8; and Theorem 5.4.7.

Let u be an L^p-solution of the problem (6.54) with $g \equiv 0$. Put $f = [n^{\Omega} \cdot \nabla u + u]_{\Omega}$. Then u is an L^p-solution of the Robin problem $\Delta u = 0$ in Ω and $\partial u/\partial n + u = f$ on $\partial\Omega$. According to Theorem 5.11.5, there exists $\psi \in L^p(\partial\Omega)$ such that $u = \mathcal{S}^{\Omega}(I-P)\psi + P\psi$. Since u is an L^p-solution of the problem (6.54) with $g \equiv 0$, we obtain $T\psi = 0$. Thus, $\psi \equiv 0$ and $u = \mathcal{S}^{\Omega}(I-P)\psi + P\psi \equiv 0$.

Equation (6.59) is a consequence of Propositions 5.2.2, 5.2.3, and 5.2.8.

6.20 The Dirichlet–Robin Problem

In this section, we study the mixed Dirichlet–Robin problem of the Poisson equation and discuss regularity of classical solutions.

Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $g \in \mathcal{C}^0(\Gamma_D)$ and $f, h \in \mathcal{C}^0(\Gamma_R)$. We shall study the mixed problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = f \text{ on } \Gamma_R. \tag{6.60}$$

Lemma 6.20.1 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, Γ_D, and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $g \in \mathcal{C}^0(\Gamma_D)$ and $f, h \in \mathcal{C}^0(\Gamma_R)$. Let u be a classical solution of the mixed Dirichlet–Robin problem (6.60). If Ω is unbounded, suppose moreover that $u(x) = O(1)$ as $|x| \to \infty$. Then u is a very weak L^2-solution of the mixed problem (6.60).*

Proof Let $\varphi \in \mathcal{C}_c^\infty(R^m \setminus \Gamma_D)$. Choose a bounded open set U with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_D \subset U$, $\Gamma_R \cap \overline{U} = \emptyset$, and $\varphi = 0$ on $\overline{U}$. Define $\omega = \Omega \setminus \overline{U}$, $h = 0$, and $f = \partial u/\partial n^\omega$ on $\partial\omega \setminus \Gamma_R$. Then u is a classical solution of the Robin problem

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = f \text{ on } \partial\omega.$$

Corollary 6.9.3 gives that u is an L^2-solution of the problem. Since u is a very weak L^2-solution of the problem by Lemma 5.20.1, we have

$$\int_\Omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1}$$

$$= \int_\omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1} = \int_{\Gamma_R} f\varphi\,\mathrm{d}\mathcal{H}_{m-1}.$$

Thus, u is a very weak L^2-solution of the mixed problem (6.60).

Corollary 6.20.2 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, Γ_D, and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $h \in \mathcal{C}^0(\Gamma_R)$ and $h \geq 0$. Let $u \in \mathcal{C}^{0,0}(\overline{\Omega})$ be a classical solution of the mixed Dirichlet–Robin problem (6.60) with $g \equiv 0$ and $f \equiv 0$. If Ω is unbounded and $m > 2$, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then $u \equiv 0$.*

Proof Lemma 6.20.1 gives that u is a very weak L^2-solution of the problem (6.60). Proposition 5.25.2 forces that $u \equiv 0$.

Proposition 6.20.3 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, Γ_D, and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. If v is a function defined on $D(v) \supset \Gamma_D$ and w is a function defined on $D(w) \supset \Gamma_R$, define*

$$P_D v(x) = \begin{cases} v(x) & x \in \Gamma_D, \\ 0 & x \in R^m \setminus \Gamma_D, \end{cases} \tag{6.61}$$

$$P_R w(x) = \begin{cases} w(x) & x \in \Gamma_R, \\ 0 & x \in R^m \setminus \Gamma_R. \end{cases} \tag{6.62}$$

Let $h \in \mathcal{C}^0(\Gamma_R)$. For $\psi \in \mathcal{C}^0(\partial\Omega)$, define

$$T\psi = \begin{cases} \frac{\psi}{2} + K_\Omega P_D\psi + \mathcal{S}^\Omega\psi & \text{on } \Gamma_D, \\ \frac{\psi}{2} - K'_\Omega\psi + H^\Omega P_D\psi + h(\mathcal{S}^\Omega\psi + \mathcal{D}^\Omega P_D\psi) & \text{on } \Gamma_R. \end{cases} \tag{6.63}$$

Then T is a Fredholm operator with index 0 *on $\mathcal{C}^0(\partial\Omega)$. If $\psi \in \mathcal{C}^0(\partial\Omega)$, then*

$$u = \mathcal{S}^\Omega\psi + \mathcal{D}^\Omega P_D\psi \tag{6.64}$$

is a classical solution of the mixed problem (6.60) if and only if $T\psi = P_Dg + P_Rf$. If $h \geq 0$ and $m > 2$, then T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$.

Proof The operator P_DTP_D is a Fredholm operator with index 0 on $\mathcal{C}^0(\Gamma_D)$ by Propositions 6.6.2 and 6.7.1 and Theorem 1.8.3. The operator P_RTP_R is a Fredholm operator with index 0 on $\mathcal{C}^0(\Gamma_R)$ by Theorems 6.9.4, 6.9.5, and 1.8.3. Thus, $P_DTP_D + P_RTP_R$ is a Fredholm operator with index 0 in $\mathcal{C}^0(\partial\Omega)$. The operator $T - (P_DTP_D + P_RTP_R)$ is a bounded linear operator from $\mathcal{C}^0(\partial\Omega)$ to $\mathcal{C}^{0,1}(\partial\Omega)$ by Theorem 1.13.7. Since $T-(P_DTP_D+P_RTP_R)$ is a compact operator on $\mathcal{C}^0(\partial\Omega)$ (see Proposition 1.16.5), the operator T is a Fredholm operator with index 0 on $\mathcal{C}^0(\partial\Omega)$ (see Theorem 1.8.3).

Let $\psi \in \mathcal{C}^0(\partial\Omega)$. Then u given by (6.64) is a classical solution of the mixed problem (6.60) if and only if $T\psi = P_Dg + P_Rf$. (See Propositions 6.6.3, 6.7.1, and 6.7.2.)

Let now $h \geq 0$ and $m > 2$. Let $\psi \in \mathcal{C}^0(\partial\Omega)$ and $T\psi = 0$. Then u given by (6.64) is a classical solution of the mixed problem (6.60) with $f \equiv 0$ and $g \equiv 0$. So, $u = 0$ in Ω by Corollary 6.20.2. Let ω be a component of $R^m \setminus \overline{\Omega}$ such that $\partial\omega \subset \Gamma_R$. The continuity of a single layer potential (Proposition 6.7.1) gives that u is a classical solution of the Dirichlet problem $\Delta u = 0$ in ω, $u = 0$ on $\partial\omega$. If ω is unbounded, then $u(x) \to 0$ as $|x| \to \infty$. So, $u = 0$ in ω (see Theorems 6.4.2 and 6.5.1). We have on $\partial\omega$

$$0 = \left[\frac{\partial u}{\partial n}\right]_\Omega - \left[\frac{\partial u}{\partial n}\right]_\omega = \left(\frac{\psi}{2} - K'_\Omega\psi\right) - \left(-\frac{\psi}{2} - K'_\Omega\psi\right) = \psi.$$

(See Theorem 5.4.8.) Thus, $\psi = 0$ on Γ_R. Since $\mathcal{S}^\Omega\psi + \mathcal{D}^\Omega\psi = 0$ in Ω, we have $\psi/2 + K^\Omega\psi + \mathcal{S}^\Omega\psi = 0$ on $\partial\Omega$ (see Propositions 6.6.3 and 6.7.1). Proposition 5.14.7 gives that $\psi = 0$. Since T is a Fredholm operator with index 0, we have $T(\mathcal{C}^0(\partial\Omega)) = \mathcal{C}^0(\partial\Omega)$. So, T is an isomorphism by Theorem 1.6.3.

Proposition 6.20.4 *Let $\Omega \subset R^2$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, Γ_D, and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let P_D and P_R be defined by (6.61) and (6.62) and T be prescribed* *by (6.63). Let* $h \in \mathcal{C}^0(\Gamma_R)$ *and* $h \geq 0$. *For* $\psi \in \mathcal{C}^0(\partial\Omega)$, *define*

$$P\psi = \oint_{\partial\Omega} \psi \, d\mathcal{H}_1, \tag{6.65}$$

$$\tilde{T}\psi = \begin{cases} T(I-P)\psi + P\psi & \text{on } \Gamma_D, \\ T(I-P)\psi + hP\psi & \text{on } \Gamma_R. \end{cases} \tag{6.66}$$

Then $\tilde{T}$ *is an isomorphism on* $\mathcal{C}^0(\partial\Omega)$. *If* $\psi \in \mathcal{C}^0(\partial\Omega)$, *then*

$$u = \mathcal{S}^\Omega(I-P)\psi + \mathcal{D}^\Omega P_D(I-P)\psi + P\psi \tag{6.67}$$

is a classical solution of the mixed problem (6.60) if and only if $\tilde{T}\psi = P_D g + P_R f$.

Proof Let $\psi \in \mathcal{C}^0(\partial\Omega)$. Then u given by (6.67) is a classical solution of the mixed problem (6.60) if and only if $T\psi = P_D g + P_R f$ by Proposition 6.20.3.

The operator $\tilde{T}$ is a Fredholm operator with index 0 on $\mathcal{C}^0(\Gamma_D)$ by Proposition 6.20.3 and Theorem 1.8.3. Let $\psi \in \mathcal{C}^0(\partial\Omega)$ and $\tilde{T}\psi = 0$. Then u given by (6.67) is a classical solution of the mixed problem (6.60) with $f \equiv 0$ and $g \equiv 0$. So, $u = 0$ in Ω by Corollary 6.20.2. Let ω be a component of $R^m \setminus \overline{\Omega}$ such that $\partial\omega \subset \Gamma_R$. The continuity of a single layer potential (Proposition 6.7.1) gives that u is a bounded classical solution of the Dirichlet problem $\Delta u = 0$ in ω, $u = 0$ on $\partial\omega$. So, $u = 0$ in ω (see Theorems 6.4.2 and 6.5.1). We have on $\partial\omega$

$$0 = \left[\frac{\partial u}{\partial n}\right]_\Omega - \left[\frac{\partial u}{\partial n}\right]_\omega = \left[\left(\frac{1}{2}I - K'_\Omega\right) - \left(-\frac{1}{2} - K'_\Omega\right)\right](I-P)\psi = (I-P)\psi.$$

(See Theorem 5.4.8.) Thus, $\psi = 0$ on Γ_R. Since $\mathcal{S}^\Omega(I-P)\psi + \mathcal{D}^\Omega(I-P)\psi + P\psi = 0$ in Ω, we have $[\frac{1}{2}I + K^\Omega + \mathcal{S}^\Omega](I-P)\psi + P\psi = 0$ on $\partial\Omega$ (see Propositions 6.6.3 and 6.7.1). Proposition 5.14.8 gives that $\psi = 0$. Since $\tilde{T}$ is a Fredholm operator with index 0, we have $\tilde{T}(\mathcal{C}^0(\partial\Omega)) = \mathcal{C}^0(\partial\Omega)$. So, $\tilde{T}$ is an isomorphism by Theorem 1.6.3.

Theorem 6.20.5 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha, \beta < 1$, $0 < \lambda < \alpha$, Γ_D, *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_R \cup \Gamma_D$. *Let* $h \in \mathcal{C}^0(\partial\Omega)$, $h \geq 0$, $g \in \mathcal{C}^0(\Gamma_D)$, $f \in \mathcal{C}^0(\Gamma_R)$, *and* $F \in \mathcal{C}^{0,\beta}_{loc}(\Omega) \cap L^p(\Omega)$ *with* $p > m$. *Then there exists a unique classical solution of the mixed problem*

$$-\Delta u = F \text{ in } \Omega, \quad u = g \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = f \text{ on } \Gamma_R. \tag{6.68}$$

Moreover,

$$\|u\|_{\mathcal{C}(\Omega)} \leq C_1 \left[\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}(\Gamma_D)} + \|F\|_{L^p(\Omega)}\right] \tag{6.69}$$

where a constant C_1 does not dependent on g, f, and F.

- *If $g \in \mathcal{C}^{0,\lambda}(\Gamma_D)$, then $u \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$ and*

$$\|u\|_{\mathcal{C}^{0,\lambda}(\Omega)} \leq C_2 \left[\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}^{0,\lambda}(\Gamma_D)} + \|F\|_{L^p(\Omega)}\right] \tag{6.70}$$

  where a constant C_2 does not depend on g, f, and F.
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$ and $\lambda \leq 1 - m/p$, then there exists a neighborhood ω of Γ_D such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*
- *If $h, f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$ and $\lambda \leq 1 - m/p$, then there exists a neighborhood ω of Γ_R such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, $h, f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, and $\lambda \leq 1 - m/p$, then $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\Omega)} \leq C_3 \left[\|f\|_{\mathcal{C}^{0,\lambda}(\Gamma_R)} + \|g\|_{\mathcal{C}^{1,\lambda}(\Gamma_D)} + \|F\|_{L^p(\Omega)}\right] \tag{6.71}$$

  where a constant C_3 does not depend on g, f, and F.

Proof The uniqueness of a solution follows from Corollary 6.20.2.

Put $F = 0$ on $R^m \setminus \Omega$. Then $VF \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(R^m)$ and

$$\|VF\|_{\mathcal{C}^{1,0}(\overline{\Omega})} \leq c_1 \|F\|_{L^p(\Omega)}$$

with a constant c_1 dependent only on Ω and p (see Theorem 3.14.2 and Proposition 3.15.1). If $\lambda \leq 1 - m/p$, then $VF \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and

$$\|VF\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega})} \leq c_2 \|F\|_{L^p(\Omega)}$$

with a constant c_2 dependent only on Ω, λ, and p. (See Proposition 3.15.1.) Theorem 3.1.5 gives $-\Delta VF = F$. Put $\tilde{f} = f - \partial VF/\partial n - hVF$ on Γ_R and $\tilde{g} = g - VF$ on Γ_D. We find a classical solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + hv = \tilde{f} \text{ on } \Gamma_R, \quad v = \tilde{g} \text{ on } \Gamma_D. \tag{6.72}$$

Let P_D, P, T, and $\tilde{T}$ be given by (6.61), (6.65), (6.63), and (6.66). If $m > 2$, then T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$, and

$$v = \mathcal{S}^\Omega T^{-1}[\tilde{g}, \tilde{f}] + \mathcal{D}^\Omega P_D T^{-1}[\tilde{g}, \tilde{f}]$$

is a classical solution of the problem (6.72). (See Proposition 6.20.3.) If $m = 2$, then $\tilde{T}$ is an isomorphism on $\mathcal{C}^0(\partial\Omega)$, and

$$v = \mathcal{S}^\Omega (I - P)\tilde{T}^{-1}[\tilde{g}, \tilde{f}] + \mathcal{D}^\Omega P_D (I - P)\tilde{T}^{-1}[\tilde{g}, \tilde{f}] + P\tilde{T}^{-1}[\tilde{g}, \tilde{f}]$$

is a classical solution of the problem (6.72). (See Proposition 6.20.4.) Clearly, $u = v + Vf$ is a classical solution of the problem (6.68). The estimate (6.69) is a consequence of Propositions 6.7.1 and 6.6.3.

Let $g \in \mathcal{C}^{0,\lambda}(\Gamma_D)$. Then $\tilde{g} \in \mathcal{C}^{0,\lambda}(\Gamma_D)$. Choose a bounded open set ω with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset \omega$ and $\Gamma_D \cap \omega = \emptyset$. Denote $\omega_R = \omega \cap \Omega$ and $\omega_D = \Omega \setminus \overline{\omega}$. Put $h = 1, \tilde{f} = \partial v/\partial n^\omega + v$ on $\partial\omega_R \setminus \partial\omega$. Then v is a classical solution of the Robin problem $\Delta v = 0$ in ω_R, $\partial v/\partial n + hv = \tilde{f}$ on $\partial\omega_R$. Theorem 6.9.4 gives that $v \in \mathcal{C}^{0,\lambda}(\overline{\omega}_R)$. Denote $\tilde{g} = v$ on $\partial\omega_D \setminus \Gamma_D$. Then v is a classical solution of the Dirichlet problem $\Delta v = 0$ in ω_D, $v = \tilde{g}$ on $\partial\omega_D$. Since $\tilde{g} \in \mathcal{C}^{0,\lambda}(\partial(\omega_D)$, Theorem 6.8.1 gives that $v \in \mathcal{C}^{0,\lambda}(\overline{\omega}_D)$. If $\tilde{g} \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, then $v \in \mathcal{C}^{1,\lambda}(\overline{\omega}_D)$ by Theorem 6.11.3. Since $v \in \mathcal{C}^\infty(\Omega)$, we infer that $v \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$.

Denote $v_{\tilde{f},\tilde{g}} = v$, i.e., our solution of the mixed problem (6.72). Then $[\tilde{f}, \tilde{g}] \mapsto v_{\tilde{f},\tilde{g}}$ is a linear operator from $\mathcal{C}^0(\Gamma_R) \times \mathcal{C}^{0,\lambda}(\Gamma_D)$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$. The relation (6.69) shows that this operator is closed. So, it is bounded by closed-graph theorem (Theorem 1.4.15). Hence, the relation (6.70) holds.

Let now $f, h \in \mathcal{C}^{0,\lambda}(\Gamma_R)$ and $\lambda \leq 1 - m/p$. Then $\tilde{f} \in \mathcal{C}^{0,\lambda}(\Gamma_R)$. Choose a bounded open set ω with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset \omega$ and $\partial\omega \cap \partial\Omega = \emptyset$. Denote $h = 1$ and $\tilde{f} = \partial v/\partial n^\omega + v$ on $\Omega \cap \partial\omega$. Then v is a classical solution of the Robin problem $\Delta v = 0$ in $\Omega \cap \omega$, $\partial v/\partial n + hv = \tilde{f}$ on $\partial(\omega \cap \Omega)$. Since $\tilde{f} \in \mathcal{C}^{0,\lambda}(\partial(\Omega \cap \omega))$, Theorem 6.10.10 gives that $\tilde{v} \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.

Let now $g \in \mathcal{C}^{1,\lambda}(\Gamma_D), f, h \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, and $\lambda \leq 1 - m/p$. We have proved that there exists a neighborhood ω of $\partial\Omega$ such that $v \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$. Since $v \in \mathcal{C}^\infty(\Omega)$, we deduce that $v \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. $[\tilde{f}, \tilde{g}] \mapsto v_{\tilde{f},\tilde{g}}$ is a linear operator from $\mathcal{C}^{0,\lambda}(\Gamma_R) \times \mathcal{C}^{1,\lambda}(\Gamma_D)$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. The relation (6.69) shows that this operator is closed. So, it is bounded by closed-graph theorem (Theorem 1.4.15). Hence, the relation (6.71) holds.

Theorem 6.20.6 *Let $\Omega \subset R^2$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$, Γ_D, and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $h \in \mathcal{C}^0(\partial\Omega)$, $h \geq 0$, $g \in \mathcal{C}^0(\Gamma_D)$, and $f \in \mathcal{C}^0(\Gamma_R)$. Let P_D, P, T, and $\tilde{T}$ be given by (6.61), (6.65), (6.63), and (6.66). Then $\tilde{T}$ is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. Put $\psi = \tilde{T}^{-1}[g,f]$. Then u given by (6.67) is a unique bounded classical solution of the mixed Dirichlet–Robin problem (6.60). Moreover,*

$$\|u\|_{\mathcal{C}(\Omega)} \leq C_1 \left(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}(\Gamma_D)}\right) \tag{6.73}$$

where a constant C_1 does not dependent on g and f.

- *If $g \in \mathcal{C}^{0,\lambda}(\Gamma_D)$, then $u \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$ and*

$$\|u\|_{\mathcal{C}^{0,\lambda}(\Omega)} \leq C_2 \left(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}^{0,\lambda}(\Gamma_D)}\right) \tag{6.74}$$

  where a constant C_2 does not depend on g and f.
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, then there exists a neighborhood ω of Γ_D such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*

- *If $h, f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, then there exists a neighborhood ω of Γ_R such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, and $h, f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, then $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and*

$$\|u\|_{\mathcal{C}^{1,\lambda}(\Omega)} \leq C_3 \left(\|f\|_{\mathcal{C}^{0,\lambda}(\Gamma_R)} + \|g\|_{\mathcal{C}^{1,\lambda}(\Gamma_D)} \right) \tag{6.75}$$

where a constant C_3 does not depend on g and f.

Proof The function u given by (6.67) is a unique bounded classical solution of the mixed Dirichlet–Robin problem (6.60) by Proposition 6.20.4 and Corollary 6.20.2. The estimate (6.73) is a consequence of Propositions 6.7.1 and 6.6.3.

Fix $r \in (1, \infty)$ such that $\partial\Omega \subset B(0; r)$. Then $u \in \mathcal{C}^2(R^2 \setminus B(0; r))$ and

$$\|u\|_{\mathcal{C}^2(R^2 \setminus B(0;r))} \leq c_0 \left(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}(\Gamma_D)} \right), \tag{6.76}$$

where c_0 does not depend on g and f. (This is a consequence of Theorem 1.13.7 and Corollary 2.9.6.) Let now $x, y \in R^2 \setminus B(0; 2r)$. If $|x - y| \geq 1$, then

$$|u(x) - u(y)| + |\partial_j u(x) - \partial_j u(y)| \leq 4\|u\|_{\mathcal{C}^2(R^2 \setminus B(0;r))} |x - y|^{\lambda}. \tag{6.77}$$

Let now $|x - y| < 1$. Then there exist $\theta_1, \theta_2 \in (0, 1)$ such that $u(x) - u(y) = (x - y) \cdot \nabla u(y + \theta_1(x - y))$ and $\partial_j u(x) - \partial_j u(y) = (x - y) \cdot \nabla \partial_j u(y + \theta_2(x - y))$. Thus (6.77) holds.

Define $g = u$ in Ω. Then u is a classical solution of the mixed problem $\Delta u = 0$ in $\Omega \cap B(0; 2r)$, $\partial u / \partial n + hu = f$ on Γ_R, and $u = g$ on $\partial(\Omega \cap B(0; 2r)) \setminus \Gamma_R$. We have proved that $g \in \mathcal{C}^{1,\lambda}(\partial B(0; r)$ and $\|g\|_{\mathcal{C}^{1,\lambda}(\partial B(0;r)} \leq 5C_0(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}(\Gamma_D)})$. The proposition of the theorem is a consequence of Theorem 6.20.5, (6.76) and (6.77).

Theorem 6.20.7 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \lambda < \alpha < 1$ and $m > 2$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_R \cup \Gamma_D$. Let $h \in \mathcal{C}^0(\partial\Omega)$, $h \geq 0$, $g \in \mathcal{C}^0(\Gamma_D)$, and $f \in \mathcal{C}^0(\Gamma_R)$. If u is a bounded classical solution of the mixed Dirichlet–Robin problem (6.60), then there exists a constant u_∞ such that*

$$u(x) \to u_\infty \quad \text{as } |x| \to \infty. \tag{6.78}$$

Let P_D and T be given by (6.61) and (6.63). Then T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$. Let $u_\infty \in R^1$ be fixed. Define

$$\psi = T^{-1}[g - u_\infty, f - hu_\infty].$$

Then

$$u = \mathcal{S}^{\Omega}\psi + \mathcal{D}^{\Omega} P_D \psi + u_\infty$$

is a unique classical solution of the mixed Dirichlet–Robin problem (6.60), (6.78). Moreover,

$$\|u\|_{\mathcal{C}(\Omega)} \le C_1 \left(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}(\Gamma_D)}\right) + |u_\infty| \tag{6.79}$$

where a constant C_1 does not dependent on g and f.

- *If $g \in \mathcal{C}^{0,\lambda}(\Gamma_D)$, then $u \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$ and*

  $$\|u\|_{\mathcal{C}^{0,\lambda}(\Omega)} \le C_2 \left(\|f\|_{\mathcal{C}(\Gamma_R)} + \|g\|_{\mathcal{C}^{0,\lambda}(\Gamma_D)}\right) + |u_\infty| \tag{6.80}$$

  where a constant C_2 does not depend on g and f.
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, then there exists a neighborhood ω of Γ_D such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*
- *If $h,f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, then there exists a neighborhood ω of Γ_R such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$.*
- *If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, and $h,f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, then $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ and*

  $$\|u\|_{\mathcal{C}^{1,\lambda}(\Omega)} \le C_3 \left(\|f\|_{\mathcal{C}^{0,\lambda}(\Gamma_R)} + \|g\|_{\mathcal{C}^{1,\lambda}(\Gamma_D)}\right) + |u_\infty| \tag{6.81}$$

  where a constant C_3 does not depend on g and f.

Proof Let u be a bounded classical solution of the mixed problem (6.60). According to Proposition 2.17.1, there exists a constant u_∞ such that (6.78) holds.

Let u_∞ be given. Proposition 6.20.3 and Corollary 6.20.2 give that T is an isomorphism on $\mathcal{C}^0(\partial\Omega)$ and $u = \mathcal{S}^\Omega\psi + \mathcal{D}^\Omega P_D\psi + u_\infty$ is a unique classical solution of the mixed Dirichlet–Robin problem (6.60), (6.78). The estimate (6.79) is a consequence of Propositions 6.7.1 and 6.6.3.

Define $g = u$ in Ω. Fix $r \in (0,\infty)$ such that $\partial\Omega \subset B(0;r)$. Since u is a classical solution of the Dirichlet problem $\Delta u = 0$ in $\Omega\setminus\overline{B(0;r)}$, $u = g$ on $\partial(\Omega\setminus\overline{B(0;r)})$, and $g \in \mathcal{C}^{1,\lambda}(\partial(\Omega\setminus\overline{B(0;r)}))$, Theorems 6.11.4 and 6.5.1 give that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega}\setminus B(0;r))$.

Let $g \in \mathcal{C}^{0,\lambda}(\Gamma_D)$. Since u is a classical solution of the mixed problem $\Delta u = 0$ in $\Omega \cap B(0;2r)$, $\partial u/\partial n + hu = f$ on Γ_R, $u = g$ on $\partial(\Omega \cap B(0;2r)) \setminus \Gamma_R$, and $g \in \mathcal{C}^{0,\lambda}(\partial(\Omega \cap B(0;2r)))$, Theorem 6.20.5 gives that $u \in \mathcal{C}^{0,\lambda}(\overline{\Omega \cap B(0;2r)})$. Hence, $u \in \mathcal{C}^{0,\lambda}(\overline{\Omega})$.

Define

$$A(g,f,c) = \mathcal{S}^\Omega T^{-1}[g-c,f-hc] + \mathcal{D}^\Omega P_D T^{-1}[g-c,f-hc] + c.$$

We have proved that A is a linear operator from $\mathcal{C}^0(\Gamma_R)\times\mathcal{C}^{0,\lambda}(\Gamma_D)\times R^1$ to $\mathcal{C}^{0,\lambda}(\overline{\Omega})$. The relation (6.79) shows that A is a closed operator. So, A is bounded by closed-graph theorem (Theorem 1.4.15). Hence, the relation (6.80) holds.

u is a classical solution of the mixed problem $\Delta u = 0$ in $\Omega \cap B(0;r)$, $\partial u/\partial n + hu = f$ on $\partial\Gamma_R$, and $u = g$ on $\Gamma_D \cup \partial B(0;r)$. If $f,h \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, then there exists a neighborhood ω of Γ_R such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$. If $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$, then there exists a neighborhood ω of Γ_D such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$. (See Theorem 6.20.5.)

Let now $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$ and $f, h \in \mathcal{C}^{0,\lambda}(\Gamma_R)$. We have proved that there exists a neighborhood ω of $\partial\Omega$ such that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega \cap \omega})$. Since $u \in \mathcal{C}^{\infty}(\Omega)$, and $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega} \setminus B(0;r))$, we deduce that $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$. It is enough to prove (6.81) for $u_\infty = 0$. The mapping A is a linear operator from $\mathcal{C}^{0,\lambda}(\Gamma_R) \times \mathcal{C}^{1,\lambda}(\Gamma_D) \times R^1$ to $\mathcal{C}^{1,\lambda}(\overline{\Omega})$. The relation (6.79) shows that A is a closed operator. So, A is bounded by closed-graph theorem (Theorem 1.4.15). Hence, the relation (6.81) holds.

6.21 Classical Solutions of the Mixed Transmission Problem

In this section, we find a necessary and sufficient condition for the existence of a solution $(u_1, \dots, u_k) \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_1) \times \cdots \times \mathcal{C}^{1,\lambda}(\overline{\Omega}_k)$ of the mixed transmission problem for a domain Ω divided into open sets $\Omega_1, \dots, \Omega_k$ with compact Ljapunov boundary.

Let $\Omega \subset R^m$ be a domain with nonempty compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint closed subset of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $\Omega_1, \dots, \Omega_k$ be disjoint domains with compact boundary of class $\mathcal{C}^{1,\alpha}$ such that $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \cdots \cup \overline{\Omega}_k$ and $k \geq 2$. Denote $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Suppose that for each component T of $\partial\Omega_i$, one of the following conditions is fulfilled:

1. $T \subset \Gamma_D$ and $T \cap \overline{\Omega}_j = \emptyset$ for $j \neq i$.
2. $T \subset \Gamma_R$ and $T \cap \overline{\Omega}_j = \emptyset$ for $j \neq i$.
3. There exists $j \in \{1, \dots, k\}$, and $j \neq i$, such that $T \subset \Gamma_{ij}$, $T \cap \partial\Omega = \emptyset$, and $T \cap \overline{\Omega}_l = \emptyset$ for $l \notin \{i, j\}$.

We denote by n^j the unit exterior normal of Ω_j. So, $n^i = -n^j$ on Γ_{ij} for $i \neq j$.

Let $b_1, \dots, b_k$ be positive numbers, $0 < \lambda < \alpha$, $h \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, $f \in \mathcal{C}^{0,\lambda}(\Gamma_R)$, and $g \in \mathcal{C}^{1,\lambda}(\Gamma_D)$. For $i < j$, let $f_{ij} \in \mathcal{C}^{0,\lambda}(\Gamma_{ij})$ and $g_{ij} \in \mathcal{C}^{1,\lambda}(\Gamma_{ij})$. We shall study the following mixed transmission problem:

$$\Delta u_j = 0 \quad \text{in } \Omega_j, \tag{6.82}$$

$$u_j = g \quad \text{on } \Gamma_D \cap \partial\Omega_j, \qquad \frac{\partial u_j}{\partial n^{\Omega}} + h u_j = f \quad \text{on } \Gamma_R \cap \partial\Omega_j, \tag{6.83}$$

$$u_i - u_j = g_{ij}, \qquad b_i \frac{\partial u_i}{\partial n^i} - b_j \frac{\partial u_j}{\partial n^i} = f_{ij} \quad \text{on } \Gamma_{ij}, i < j. \tag{6.84}$$

Lemma 6.21.1 *Let $h \geq 0$ and $(u_1, \dots, u_k)$ be an L^2-solution of the mixed transmission problem (6.82), (6.83), (6.84). Then $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$.*

Proof Let B be a component of $\partial\Omega_i$. Choose a bounded domain $\omega \subset R^m$ with boundary of class $\mathcal{C}^\infty$ such that $B \subset \omega$ and $\overline{\omega} \cap (\cup_j \partial\Omega_j \setminus B) = \emptyset$. If $B \subset \partial\Omega$, define $g = u_i$ on $\partial\omega \cap \Omega_i$. Since u_i is an L^2-solution of the problem $\Delta u_i = 0$ in $\omega \cap \Omega_i$, $u_i = g$ on $\partial(\omega \cap \Omega_i) \setminus \Gamma_R$, $\partial u_i / \partial n^i + h u_i = f$ on $\Gamma_R \cap \partial(\omega \cap \Omega_i)$, Theorems 5.14.9, 5.14.10, 6.11.3, 5.25.5, 5.25.6, and 6.20.5 give that $u_i \in \mathcal{C}^{1,\lambda}(\overline{\omega \cap \Omega_i})$. Let now

$B \subset \partial\Omega_i \cap \partial\Omega_j$ for $j \neq i$. Define $\omega_1 = \omega \cap \Omega_i$, $v_1 = u_i$, $\tilde{b}_1 = b_i$, $\omega_2 = \omega \cap \Omega_j$, $v_2 = u_j$, $\tilde{b}_2 = b_j$, $\omega_3 = R^m \setminus \overline{\omega_1 \cup \omega_2}$, $v_3 \equiv 0$, and $\tilde{b}_3 = 1$. Define $\tilde{g}_{rs} = v_r - v_s$ and $\tilde{f}_{rs} = \tilde{b}_r \partial v_r / \partial n^r - \tilde{b}_s \partial v_s / \partial n^r$, where n^r is the unit exterior normal to ω_r. Then $g_{rs} \in \mathcal{C}^{1,\lambda}(\partial\omega_r \cap \partial\omega_s)$ and $f_{rs} \in \mathcal{C}^{0,\lambda}(\partial\omega_r \cap \partial\omega_s)$. Since (v_1, v_2, v_3) is an L^2-solution of the transmission problem $\Delta v_s = 0$ in ω_s, $\tilde{b}_r \partial v_r / \partial n^r - \tilde{b}_s \partial v_s / \partial n^r = f_{rs}$, $v_r - v_s = g_{rs}$ on $\partial\omega_r \cap \partial\omega_s$, Theorems 5.26.6, 5.26.7, 6.17.1, and 6.17.2 give that $v_r \in \mathcal{C}^{1,\lambda}(\overline{\omega}_r)$. If Ω_i is bounded, then $u_i \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$. Let now Ω_i be unbounded. Since $M_a(u_i) \in L^2(\partial\Omega_i)$, the function u_i is bounded. According to Proposition 2.17.1, there exists $u_\infty \in R^1$ such that $u_i(x) \to u_\infty$ as $|x| \to \infty$. Proposition 2.17.3 yields $|\nabla u_i(x)| = O(|x|^{1-m})$ and $|\nabla^2 u_i(x)| = O(|x|^{-m})$ as $|x| \to \infty$. This forces $u_i \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$.

Theorem 6.21.2 *Let Ω be bounded or $m = 2$. Let $h \geq 0$. Suppose that $\Gamma_D \neq \emptyset$ or $h(z) \neq 0$ for some $z \in \Gamma_R$. Then there exist unique $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, such that $(u_1, \dots, u_k)$ is a solution of the mixed transmission problem (6.82), (6.83), and (6.84). Moreover,*

$$\|u_l\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega}_l)} \leq C\Bigg[\|g\|_{\mathcal{C}^{1,\lambda}(\Gamma_D)} + \|f\|_{\mathcal{C}^{0,\lambda}(\Gamma_R)} + \sum_{1\leq i<j\leq k} \left(\|g_{ij}\|_{\mathcal{C}^{1,\lambda}(\Gamma_{ij})} + \|f_{ij}\|_{\mathcal{C}^{0,\lambda}(\Gamma_{ij})}\right)\Bigg]$$

where C does not depend on $[g_{ij}, f_{ij}]$, g, and f.

Proof According to Theorem 5.27.4, there exists a unique L^2-solution $(u_1, \dots, u_k)$ of the problem (6.82), (6.83), and (6.84). Lemma 6.21.1 gives that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$.

Denote by $T(g, f, g_{ij}, f_{ij})$, the solution of the mixed transmission problem (6.82), (6.83), and (6.84). We have proved that

$$T : \mathcal{C}^{1,\lambda}(\Gamma_D) \times \mathcal{C}^{0,\lambda}(\Gamma_R) \times \prod_{1\leq i<j\leq k} \mathcal{C}^{1,\lambda}(\Gamma_{ij}) \times \mathcal{C}^{0,\lambda}(\Gamma_{ij}) \to \prod_{i=1}^{k} \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$$

is a linear operator. The operator T is closed by Theorem 5.27.4. So, it is bounded by Theorem 1.4.15.

Theorem 6.21.3 *Let Ω be bounded, $\Gamma_D = \emptyset$, and $h \equiv 0$. Then there exist $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, such that $(u_1, \dots, u_k)$ is a solution of the mixed transmission problem (6.82), (6.83), (6.84) if and only if*

$$\sum_{1\leq i<j\leq k} \int_{\partial\Omega_i \cap \partial\Omega_j} f_{ij} \, \mathrm{d}\mathcal{H}_{m-1} + \sum_{i=1}^{k} \int_{\partial\Omega_i \cap \partial\Omega} b_i f \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{6.85}$$

If $v_j, u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, *and* $(u_1, \dots, u_k)$ *is a solution of the problem (6.82), (6.83), (6.84), then* $(v_1, \dots, v_k)$ *is a solution of the problem (6.82), (6.83), (6.84) if and only if* $u_j = v_j + c$, $j = 1, \dots, k$, *where c is a constant.*

Proof According to Theorem 5.27.5, there is an L^2-solution $(u_1, \dots, u_k)$ of (6.82), (6.83), (6.84) if and only if (6.85) holds. If $(u_1, \dots, u_k)$ is an L^2-solution of the problem (6.82), (6.83), (6.84), then the general form of L^2-solution of the problem (6.82), (6.83), (6.84) is $(u_1 + c, \dots, u_k + c)$ where c is a constant. Lemma 6.21.1 gives that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$.

Theorem 6.21.4 *Let* Ω_1 *be unbounded,* $m > 2$, *and* $h \ge 0$. *If* $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, *solve the mixed transmission problem (6.82), (6.83), (6.84), then there exists* $u_\infty \in R^1$ *such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *If* $u_\infty \in R^1$ *is given, then there exist unique* $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, *such that* $(u_1, \dots, u_k)$ *is a solution of the problem (6.82), (6.83), (6.84), and* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover*

$$\|u_l\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega}_l)} \le C\Bigg[\|f\|_{\mathcal{C}^{0,\lambda}(\Gamma_R)} + \|g\|_{\mathcal{C}^{1,\lambda}(\Gamma_D)}$$

$$+ \sum_{1 \le i < j \le k} \left(\|f_{ij}\|_{\mathcal{C}^{0,\lambda}(\Gamma_{ij})} + \|g_{ij}\|_{\mathcal{C}^{1,\lambda}(\Gamma_{ij})} \right) + |u_\infty| \Bigg]$$

where C does not depend on g, f, g_{ij}, f_{ij}, *and* u_∞.

Proof Let $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, and solve the mixed transmission problem (6.82), (6.83), (6.84). According to Theorem 5.27.6, there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

Let $u_\infty \in R^1$ be given. According to Theorem 5.27.6, there is a unique L^2-solution $(u_1, \dots, u_k)$ of the problem (6.82), (6.83), (6.84) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. Lemma 6.21.1 gives that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$.

Denote by $T(u_\infty, g, f, g_{ij}, f_{ij})$ the solution of the mixed transmission problem (6.82), (6.83), (6.84) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. We have proved that

$$T : R^1 \times \mathcal{C}^{1,\lambda}(\Gamma_D) \times \mathcal{C}^{0,\lambda}(\Gamma_R) \times \prod_{1 \le i < j \le k} \mathcal{C}^{1,\lambda}(\Gamma_{ij}) \times \mathcal{C}^{0,\lambda}(\Gamma_{ij}) \to \prod_{i=1}^{k} \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$$

is a linear operator. The operator T is closed by Theorem 5.27.6. So, it is bounded by Theorem 1.4.15.

Theorem 6.21.5 *Let* Ω_1 *be unbounded,* $m = 2$, $h \equiv 0$, *and* $\Gamma_D = \emptyset$. *If* $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, *solve the mixed transmission problem (6.82), (6.83), (6.84), and then (6.85) holds, and there exists* $u_\infty \in R^1$ *such that* $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *If (6.85) holds and* $u_\infty \in R^1$ *is given, then there exist unique* $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1, \dots, k$, *such that* $(u_1, \dots, u_k)$ *is a solution of the problem (6.82), (6.83), and*

(6.84) and $u_1(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover*

$$\|u_l\|_{\mathcal{C}^{1,\lambda}(\overline{\Omega}_l)} \le C\Bigg[\|f\|_{\mathcal{C}^{0,\lambda}(\partial\Omega)} + \sum_{1\le i<j\le k}\Bigg(\|f_{ij}\|_{\mathcal{C}^{0,\lambda}(\Gamma_{ij})} + \|g_{ij}\|_{\mathcal{C}^{1,\lambda}(\Gamma_{ij})}\Bigg) + |u_\infty|\Bigg]$$

where C does not depend on f, g_{ij}, f_{ij}, *and* u_∞.

Proof Let $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1,\dots,k$, and solve the mixed transmission problems (6.82), (6.83), and (6.84). According to Theorem 5.27.7, the relation (6.85) holds, and there exists $u_\infty \in R^1$ such that $u_1(x) \to u_\infty$ as $|x| \to \infty$.

Let (6.85) hold and $u_\infty \in R^1$ be given. According to Theorem 5.27.7, there is a unique L^2-solution $(u_1,\dots,u_k)$ of the problems (6.82), (6.83) and (6.84) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. Lemma 6.21.1 gives that $u_j \in \mathcal{C}^{1,\lambda}(\overline{\Omega}_j)$, $j = 1,\dots,k$.

Denote by Z the space of all $[f, g_{ij}, f_{ij}]$ satisfying (6.85), where $f \in \mathcal{C}^{0,\lambda}(\partial\Omega)$, $g_{ij} \in \mathcal{C}^{1,\lambda}(\Gamma_{ij})$, and $f_{ij} \in \mathcal{C}^{0,\lambda}(\Gamma_{ij})$, $i < j$. Denote by $T(u_\infty, f, g_{ij}, f_{ij})$ the solution of the mixed transmission problems (6.82), (6.83), and (6.84) such that $u_1(x) \to u_\infty$ as $|x| \to \infty$. We have proved that

$$T : R^1 \times Z \to \prod_{i=1}^{k} \mathcal{C}^{1,\lambda}(\overline{\Omega}_i)$$

is a linear operator. The operator T is closed by Theorem 5.27.7. So, it is bounded by Theorem 1.4.15.

6.22 Eigenvalue Problems

This section is devoted to eigenvalue problems. First we prove that all eigenvalues for the Dirichlet problem, for the Robin problem, and for the mixed problem are positive. Then we study the eigenvalue problem $\Delta^k u = \lambda u$ in Ω, $0 = u = \cdots = \Delta^{k-1} u$ on $\partial\Omega$. We show that all eigenvalues form a sequence of positive numbers such that $\lambda_j \to \infty$ and corresponding eigenfunctions form an orthonormal basis in $L^2(\Omega)$. Moreover, u is an eigenfunction of the problem if and only if u is an eigenfunction of the problem $\Delta u = \mu u$ on Ω and $u = 0$ on $\partial\Omega$ where $\lambda = \mu^k$.

Definition 6.22.1 Let $\Omega \subset R^m$ be a bounded domain, $m \ge 2$, $\Gamma_D \subset \partial\Omega$, $\theta = (\theta_1,\dots,\theta_m)$ be a nonzero vector function on $\partial\Omega\setminus\Gamma_D$, and h be a function on $\partial\Omega\setminus\Gamma_D$. We say that the number λ is an eigenvalue for the Poisson equation corresponding to a boundary value problem $u = 0$ on Γ_D and $\partial u/\partial\theta + hu = 0$ on $\partial\Omega \setminus \Gamma_D$, if there exists a nontrivial classical solution u of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \text{ and } \quad \partial u/\partial\theta + hu = 0 \text{ on } \partial\Omega \setminus \Gamma_D. \tag{6.86}$$

A nontrivial solution u of the problem is called an eigenfunction corresponding to λ.

Proposition 6.22.2 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$, $\Gamma_D \subset \partial\Omega$, $\theta = (\theta_1, \dots, \theta_m)$ be an outward-pointing vector function on $\partial\Omega \setminus \Gamma_D$, and h be a nonnegative function on $\partial\Omega \setminus \Gamma_D$. Let λ be an eigenvalue for the Poisson equation corresponding to a boundary value problem $u = 0$ on Γ_D, $\partial u/\partial\theta + hu = 0$ on $\partial\Omega \setminus \Gamma_D$, and u be a corresponding eigenfunction. Then $u \in \mathcal{C}^\infty(\Omega)$. Let now $h \geq 0$. Suppose $\Gamma_D \neq \emptyset$ or $h \not\equiv 0$. Assume moreover that for each $x \in \partial\Omega \setminus \overline{\Gamma}_D$, we have $h(x) > 0$ or Ω has boundary of class $\mathcal{C}^2$ in a neighborhood of x. Then $\lambda > 0$.*

Proof $u \in \mathcal{C}^0(\Omega)$. If $u \in \mathcal{C}^k(\Omega)$, then $u \in \mathcal{C}^{k+1}(\Omega)$ by Proposition 3.18.1. Thus $u \in \mathcal{C}^\infty(\Omega)$.

Suppose now that $h(x) > 0$ or Ω has boundary of class $\mathcal{C}^2$ in a neighborhood of x for each $x \in \partial\Omega \setminus \overline{\Gamma}_D$. Let $\lambda \leq 0$. For $x \in \Omega$, and $t \in R^1$, put $f(x,t) = -\lambda t$. Then $\Delta u = f(x,u)$, $u = 0$ on Γ_D, and $\partial u/\partial\theta + hu = 0$ on $\partial\Omega \setminus \Gamma_D$. Since $v \equiv 0$ is a classical solution of the same problem, Theorem 6.3.1 gives that $u \equiv v \equiv 0$. That is a contradiction.

Theorem 6.22.3 *Let $\Omega \subset R^m$, $m \geq 2$, be a bounded regular domain (e.g., a domain with Lipschitz boundary). Then there exist a sequence of positive numbers $\{\lambda_k\}$ and a complete orthonormal sequence $\{u_k\}$ in $L^2(\Omega)$ such that $u_k \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, $\lambda_k \leq \lambda_{k+1}$, and $\lambda_k \to \infty$ as $k \to \infty$, λ_k is an eigenvalue of the Dirichlet problem for the Poisson equation, and u_k is a corresponding eigenfunction. Moreover, $\lambda_1 < \lambda_2$ and $u_1 > 0$ in Ω. If λ is an eigenvalue of the Dirichlet problem for the Poisson equation with a corresponding eigenfunction u, then there exist $k(1), k(2) \in N$ and real numbers a_k such that $\lambda = \lambda_k$ for each $k(1) \leq k \leq k(2)$ and*

$$u = \sum_{k=k(1)}^{k(2)} a_k u_k. \tag{6.87}$$

If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ and $0 < \beta < \alpha < 1$, then $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$.

Proof Let G_Ω be the Green function of Ω. According to Theorem 3.20.1, there exist a sequence of real numbers μ_k and a complete orthonormal sequence $\{u_k\}$ in $L^2(\Omega)$ such that $u_k \in \mathcal{C}^{0,0}(\Omega) \cap \mathcal{C}^\infty(\Omega)$, $G_\Omega u_k = \mu_k u_k$, $|\mu_k| \geq |\mu_{k+1}|$, and $\mu_k \to 0$ as $k \to \infty$. Moreover, $|\mu_1| > |\mu_2|$ and $u_1 > 0$ in Ω. If $u \in L^2(\Omega)$ is nontrivial and $G_\Omega u = \mu u$ in Ω, then there exist $k(1), k(2) \in N$ and real numbers a_k such that $\mu = \mu_k$ for each $k(1) \leq k \leq k(2)$ and (6.87) holds true.

Put $\lambda_k = 1/\mu_k$. Then $-\Delta u_k = -\Delta \lambda_k G_\Omega u_k = \lambda_k u_k \in \mathcal{C}^0(\Omega)$ by Theorem 3.11.5. The induction argument and Proposition 3.18.1 give that $u_k \in \mathcal{C}^\infty(\Omega)$. Since $u_k = \lambda_k G_\Omega u_k$ and Ω is regular, Theorem 4.6.3 gives that $u_k \in \mathcal{C}^0(\overline{\Omega})$ and $u_k = 0$ on

$\partial\Omega$. (If Ω has Lipschitz boundary, then it is regular by Theorems 4.5.7 and 4.5.8.) Proposition 6.22.2 gives that $\lambda_k > 0$.

Suppose now that λ is an eigenvalue of the Dirichlet problem with a corresponding eigenfunction u. Proposition 6.22.2 gives that $u \in \mathcal{C}^\infty(\Omega)$ and $\lambda > 0$. Proposition 6.4.1 gives that $u = G_\Omega \lambda u$. Thus, $\mu = 1/\lambda$ is an eigenvalue of G_Ω with the eigenfunction u. If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ and $0 < \beta < \alpha < 1$, then $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ by Theorem 6.11.3.

Definition 6.22.4 Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$, $k \in N$, and $k > 1$. We say that the number λ is an eigenvalue for the Navier problem

$$(-\Delta)^k u = \lambda u \quad \text{in } \Omega, \qquad u = \Delta u = \cdots = \Delta^{k-1} u = 0 \quad \text{on } \partial\Omega, \tag{6.88}$$

if there exists a nontrivial $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^{2k}(\Omega)$ with $\Delta^j u \in \mathcal{C}^0(\overline{\Omega})$ for $j = 1, \dots, k-1$ such that (6.88) holds. This function u is called an eigenfunction corresponding to the eigenvalue λ.

Theorem 6.22.5 *Let $\Omega \subset R^m$ be a bounded regular domain, $m \geq 2$, $k \in N$, and $k > 1$. If λ is an eigenvalue of the Dirichlet problem for the Poisson equation and u is a corresponding eigenfunction, then $\mu = \lambda^k$ is an eigenvalue of the Navier problem (6.88), and u is a corresponding eigenfunction. On the other hand, if μ is an eigenvalue of the Navier problem (6.88) and u is a corresponding eigenfunction, then $\mu > 0$, $\lambda = \mu^{1/k}$ is an eigenvalue of the Dirichlet problem for the Poisson equation, and u is a corresponding eigenfunction.*

Proof Suppose first that λ is an eigenvalue of the Dirichlet problem for the Poisson equation and u is a corresponding eigenfunction. Then $u \in \mathcal{C}^\infty(\Omega)$ by Theorem 6.22.3. Easy calculation gives that $(-\Delta)^k u = \lambda^k u$ in Ω and $u = \Delta u = \cdots = \Delta^{k-1} u = 0$ on $\partial\Omega$.

Let now μ be an eigenvalue of the Navier problem (6.88) with an eigenfunction u. If $\mu = 0$, then $\Delta^{k-1} u$ is a classical solution of the problem $\Delta v = 0$ in Ω and $v = 0$ on $\partial\Omega$. So, $\Delta^{k-1} u \equiv 0$ (see Theorem 6.3.1). By the induction argument, we obtain that $u \equiv 0$. That is a contradiction. Thus, $\mu \neq 0$. Let G_Ω be the Green function of Ω. Proposition 6.4.1 and Theorem 6.3.1 give that $u = G_\Omega(-\Delta u)$ and $(-\Delta)^j u = G_\Omega(-\Delta)^{j+1} u$ for $j = 1, \dots, k-1$. Thus, $u = G^k_\Omega(-\Delta)^k u = G^k_\Omega \mu u$. Since $G^k_\Omega u = \mu^{-1} u$, we have $\mu^{-1} \in \sigma(G^k_\Omega)$. Spectral mapping theorem (Proposition 1.6.8) gives that there exists $\nu \in \sigma(G_\Omega)$ such that $\mu^{-1} = \nu^k$. The operator G_Ω is a compact operator in $L^\infty(\Omega)$ by Theorem 3.20.1. Since $\nu \neq 0$, there exists a nontrivial $\varphi \in L^\infty(\Omega, C)$ such that $G_\Omega \varphi = \nu\varphi$ (see Theorem 1.7.7). According to Theorem 3.20.1, we have $\nu \in R$, and we can choose $\varphi \in \mathcal{C}^\infty(\Omega)$. Proposition 6.4.1 gives that $\varphi = G_\Omega \nu^{-1} \varphi \in \mathcal{C}^0(\overline{\Omega})$ and $\varphi = 0$ on Ω. Moreover, $-\Delta\varphi = \nu^{-1}\varphi$ in Ω. Thus, $\lambda = \nu^{-1}$ is an eigenvalue of the Dirichlet problem for the Poisson equation. Theorem 6.22.3 gives that $\lambda > 0$. Clearly, $\mu = \lambda^k > 0$. According to Theorem 6.22.3, there exists a sequence of eigenfunctions $u_j \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ of the Dirichlet problem for the Poisson equation with eigenvalues $\lambda_j > 0$ such that $\{u_j\}$ is a complete orthonormal sequence in $L^2(\Omega)$. Moreover, there are $k(1), k(2) \in N$ such that $\lambda_j = \lambda$ if and only if $k(1) \leq j \leq k(2)$. We have

$$u = \sum_{j=1}^{\infty} c_j u_j, \quad c_j = \int_{\Omega} u u_j \, \mathrm{d}\mathcal{H}_m$$

by Theorem 1.9.9. Fix $j \in N$ such that $\lambda_j \neq \lambda$. We have proved that λ_j^k is an eigenvalue of the Navier problem (6.88) and u_j is a corresponding eigenfunction. Thus, $G_\Omega^k u_j = \lambda_j^{-k} u_j$. Since $G_\Omega(x, y) = G_\Omega(y, x)$ by Proposition 3.9.12, the operator G_Ω is self-adjoint in $L^2(\Omega, C)$. So, G_Ω^k is a self-adjoint operator in $L^2(\Omega, C)$, too (see Lemma 1.9.14). Since $G_\Omega^k u = \lambda^{-k} u$ and $\lambda^{-k} \neq \lambda_j^{-k}$, Proposition 1.9.17 gives

$$c_j = \int_{\Omega} u u_j \, \mathrm{d}\mathcal{H}_m = 0.$$

Thus

$$u = \sum_{j=k(1)}^{k(2)} c_j u_j$$

is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ.

6.23 Neumann Problem on the Halfspace

In this section, we show then a solution of the Neumann problem for the Laplace equation on the halfspace that is given by the single layer potential corresponding to the boundary condition.

Denote $R_+^m = \{[x_1, \ldots, x_m]; x_m > 0\}$, $R_-^m = \{[x_1, \ldots, x_m]; x_m < 0\}$.

Proposition 6.23.1 *Let u be a classical solution of the Neumann problem $\Delta u = 0$ in R_+^m and $\partial u/\partial n = 0$ on ∂R_+^m. If $u(x) = o(1)$ as $|x| \to \infty$, then $u \equiv 0$.*

Proof According Corollary 2.4.3, there exists $x \in \partial R_+^m$ such that $|u(x)| = M \equiv \sup |u|$. Without loss of generality, we can suppose that $u(x) = M$. If $u < M$ on R_+^m, then $\partial u(x)/\partial n > 0$ by Proposition 3.5.1, what is a contradiction. Thus, there exists $y \in R_+^m$ such that $u(y) = M$. Theorem 2.4.2 gives that $u \equiv M$. Since $u(z) = o(1)$ as $|z| \to \infty$, we infer that $u \equiv 0$.

Theorem 6.23.2 *Let* $g \in \mathcal{C}^0(\partial R^m_+)$ *and* $g(x) = O(|x|^{-1-\alpha})$ *as* $|x| \to \infty$ *with* $0 < \alpha < 1$. *If* $m = 2$, *suppose, moreover, that* g *has a compact support and*

$$\int\limits_{\partial R^m_+} g \, \mathrm{d}\mathcal{H}_1 = 0. \tag{6.89}$$

Put

$$\mathcal{S}_+ g(x) = \int\limits_{\partial R^m_+} h_m(x-y) g(y) \, \mathrm{d}\mathcal{H}_{m-1}(y). \tag{6.90}$$

Then $u = 2\mathcal{S}_+ g$ *is unique classical solution of the Neumann problem* $\Delta u = 0$ *in* R^m_+, $\partial u / \partial n (= -\partial_m u) = g$, *and* $u(x) = o(1)$ *as* $|x| \to \infty$. *If* $m = 2$, *then* $u(x) = O(|x|^{-1})$ *as* $|x| \to \infty$. *If* $m > 2$, *then* $u(x) = O(|x|^{-\alpha})$ *as* $|x| \to \infty$. *If* $g \in \mathcal{C}^{0.\lambda}_{loc}(\partial R^m_+)$, *and* $0 < \lambda < 1$, *then* $u \in \mathcal{C}^{1,\lambda}_{loc}(\overline{R}^m_+)$.

Proof $u \in H(R^m_+)$ by Theorem 3.10.3 and Propositions 3.9.2 and 3.11.2. Fix $z \in \partial R^m_+$. Choose $\varphi \in \mathcal{C}^\infty_c(R^m)$ such that $\varphi = 1$ on a neighborhood ω of z. Theorem on differentiability of parametrized integrals (Theorem 1.13.7) gives that $\mathcal{S}_+ g(1-\varphi) \in \mathcal{C}^\infty(\omega)$ and

$$\frac{\partial \mathcal{S}_+[g(1-\varphi)](z)}{\partial n} = \int\limits_{\partial R^m_+} \frac{z_m g(y)[1-\varphi(y)]}{\mathcal{H}_{m-1}(\partial B(0;1))|z-y|^m} \, \mathrm{d}\mathcal{H}_{m-1}(y) = 0.$$

Choose a bounded domain $\Omega \subset R^m_+$ with boundary of class $\mathcal{C}^\infty$ such that $\partial R^m_+ \cap \operatorname{spt} \varphi \subset \partial \Omega$. Put $g = 0$ on $R^m \setminus \partial R^m_+$. Since $\mathcal{S}_+(g\varphi) = \mathcal{S}^\Omega(g\varphi)$, Proposition 6.7.1 gives that $\mathcal{S}_+(g\varphi)$ is continuous in R^m. According to Proposition 6.7.2

$$\frac{\partial \mathcal{S}_+(\varphi g)(z)}{\partial n} = \frac{\partial \mathcal{S}^\Omega(\varphi g)(z)}{\partial n} = \frac{1}{2} g(z) - K'_\Omega g(z) = \frac{1}{2} g(z).$$

Thus, $\partial u(z)/\partial n = 2\partial \mathcal{S}_+ g(z)/\partial n = g(z)$. If $g \in \mathcal{C}^{1,\lambda}_{loc}(\partial R^m_+)$, then $\mathcal{S}_+(\varphi g) = \mathcal{S}^\Omega(\varphi g) \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ by Proposition 6.10.4, and thus $u \in \mathcal{C}^{1,\lambda}_{loc}(\overline{R}^m_+)$.

If $m = 2$, then $\mathcal{S}_+ g = O(|x|^{-1})$ as $|x| \to \infty$ by Proposition 3.1.6. Let now $m > 2$. According to Proposition 3.1.6

$$\int\limits_{\partial R^m_+ \cap B(0;1)} g(y) h_m(x-y) \, \mathrm{d}\mathcal{H}_{m-1}(y) = O(|x|^{2-m}) \quad \text{as } |x| \to \infty.$$

Fix a constant C such that $|h(y)| \le C|y|^{-1-\alpha}$ for $|y| \ge 1$. Put $c = \mathcal{H}_{m-2}(\partial B(0;1) \cap \partial R^m_+)$ and $d = (m-2)\mathcal{H}_{m-1}(\partial B(0;1))$. Let $x \in R^m$ and $|x| > 2$. Then

$$\int_{\partial R^m_+ \cap B(x;|x|/2)} |g(y)h_m(x-y)| \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le \int_{\partial R^m_+ \cap B(x;|x|/2)} C\left(\frac{|x|}{2}\right)^{-1-\alpha} \frac{|x-y|^{2-m}}{d} \, \mathrm{d}\mathcal{H}_{m-1}(y) = \frac{C2^\alpha c}{d}|x|^{-\alpha},$$

$$\int_{\{y \in \partial R^m_+; 1 \le |y| \le 2|x|\} \setminus B(x;|x|/2)} |g(y)h_m(x-y)| \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le \int_{\{y \in \partial R^m_+; 1 \le |y| \le 2|x|\} \setminus B(x;|x|/2)} C|y|^{-1-\alpha} \frac{|x|^{2-m}}{d2^{2-m}} \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

$$\le \frac{Cc2^{2m}}{(m-2-\alpha)d}|x|^{-\alpha},$$

$$\int_{\partial R^m_+ \setminus B(x;2|x|)} |g(y)h_m(x-y)| \, \mathrm{d}\mathcal{H}_{m-1}$$

$$\le \int_{\partial R^m_+ \setminus B(x;2|x|)} C|y|^{-1-\alpha} \frac{|y|^{2-m}}{d2^{2-m}} \, \mathrm{d}\mathcal{H}_{m-1} \le \frac{Cc2^m}{d\alpha}|x|^{-\alpha}.$$

Thus, $u(x) = O(|x|^{-\alpha})$ as $|x| \to \infty$.

The uniqueness follows from Proposition 6.23.1.

Proposition 6.23.3 *Let $u \in \mathcal{C}^1(\overline{R}^m_+) \cap \mathcal{C}^2(R^m_+)$,*

$$\Delta u = 0 \quad \text{in } R^m_+, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial R^m_+.$$

If $|\nabla u(x)| = o(1)$ or $u(x) = O(1)$ as $|x| \to \infty$, then u is constant.

Proof Put $u(x_1, \dots, x_m) = u(x_1, \dots, x_{m-1}, -x_m)$ for $x_m < 0$. Then $u \in H(R^m)$ by Proposition 2.19.2. The rest is an easy consequence of Corollary 2.16.4.

Proposition 6.23.4 *Let $g \in \mathcal{C}^0(\partial R^m_+)$ and $g(x) = O(|x|^{-1-\alpha})$ as $|x| \to \infty$ with $0 < \alpha < 1$. If $m = 2$, suppose, moreover, that g has a compact support and (6.89) holds. Let $\mathcal{S}_+ g$ be given by (6.90). Then $u = 2\mathcal{S}_+ g + c$ with a constant c is a* *general form of a bounded solution of the Neumann problem* $u \in H(R^m_+) \cap \mathcal{C}^1(\overline{R}^m_+)$ *and* $\partial u/\partial n = g$ *on* ∂R^m_+.

Proof The proposition is an easy consequence of Theorem 6.23.2 and Proposition 6.23.3.

6.24 Symmetry and the Dirichlet Problem

In this section, we show that a solution of the Dirichlet problem for the nonlinear Poisson equation with a symmetric boundary condition is also symmetric.

Theorem 6.24.1 *Let* $a, b \in R^m$, $|b| = 1$, $m \geq 2$, $c \in R^1$, *and* $L = \{x \in R^m; (x-a)\cdot b = 0\}$ *be a hyperplane. Denote* $Tx = a - 2[(x-a)\cdot b]b$ *the reflexion mapping with respect to L. Let* $\Omega \subset R^m$ *be a domain symmetric with respect to L, i.e.,* $T(\Omega) = \Omega$. *Let f be a measurable function on* $\Omega \times R^1$ *such that* $f(x, \cdot)$ *is non-decreasing and* $f(Tx, \cdot) = f(x, \cdot)$ *for each* $x \in \Omega$. *Let* $u \in \mathcal{C}^0(\overline{\Omega})$ *and* $u(Tx) = u(x)$ *for* $x \in \partial\Omega$. *If* Ω *is unbounded, suppose moreover* $u(x) \to c$ *as* $|x| \to \infty$. *If* $\Delta u = f(x, u)$ *in* Ω *in the sense of distributions, then u is symmetric with respect to L, i.e.,* $u(Tx) = u(x)$ *for each* $x \in \Omega$.

Proof We can change the coordinate system such that $L = \{x = [x_1, \ldots, x_m]; x_m = 0\}$ (see Lemma 2.1.2). Then the reflexion mapping is $T(x_1, \ldots, x_m) = [x_1, \ldots, x_{m-1}, -x_m]$. Put $v(x) = u(Tx)$. Easy calculation yields that $\Delta v(x) = \Delta u(Tx)$. Therefore, $\Delta v(x) = \Delta u(Tx) = f(Tx, u(Tx)) = f(x, v(x))$. If $x \in \partial\Omega$, then $u(x) = v(x)$. If Ω is unbounded, then $v(x) \to c$ as $|x| \to \infty$. Thus, $u \equiv v$ by Proposition 6.2.2.

Theorem 6.24.2 *Let* $a, b \in R^2$, $|b| = 1$ *and* $L = \{x \in R^m; (x-a)\cdot b = 0\}$ *be a line. Denote* $Tx = a - 2[(x-a)\cdot b]b$ *the reflexion mapping with respect to L. Let* $\Omega \subset R^2$ *be an unbounded domain symmetric with respect to L, i.e.,* $T(\Omega) = \Omega$, *such that* $\overline{\Omega} \neq R^2$. *Let f be a measurable function on* $\Omega \times R^1$ *such that* $f(x, \cdot)$ *is non-decreasing and* $f(Tx, \cdot) = f(x, \cdot)$ *for each* $x \in \Omega$. *Let* $u \in \mathcal{C}^0(\overline{\Omega})$, $u(Tx) = u(x)$ *for* $x \in \partial\Omega$, *and* $u(x) = O(1)$ *as* $|x| \to \infty$. *If* $\Delta u = f(x, u)$ *in* Ω *in the sense of distributions, then u is symmetric with respect to L, i.e.,* $u(Tx) = u(x)$ *for each* $x \in \Omega$.

Proof We can change the coordinate system such that $L = \{x = [x_1, x_2]; x_2 = 0\}$ (see Lemma 2.1.2). Then $T(x_1, x_2) = [x_1, -x_2]$. Put $v(x) = u(Tx)$. Easy calculation yields that $\Delta v(x) = \Delta u(Tx)$. Therefore, $\Delta v(x) = \Delta u(Tx) = f(Tx, u(Tx)) = f(x, v(x))$. If $x \in \partial\Omega$, then $u(x) = v(x)$. Since $v(x) - u(x) = O(1)$ as $|x| \to \infty$, Proposition 6.2.3 gives $u \equiv v$.

Example 6.24.3 The condition "$f(x, \cdot)$ is non-decreasing" in Theorems 6.24.1, and 6.24.2 is essential as this example shows.

Put $u(x,y) = (\sin x)(\sin y)$, $\Omega = (-\pi,\pi)\times(0,3\pi)$, $\tilde{\Omega} = R^2\setminus\overline{\Omega}$, $L_1 = \{[0,t]; t \in R^1\}$, and $L_2 = \{[t,3\pi/2]; t \in R^1\}$. Then Ω and $\tilde{\Omega}$ are symmetric with respect to L_1 and L_2, $u = 0$ on $\partial\Omega = \partial\tilde{\Omega}$, and $u(x,y) = O(1)$ as $|[x,y]| \to \infty$. But u is not symmetric with respect to L_1 and L_2 because $\Delta u = f(u)$, where $f(t) = -2t$ is decreasing.

Theorem 6.24.4 *Let $z \in R^m$, $r > 0$, $m \geq 2$, and $c \in R^1$. Suppose that f is a function on $\langle 0,\infty) \times R^1$ such that $f(t,\cdot)$ is non-decreasing for each $t \geq 0$. If $u \in \mathcal{C}^0(\overline{B(z;r)})$, $u = c$ on $\partial B(z;r)$, and $\Delta u(x) = f(|x-z|, u(x))$ in $B(z;r)$ in the sense of distributions, then there exists $\varphi \in \mathcal{C}^0(\langle 0;r\rangle)$ such that $u(x) = \varphi(|x-z|)$. Suppose, moreover, that $f(t,s) = g(s) \in \mathcal{C}^{1,\lambda}_{loc}(R^1)$. Then φ is a monotone function, $|\varphi'|$ is non-decreasing, $u \in \mathcal{C}^1(\overline{B(z;r)})$, and one of the following cases occurs:*

- *$u \equiv c$.*
- *$u > c$ in $B(z;r)$ and $\partial u/\partial n < 0$ on $\partial B(z;r)$.*
- *$u < c$ in $B(z;r)$ and $\partial u/\partial n > 0$ on $\partial B(z;r)$.*

Proof Fix $x, y \in B(z;r)$ such that $|x-z| = |y-z|$ and $x \neq y$. Put $L = \{w; (w-z)\cdot(x-y) = 0\}$. Then x and y are symmetric with respect to the hyperplane L. Since u is symmetric with respect to the hyperplane L (see Theorem 6.24.1), we have $u(x) = u(y)$. Therefore there exists $\varphi \in \mathcal{C}^0(\langle 0;r\rangle)$ such that $u(x) = \varphi(|x-z|)$.

Let now $f(t,s) = g(s) \in \mathcal{C}^{1,\lambda}_{loc}(R^1)$. Put $h(x) = g(u(x))$ for $|x-z| < r$, $h(x) = 0$ for $|x-z| > r+1$, and $h(x) = g(c)(r+1-|x-z|)$ for $r \leq |x-z| \leq r+1$. Then h is a continuous function with compact support. Theorem 3.13.4 gives that $Vh \in \mathcal{C}^1(R^m)$. Due to the symmetry of h, there exists a constant d such that $Vh = d$ on $\partial B(z;r)$. Since $-\Delta Vh = h = g(u)$ in $B(z;r)$ in the sense of distributions (see Theorem 3.1.5), $\Delta(u+Vh) = 0$ in $B(z;r)$ in the sense of distributions. Therefore, $u + Vh \in H(B(z;r))$ (see Corollary 2.18.3). Moreover, $u + Vh \in \mathcal{C}^0(\overline{B(z;r)})$, and $u+Vh = c+d$ on $\partial B(z;r)$. The uniqueness of the Dirichlet problem for the Laplace equation (Theorem 6.3.1) gives that $u+Vh \equiv c+d$ in $\overline{B(z;r)}$. Thus, $u = -Vh+c+d \in \mathcal{C}^1(\overline{B(z;r)})$. Proposition 3.21.1 gives that $|\varphi'(|x-z|)| = |\nabla u(x)| \leq |\nabla u(y)| = |\varphi'(|y-z|)|$ for $|x-z| < |y-z| < r$. Denote $M = \{t \in (0;r); \varphi'(t) = 0\}$. Then $M = (0,t)$ for some $t \in \langle 0,r\rangle$. Therefore, φ' does not change the sign, and φ is a monotone function. If $M = (0,r)$, then $u \equiv c$. In other cases, $u < c$ or $u > c$ in $B(z;r)$, and $\partial u/\partial n = \varphi' \neq 0$ on $\partial B(z;r)$.

6.25 Symmetry of the Robin Problem

In this section, we prove the monotony principle for the Robin problem of the nonlinear Poisson equation. As a consequence, we obtain that a solution of the Robin problem for the nonlinear Poisson equation in the ball depends only on the distance of the center of the ball provided the boundary condition is constant.

Proposition 6.25.1 *Let $\Omega \subset R^m$, $m \geq 2$, be a domain with boundary of class $\mathcal{C}^1$ and f be a measurable function on $\Omega \times R^1$ such that $f(x, \cdot)$ is non-decreasing for each $x \in \Omega$, $h \in \mathcal{C}^0(\partial\Omega)$, $h \geq 0$, $\theta \in \mathcal{C}^0(\partial\Omega, R^m)$, and $\theta(x) \cdot n^\Omega(x) > 0$ on Ω. Let $u, v \in \mathcal{C}^1(\overline{\Omega})$, $(\partial u/\partial\theta) + hu \geq (\partial v/\partial\theta) + hv$ on $\partial\Omega$ and $-\Delta u(x) + f(x, u) \geq -\Delta v(x) + f(x, v)$ in Ω in the sense of distributions. If Ω is unbounded, suppose moreover*

$$\liminf_{\substack{|x| \to \infty \\ x \in \Omega}} (u(x) - v(x)) \geq 0.$$

- *If $h > 0$ on $\partial\Omega$, then $u \geq v$ in Ω.*
- *Suppose that $\partial\Omega$ is of class $\mathcal{C}^2$. Then $u - v$ is constant or $u \geq v$ in Ω. If $h \not\equiv 0$, then $u \geq v$ in Ω.*

Proof Denote $V = \{x \in \Omega; u(x) < v(x)\}$. Suppose that $V \neq \emptyset$. According to Lemma 1.20.3, there exists a nonnegative measure μ such that $-\Delta(u-v) + f(\cdot, u) - f(\cdot, v) = \mu$. Since $-\Delta(u - v) = \mu - f(\cdot, u) + f(\cdot, v) \geq 0$ on V, the function $u - v$ is superharmonic in V by Corollary 3.19.5.

Put $M = \inf\{u(x) - v(x); x \in \overline{\Omega}\}$. Then $M < 0$, and $K = \{x \in \overline{\Omega}; u(x) - v(x) = M\}$ is a nonempty closed set. Suppose first that there exists $x \in K \cap \Omega$. Then there exists $r > 0$ such that $B(x; r) \subset V$. The superharmonic function $u - v$ attains its minimum in x, so $u - v$ is constant in $B(x; r)$ by Proposition 3.3.5. Since $K \cap \Omega$ is a nonempty open closed subset of the connected metric space Ω, we deduce $K \cap \Omega = \Omega$. Thus, $u = v + M$. If there exists $z \in \partial\Omega$ such that $h(z) > 0$, then $(\partial u(z)/\partial\theta) + h(z)u(z) - [(\partial v(z)/\partial\theta) + h(z)v(z)] = h(z)M < 0$, what is a contradiction.

Suppose now that $K \cap \Omega = \emptyset$. We can choose $z \in \partial\Omega$ such that $u(z) - v(z) = M < u(x) - v(x)$ for all $x \in \Omega$. Then $\partial(u - v)/\partial\theta \leq 0$ in z. If $h(z) > 0$, then $(\partial u(z)/\partial\theta) + h(z)u(z) - [(\partial v(z)/\partial\theta) + h(z)v(z)] \leq h(z)M < 0$, what is a contradiction. Let now $h(z) = 0$, and $\partial\Omega$ is of class $\mathcal{C}^2$. Since $u(z) - v(z) < 0$, there exists $\delta > 0$ such that $\Omega \cap B(z; \delta) \subset V$. Since there exists an open ball $B \subset \Omega \cap B(z; \delta)$ such that $z \in \partial B$ (see Lemma 1.17.11), Proposition 3.5.1 gives $\partial u(z)/\partial\theta + h(z)u(z) - [\partial v(z)/\partial\theta + h(z)v(z)] = \partial[u(z) - v(z)]/\partial\theta < 0$, what is a contradiction.

Theorem 6.25.2 *Let $z \in R^m$, $r > 0$, $m \geq 2$, $a, b \in R^1$, and $a \geq 0$. Suppose that f is a function on $\langle 0, \infty) \times R^1$ such that $f(t, \cdot)$ is non-decreasing for each $t \geq 0$. If $u \in \mathcal{C}^2(B(z; r)) \cap \mathcal{C}^1(\overline{B}(z; r))$, $\partial u/\partial n^{B(z;r)} + au = b$ on $\partial B(z; r)$, and $\Delta u = f(|x - z|, u)$ in $B(z; r)$, then there exists $\varphi \in \mathcal{C}^1(\langle 0; r\rangle)$ such that $u(x) = \varphi(|x - z|)$.*

Proof Without loss of generality, we can suppose that $z = 0$. Let Ax be a rotation with center 0. Put $v(x) = u(Ax)$. Then the mapping A is representable by a matrix A such that $(Ax) \cdot (Ax) = x \cdot y$. Lemma 2.1.2 gives $\Delta v(x) = f(|x|, v(x))$ in $B(0; r)$. Clearly, $\partial v/\partial n^{B(z;r)} + av = b$ on $\partial B(0; r)$. According to Proposition 6.25.1, there exists a constant M such that $v = u + M$. From the fact

$$\inf_{x \in B(0;r)} u(x) = \inf_{x \in B(0;r)} v(x) = \inf_{x \in B(0;r)} u(x) + M$$

we infer that $M = 0$. Since $u(x) = u(Ax)$ for each rotation Ax, there exists a function φ such that $u(x) = \varphi(|x|)$.

Chapter 7
Solutions in Sobolev and Besov Spaces

Abstract Chapter 7 studies boundary value problems for the Poisson equation in Sobolev and Besov spaces on bounded domains and also in homogeneous Sobolev spaces on unbounded domains.

Chapter 7 studies boundary value problems for the Poisson equation in Sobolev and Besov spaces on bounded domains and also in homogeneous Sobolev spaces on unbounded domains. First, sections are devoted to the Dirichlet problem. We prove that there exists a unique solution of the Dirichlet problem in $\tilde{W}^{1,2}(\Omega)$ where $\Omega \subset R^m$ is an open set. If $m > 2$, then Ω can be arbitrary. If $m = 2$, we suppose that Ω is not dense in R^2. We show that there exists a unique solution of the Dirichlet problem for the Poisson equation in the Sobolev space $W^{s,p}(\Omega)$ or in the Besov space $B_s^{p,q}(\Omega)$, where Ω is a bounded domain with Ljapunov boundary and $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. Then, we study a solution of the Dirichlet problem in the space $W^{2+k,p}(\Omega)$ where $k \in N_0$ and Ω is a bounded domain with boundary of class $\mathcal{C}^{k+2}$. For bounded domains, we compare Perron-Wiener-Brelot solution of the Dirichlet problem and solutions of this problem in Sobolev and Besov spaces. We are also interested in the Dirichlet problem for the Poisson equation in homogeneous Sobolev spaces $D^{k,p}(\Omega)$ on exterior domains. We prove the existence of a solution and characterize the set of all solutions.

Further, we study the Neumann problem, the Robin problem, and the derivative oblique problem. We find a necessary and sufficient condition for the existence of a solution in $\tilde{W}^{1,2}(\Omega)$ for a domain with compact Lipschitz boundary and discuss the uniqueness. For domains with compact boundary, we study the Neumann problem and the Robin problem in $D^{1,p}(\Omega)$ and in $D^{2,p}(\Omega)$. For planar domains, we are able to treat these problems in $D^{k,p}(\Omega)$ for $k \in N$.

In the rest of the sections, we study the transmission problem in $D^{1,p}(\Omega_\pm)$, the obstacle problem in $W^{1,2}(\Omega)$ for Ω bounded, and mixed problems in $\tilde{W}^{1,2}(\Omega)$, $W^{s,p}(\Omega)$, $B_s^{p,q}(\Omega)$, and $D^{1,p}(\Omega)$. At the end of this chapter, we study eigenvalue problem for the Dirichlet problem, for the Neumann problem, for the Robin problem, and for the mixed problem. We compare eigenvalues of these problems.

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7.1 The Dirichlet Problem in $\tilde{W}^{1,2}(\Omega)$

In this section, we prove that there exists a unique solution of the Dirichlet problem in $\tilde{W}^{1,2}(\Omega)$ where $\Omega \subset R^m$ is an open set. If $m > 2$, then Ω can be arbitrary. If $m = 2$, we suppose that Ω is not dense in R^2.

Let $1 \le p \le \infty$. The space $D^{1,p}(R^m)$ consists of those functions in $L^p_{loc}(R^m)$ for which all generalized derivatives of the order 1 are in $L^p(R^m)$. Fix a bounded open set ω. Then $D^{1,p}(R^m)$ is a Banach space with the norm

$$\|u\|_{D^{1,p}(R^m)} = \|u\|_{L^p(\omega)} + \| \, |\nabla u| \, \|_{L^p(R^m)}.$$

Moreover, different choices of ω give equivalent norms.

Let $\Omega \subset R^m$ be an open set, $1 \le p, p' \le \infty$, $1/p + 1/p' = 1$. Denote by $\overset{\circ}{D}{}^{1,p}(\Omega)$ the closure of $\mathcal{C}^\infty_c(\Omega)$ in $D^{1,p}(R^m)$ and by $D^{-1,p'}(\Omega)$ its dual space. Denote $\tilde{W}^{1,p}(\Omega) = \{\varphi|\Omega; \varphi \in \overset{\circ}{D}{}^{1,p}(R^m)\}$ equipped with the norm

$$\|f\|_{\tilde{W}^{1,p}(\Omega)} = \inf\{\|\varphi\|_{D^{1,p}(R^m)}; \varphi \in \overset{\circ}{D}{}^{1,p}(R^m), f = \varphi|\Omega\}.$$

Denote $W^{1,p}(\Omega) = \{u \in L^p(\Omega); \nabla u \in L^p(\Omega, R^m)\}$ equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \| \, |\nabla u| \, \|_{L^p(\Omega)}.$$

If Ω is a bounded domain with Lipschitz boundary, then $\tilde{W}^{1,p}(\Omega) = W^{1,p}(\Omega)$ and both norms are equivalent.

If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary, then $u \in W^{1,2}(\Omega)$ is a solution of the Dirichlet problem for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{7.1}$$

if $-\Delta u = f$ in the sense of distributions and the trace $\gamma_\Omega u$ of u on $\partial\Omega$ is equal to g. Clearly, g is the trace of some function $g \in \tilde{W}^{1,2}(\Omega)$. Theorem 3.19.3 gives that $f \in D^{-1,2}(\Omega)$. Theorem 3.19.3 and Proposition 1.24.3 give that $u \in W^{1,2}(\Omega) = \tilde{W}^{1,2}(\Omega)$ is a solution of the problem (7.1) if and only if $u - g \in \overset{\circ}{D}{}^{1,2}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{C}^\infty_c(\Omega) \tag{7.2}$$

(or equivalently for all $\varphi \in \overset{\circ}{D}{}^{1,2}(\Omega)$).

This motivates the definition of a solution of the Dirichlet problem (7.1) in $\tilde{W}^{1,2}(\Omega)$ for a general open set $\Omega \subset R^m$. Instead of the problem (7.1), we shall study a more general problem. Let $a_{ij} \in L^\infty(R^m)$, $i,j = 1,\dots,m$, be such that $a_{ij} = a_{ji}$ and there exists a positive constant ν such that

$$\nu |y|^2 \le \sum_{i,j=1}^m a_{ij}(x) y_i y_j \qquad \forall x, y \in R^m. \tag{7.3}$$

Suppose moreover that $c \in L^\infty(R^m)$ has compact support, $c \ge 0$. If $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$, we say that $u \in \tilde{W}^{1,2}(\Omega)$ is a solution of the problem

$$-\sum_{i,j=1}^m \partial_i(a_{ij}\partial_j u) + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{7.4}$$

in $\tilde{W}^{1,2}(\Omega)$ if $u - g \in \overset{\circ}{D}{}^{1,2}(\Omega)$ and

$$\int_\Omega \left[\sum_{i,j=1}^m a_{ij}(\partial_j u)(\partial_i \varphi) + cu\varphi \right] \mathrm{d}\mathcal{H}_m = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega) \tag{7.5}$$

(or equivalently for all $\varphi \in \overset{\circ}{D}{}^{1,2}(\Omega)$).

Lemma 7.1.1 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$. If $m = 2$, suppose moreover that $\overline{\Omega} \ne R^2$. Let $a_{ij} \in L^\infty(R^m)$, $i,j = 1,\dots,m$, be such that $a_{ij} = a_{ji}$ and there exists a positive constant ν such that (7.3) holds. Suppose that $c \in L^\infty(R^m)$ has compact support, $c \ge 0$. Denote*

$$(\varphi, \psi) = \int_\Omega \left[\sum_{i,j=1}^m a_{ij}(\partial_j \varphi)(\partial_i \psi) + c\varphi\psi \right] \mathrm{d}\mathcal{H}_m. \tag{7.6}$$

Then $\|\varphi\| = \sqrt{(\varphi, \varphi)}$ is an equivalent norm on $\overset{\circ}{D}{}^{1,2}(\Omega)$.

Proof According to (7.3)

$$\|\varphi\|^2 = (\varphi, \varphi) \ge \nu \|\nabla \varphi\|^2_{L^2(\Omega)}.$$

Since $a_{ij}, c \in L^\infty(R^m)$ and c has compact support, Lemma 1.25.4 and Proposition 1.25.8 give that $\| \; \|$ is an equivalent norm on $\overset{\circ}{D}{}^{1,2}(\Omega)$.

Theorem 7.1.2 *Let $\Omega \subset R^m$ be an open set, $m \ge 2$. If $m = 2$, suppose moreover that $\overline{\Omega} \ne R^2$. Let $a_{ij} \in L^\infty(R^m)$, $i,j = 1,\dots,m$, be such that $a_{ij} = a_{ji}$ and there exists a positive constant ν such that (7.3) holds. Suppose that $c \in L^\infty(R^m)$ has* *compact support,* $c \geq 0$. *If* $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$ *then there exists a unique solution* u *of the Dirichlet problem (7.4) in* $\tilde{W}^{1,2}(\Omega)$. *Moreover*

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\left[\|f\|_{D^{-1,2}(\Omega)} + \|g\|_{\tilde{W}^{1,2}(\Omega)}\right]$$

where a constant C *depends only on* Ω, a_{ij}, *and* c. *Let* $(\ ,\ )$ *be given by (7.6),* $\|\varphi\| = \sqrt{(\varphi, \varphi)}$. *If* $g \equiv 0$, *then*

$$\|u\| = \sup\{\langle f, v\rangle; v \in \overset{\circ}{D}^{1,2}(\Omega), \|v\| \leq 1\}.$$

Proof $\overset{\circ}{D}^{1,2}(\Omega)$ is a Hilbert space with the inner product $(\ ,\ )$ by Lemma 7.1.1. Define

$$\langle F, \varphi\rangle = \langle f, \varphi\rangle - (g, \varphi), \qquad \varphi \in \overset{\circ}{D}^{1,2}(\Omega).$$

Then $F \in D^{-1,2}(\Omega)$. Riesz representation theorem (Theorem 1.9.10) gives that there exists unique $v \in \overset{\circ}{D}^{1,2}(\Omega)$ such that $(v, \varphi) = \langle F, \varphi\rangle$ for each $\varphi \in \overset{\circ}{D}^{1,2}(\Omega)$ and $(v, v) = \|F\|^2$. Clearly, $u = g + v$ is a solution of the problem (7.1).

If $f = 0$, $g = 0$, then $F = 0$ and thus $u = v = 0$.

7.2 The Dirichlet Problem and Approximation of Domains

Theorem 7.2.1 *Let* $\Omega \subset R^m$ *be an open set,* $m \geq 2$. *If* $m = 2$, *suppose moreover that* $\overline{\Omega} \neq R^2$. *Let* $a_{ij} \in L^\infty(R^m)$, $i, j = 1, \dots, m$, *be such that* $a_{ij} = a_{ji}$ *and there exists a positive constant* ν *such that (7.3) holds. Suppose that* $c \in L^\infty(R^m)$ *has compact support,* $c \geq 0$. *Let* Ω_k *be an increasing sequence of open sets such that* $\Omega = \cup\Omega_k$. *Let* $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$. *Let* u *be a solution of the problem (7.4) in* $\tilde{W}^{1,2}(\Omega)$ *and* u_k *be a solution of the problem (7.4) in* $\tilde{W}^{1,2}(\Omega_k)$. *Extend* u_k *by* g *on* $\Omega \setminus \Omega_k$. *Then* $u_k \in \tilde{W}^{1,2}(\Omega)$ *and* $u_k \to u$ *in* $\tilde{W}^{1,2}(\Omega)$ *as* $k \to \infty$.

Proof Let $(\ ,\ )$ be given by (7.6), $\|\varphi\| = \sqrt{(\varphi, \varphi)}$. Then $\overset{\circ}{D}^{1,2}(\Omega)$ is a Hilbert space with the inner product $(\ ,\ )$ by Lemma 7.1.1. Define

$$\langle F, \varphi\rangle = \langle f, \varphi\rangle - (g, \varphi), \qquad \varphi \in \overset{\circ}{D}^{1,2}(\Omega).$$

Then $F \in D^{-1,2}(\Omega)$. Put $v = u - g$, $v_k = u_k - g$. Clearly, $v \in \overset{\circ}{D}^{1,2}(\Omega)$. Since $v_k \in \overset{\circ}{D}^{1,2}(\Omega_k)$, v_k extended by 0 is in $\overset{\circ}{D}^{1,2}(R^m)$. Thus, $v_k \in \overset{\circ}{D}^{1,2}(\Omega)$. Moreover,

$$-\sum_{i,j=1}^{m} \partial_i(a_{ij}\partial_j v) + cv = F \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{7.7}$$

$$-\sum_{i,j=1}^{m} \partial_i(a_{ij}\partial_j v_k) + cv_k = F \quad \text{in } \Omega_k, \qquad v_k = 0 \quad \text{on } \partial\Omega_k. \tag{7.8}$$

Since v_k are solutions of the Dirichlet problem (7.8) and v is a solution of the Dirichlet problem (7.7), by virtue of Theorem 7.1.2, we receive

$$\|v_k\| = \sup\{\langle F, w\rangle; w \in \overset{\circ}{D}{}^{1,2}(\Omega_k), \|w\| \leq 1\}$$
$$\leq \sup\{\langle F, w\rangle; v \in \overset{\circ}{D}{}^{1,2}(\Omega), \|w\| \leq 1\} = \|v\|.$$

Moreover,

$$\|v - v_k\|^2 = (v, v) - (v, v_k) - (v_k, v) + (v_k, v_k)$$
$$= (v, v) - \langle F, v_k\rangle - (v_k, v) + \langle F, v_k\rangle = (v, v) - (v_k, v).$$

Fix $\epsilon > 0$. Then, there exists $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\|v - \varphi\| < \epsilon/(2\|v\| + 1)$. Since $\operatorname{spt}\varphi$ is compact, there exists $n \in N$ such that $\operatorname{spt}\varphi \subset \Omega_n$. If $k \geq n$, then

$$\|v - v_k\|^2 = (v, v) - (v_k, v) = (v, \varphi) - (v_k, \varphi) + (v, v - \varphi) - (v_k, v - \varphi)$$
$$\leq \langle F, \varphi\rangle - \langle F, \varphi\rangle + \|v\|\|v - \varphi\| + \|v_k\|\|v - \varphi\| \leq 2\|v\|\epsilon/(2\|v\| + 1) < \epsilon.$$

7.3 The Weak Maximum Principle

In this section, we prove the monotony principle for solutions of the Dirichlet problem for the Poisson equation in $\tilde{W}^{1,2}(\Omega)$. As a consequence, we prove the maximum principle for solutions of the Dirichlet problem for the Poisson equation in $\tilde{W}^{1,2}(\Omega)$.

Let $\Omega \subset R^m$ be an open set, $\mathcal{F}, \mathcal{G} \in \mathcal{D}'(\Omega)$. Remember that $\mathcal{F} \leq \mathcal{G}$ if $\langle \mathcal{F}, \varphi\rangle \leq \langle \mathcal{G}, \varphi\rangle$ for each nonnegative $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

If Ω is a bounded open set with Lipschitz boundary, $1 < p < \infty$ and $u, v \in W^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$, then $u \leq v$ on $\partial\Omega$ (in the sense of traces) if and only if $(u - v)^+ \in \overset{\circ}{W}{}^{1,p}(\Omega) = \overset{\circ}{D}{}^{1,p}(\Omega)$. That is a motivation of the following definition:

Definition 7.3.1 Let $\Omega \subset R^m$ be an open set, $u, v \in \tilde{W}^{1,p}(\Omega)$, $1 \leq p < \infty$. We say that $u \leq v$ on $\partial\Omega$ if $(u - v)^+ \in \overset{\circ}{D}{}^{1,p}(\Omega)$.

Theorem 7.3.2 (Weak Maximum Principle) *Let a be a bounded nonnegative measurable function in R^m with compact support and $\Omega \subset R^m$ be an open set, $m \geq 2$. If $m = 2$, suppose moreover $\overline{\Omega} \neq R^2$. If $u, v \in \tilde{W}^{1,2}(\Omega)$, $-\Delta u + au \leq -\Delta v + av$ in $\mathcal{D}'(\Omega)$ and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

Proof Put $w = (u-v)^+$. Then, $\nabla w(x) = \nabla(u-v)(x)$ if $(u-v)(x) > 0$, $\nabla w(x) = 0$ elsewhere (see Theorem 1.21.5). Since $w \in \overset{\circ}{D}{}^{1,2}(\Omega)$, $w \geq 0$, there is a sequence $\varphi_k \in \mathcal{C}_c^\infty(\Omega)$ such that $\varphi_k \geq 0$ and $\varphi_k \to w$ in $\overset{\circ}{D}{}^{1,2}(\Omega)$ as $k \to \infty$ (see Lemma 1.25.10).

Since $-\Delta(u-v)+a(u-v)\le 0$, we obtain by Theorem 3.19.3

$$\int_\Omega (|\nabla w|^2 + a|w|^2)\, \mathrm{d}\mathcal{H}_m = \int_\Omega [\nabla(u-v)\cdot\nabla w + a(u-v)w]\, \mathrm{d}\mathcal{H}_m$$
$$= \lim_{k\to\infty} \int_\Omega [\nabla(u-v)\cdot\nabla\varphi_k + a(u-v)\varphi_k]\, \mathrm{d}\mathcal{H}_m \le 0.$$

Since $|\nabla w|^2 \ge 0$, $a|w|^2 \ge 0$, we infer that $\nabla w = 0$ a.e. in Ω. Since

$$\|\nabla\phi\|_{L^2(\Omega)}$$

is an equivalent norm on $\overset{\circ}{D}{}^{1,2}(\Omega)$ by Lemma 1.25.4 and Proposition 1.25.8, and $w\in \overset{\circ}{D}{}^{1,2}(\Omega)$, we infer that $w\equiv 0$.

Corollary 7.3.3 *Let $\Omega\subset R^m$ be an open set, $m\ge 2$. If $m=2$, suppose that $\overline{\Omega}\ne R^2$. Let $u,g\in \tilde{W}^{1,2}(\Omega)$, M be a constant, $-\Delta u = f$ in $\mathcal{D}'(\Omega)$. Then, $f\in D^{-1,2}(\Omega)$. Suppose that $u=g$ on $\partial\Omega$. If $f\le 0$ and $g\le M$, then $u\le M$ in Ω.*

Proof Theorem 3.19.3 gives that $f\in D^{-1,2}(\Omega)$.

If Ω is bounded put $v=M$ and use Theorem 7.3.2.

Let now Ω be unbounded. Fix a bounded open set ω such that $\overline{\omega}\subset\Omega$. Choose an increasing sequence of bounded open sets Ω_k such that $\omega\subset\Omega_1$ and $\Omega=\cup\Omega_k$. Theorem 3.19.3 gives that $f\in D^{-1,2}(\Omega_k)$. According to Theorem 7.1.2, there exists a solution $u_k\in\tilde{W}^{1,2}(\Omega_k)$ of the problem $-\Delta u_k = f$ in Ω_k, $u_k = g$ on $\partial\Omega_k$. We have proved that $u_k\le M$. Theorem 7.2.1 forces that $u_k\to u$ in $W^{1,2}(\omega)$. According to Proposition 1.14.8, there exists a subsequence $u_{k(j)}$ such that $u_{k(j)}(x)\to u(x)$ for almost all $x\in\omega$. Thus, $u(x)\le M$ for almost all $x\in\omega$.

Corollary 7.3.4 *Let $\Omega\subset R^m$ be a bounded open set $m\ge 2$, $a\in L^\infty(\Omega)$ be a nonnegative function. If $u\in\tilde{W}^{1,2}(\Omega)$, $-\Delta u + au\le 0$ in $\mathcal{D}'(\Omega)$ and $u\le M$ on $\partial\Omega$, where M is a nonnegative constant, then $u\le M$ in Ω.*

Proof Put $v=M$. Then $-\Delta u + au \le 0\le aM = -\Delta v + av$. Now we use Theorem 7.3.2.

Corollary 7.3.5 *Let $\Omega\subset R^m$ be a bounded open set, $g\in\tilde{W}^{1,2}(\Omega)\cap\mathcal{C}^0(\overline{\Omega})$. If $u\in\tilde{W}^{1,2}(\Omega)$ is a solution of the problem $-\Delta u=0$ in Ω, $u=g$ on $\partial\Omega$ then*

$$\min_{x\in\partial\Omega} g(x)\le \inf_{x\in\Omega} u(x)\le \sup_{x\in\Omega} u(x)\le \max_{x\in\partial\Omega} g(x).$$

Proof Let $M=\max\{g(x); x\in\partial\Omega\}$. For $n\in N$, denote $g_n=\min(g, M+1/n)$. Theorem 1.21.5 and the fact that $g_n = g$ on a neighborhood of $\partial\Omega$ give that $g_n\in\tilde{W}^{1,2}(\Omega)$. Choose an open set ω_n with Lipschitz boundary such that $\overline{\omega}_n\subset\Omega$ and $g-g_n=0$ on a neighborhood of $\overline{\Omega\setminus\omega_n}$. Proposition 1.24.3 gives $g-g_n\in\overset{\circ}{W}{}^{1,2}(\omega)\subset$

$\overset{\circ}{D}{}^{1,2}(\Omega)$. Since $u - g_n = (u - g) + (g - g_n) \in \overset{\circ}{D}{}^{1,2}(\Omega)$, the function u is a solution of the problem $\Delta u = 0$ in Ω, $u = g_n$ on $\partial\Omega$. Corollary 7.3.3 gives that $u \leq M + 1/n$. Since $n \in N$ is arbitrary, we deduce that $u \leq M$. The second inequality we obtain using this inequality for $-u$.

Corollary 7.3.6 *Let $\Omega \subset R^m$ be an open set with compact boundary, $g \in \tilde{W}^{1,2}(\Omega) \cap C^0(\overline{\Omega})$. If $m = 2$, suppose moreover that $\overline{\Omega} \neq R^2$. If $u \in \tilde{W}^{1,2}(\Omega)$ is a solution of the problem $-\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then*

$$\sup_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |g(x)|.$$

Proof We can suppose that $g \in \overset{\circ}{D}{}^{1,2}(R^m)$. Let $M = \max\{|g(x)|; x \in \partial\Omega\}$. For $n \in N$, denote $g_n = \max(\min(g, M + 1/n), -M - 1/n)$. Theorem 1.21.5 gives that $g_n \in D^{1,2}(R^m)$. Proposition 3.38.4 gives $\overset{\circ}{D}{}^{1,2}(R^2) = D^{1,2}(R^2)$. If $m > 2$ then $g \in L^{2m/(m-2)}(R^m)$ by Proposition 3.38.5. Since $g_n \in L^{2m/(m-2)}(R^m)$, Proposition 3.38.5 forces $g_n \in \overset{\circ}{D}{}^{1,2}(R^m)$. Hence, $g_n \in \tilde{W}^{1,2}(\Omega)$.

Since $g - g_n \in \overset{\circ}{D}{}^{1,2}(R^m)$ and $g - g_n = 0$ on a neighborhood of $\partial\Omega$, we infer that $g - g_n \in \overset{\circ}{D}{}^{1,2}(\Omega)$. Since $u - g_n = (u - g) + (g - g_n) \in \overset{\circ}{D}{}^{1,2}(\Omega)$, the function u is a solution of the problem $\Delta u = 0$ in Ω, $u = g_n$ on $\partial\Omega$. Fix a bounded open set ω such that $\overline{\omega} \subset \Omega$. Choose an increasing sequence of bounded open sets Ω_k such that $\omega \subset \Omega_1$ and $\Omega = \cup\Omega_k$. Then there exists a unique solution $u_{n,k} \in \tilde{W}^{1,2}(\Omega_k)$ of the problem $\Delta u_{n,k} = 0$ in Ω_k, $u_{n,k} = g_n$ on Ω_k. Corollary 7.3.4 gives that $|u_{n,k}| \leq M + 1/n$. Theorem 7.2.1 forces that $u_k \to u$ in $W^{1,2}(\omega)$. According to Proposition 1.14.8, there exists a subsequence $u_{k(j),n}$ such that $u_{k(j),n}(x) \to u(x)$ for almost all $x \in \omega$. Thus, $|u(x)| \leq M + \frac{1}{n}$ for almost all $x \in \omega$. Since n was arbitrary, we infer $|u(x)| \leq M$ for almost all $x \in \omega$.

Theorem 7.3.7 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, $f \in L^\infty(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega) \cap L^\infty(\Omega)$. If u is a solution of the Dirichlet problem (7.1) in $\tilde{W}^{1,2}(\Omega)$, then $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)} + \frac{(\operatorname{diam}\Omega)^2}{2m}\|f\|_{L^\infty(\Omega)}. \tag{7.9}$$

Proof According to Theorem 7.1.2, there exist $u_1, u_2 \in \tilde{W}^{1,2}(\Omega)$ such that $-\Delta u_1 = 0$ in Ω, $u_1 = g$ on $\partial\Omega$, $-\Delta u_2 = f$ in Ω, $u_2 = 0$ on $\partial\Omega$. Corollary 7.3.3 used for u_1 and $-u_1$ gives $|u_1| \leq \|g\|_{L^\infty(\Omega)}$.

We can suppose that $0 \in \Omega$. Put $b = \operatorname{diam}\Omega$,

$$v(x) = \|f\|_{L^\infty(\Omega)}(b^2 - |x|^2)/(2m).$$

Then $v \in \tilde{W}^{1,2}(\Omega)$, $v \geq 0 = u_2$ on $\partial\Omega$ and

$$-\Delta v = \|f\|_{L^\infty(\Omega)} \geq f = -\Delta u_2.$$

Weak maximum principle (Theorem 7.3.2) forces

$$u_2 \leq v \leq b^2 \|f\|_{L^\infty(\Omega)}/(2m).$$

The same reasoning for $-u_2$ gives that $-u_2 \leq v \leq b^2\|f\|_{L^\infty(\Omega)}/(2m)$.

The uniqueness of a solution of the Dirichlet problem (Theorem 7.1.2) gives $u = u_1 + u_2$. This forces the estimate (7.9).

7.4 The Poisson Equation in R^m

In this section, we study solutions of the Poisson equation in $D^{k,p}(R^m)$ and in $\overset{\circ}{D}{}^{k,p}(R^m)$ for $k \in N$ and $1 < p < \infty$.

Denote by $\mathcal{E}_p(R^m)$ the closure of $\{\Delta\varphi; \varphi \in \mathcal{C}_c^\infty(R^m)\}$ in $D^{-1,p}(R^m)$. If $p' < m$ (i.e., if $p > m/(m-1)$) then $\mathcal{E}_p(R^m) = D^{-1,p}(R^m)$. If $p' \geq m$ (i.e., if $p \leq m/(m-1)$), then $\mathcal{E}_p(R^m) = \{v \in D^{-1,p}(R^m); \langle v, 1\rangle = 0\}$. (See Corollary 2.18.6.)

Proposition 7.4.1 *Let $1 < p < \infty$, $m \in N$, $m \geq 2$. If $u \in D^{1,p}(R^m)$ then $\Delta u \in \mathcal{E}_p(R^m)$. Fix $f \in \mathcal{E}_p(R^m)$. Then there exists $u \in \overset{\circ}{D}{}^{1,p}(R^m)$ such that $\Delta u = f$. If $p < m$, then u is unique; if $p \geq m$, then $\overset{\circ}{D}{}^{1,p}(R^m) = D^{1,p}(R^m)$. Moreover, $\{v \in D^{1,p}(R^m); \Delta v = f\} = \{u + c; c \text{ constant}\}$.*

Proof See Propositions 3.38.3 and 3.38.4.

Proposition 7.4.2 *Let $1 < p < \infty$, $m \in N$, $m \geq 2$. If $u \in D^{2,p}(R^m)$, then $\Delta u \in L^p(R^m)$. Fix $f \in L^p(R^m)$. Then there exists $u \in \overset{\circ}{D}{}^{2,p}(R^m)$ such that $\Delta u = f$. Moreover, $\{v \in D^{2,p}(R^m); \Delta v = f\} = \{u + p; p \text{ is a polynomial of the first degree}\}$. If $p < m/2$ then $\{v \in \overset{\circ}{D}{}^{2,p}(R^m); \Delta v = f\} = \{u\}$. If $m/2 \leq p < m$ then $\{v \in \overset{\circ}{D}{}^{2,p}(R^m); \Delta v = f\} = \{u+c; c \text{ constant}\}$. If $p \geq m$ then $\overset{\circ}{D}{}^{2,p}(R^m) = D^{2,p}(R^m)$.*

Proof Let $f \in L^p(R^m)$. According to Theorem 3.16.1, there exists $w \in D^{2,p}(R^m)$ such that $\Delta w = f$. By Proposition 3.16.5, there exists $u \in \overset{\circ}{D}{}^{1,p}(R^m)$ and a polynomial p of the first degree such that $w = u + p$. Hence, $\Delta u = f$. Let now $v \in D^{2,p}(R^m)$; $\Delta v = f$. Then $u - v \in D^{2,p}(R^m)$, $\Delta(u - v) = 0$. So, $u - v \in \mathcal{C}^\infty(R^m)$ by Theorem 2.18.2. Since $\partial_j\partial_k(u-v) \in L^p(R^m)$ and $\Delta\partial_j\partial_k(u-v) = 0$, Theorem 2.16.2 gives that $\partial_j\partial_k(u - v) \equiv 0$. So, $u - v$ is a polynomial of the first degree. The rest is a consequence of Proposition 3.16.5.

Proposition 7.4.3 *Let $1 < p < \infty$, $m, k \in N$, $m \geq 2$. Then there exists a solution $u \in D^{k+2,p}(R^m)$ of $\Delta u = f$ if and only if $f \in D^{k,p}(R^m)$. If $f \in \overset{\circ}{D}{}^{k,p}(R^m)$, then there exists $u \in \overset{\circ}{D}{}^{k+2,p}(R^m)$ such that $\Delta u = f$. We have $\{u \in D^{k+2,p}(R^m); \Delta u \equiv 0\} = \{P \in \tilde{P}_{k+1}(R^m); \Delta P = 0\}$, where $\tilde{P}_{k+1}(R^m)$ denotes the space of all polynomials of degree at most $k + 1$.*

Proof According to Theorem 3.42.1, there is a solution $u \in D^{k+2,p}(R^m)$ of $\Delta u = f$ if and only if $f \in D^{k,p}(R^m)$. If $f \in \mathring{D}^{k,p}(R^m)$, then there exists $u \in \mathring{D}^{k+2,p}(R^m)$ such that $\Delta u = f$. (See Theorem 3.17.2.) Clearly $\{P \in \tilde{P}_{k+1}(R^m); \Delta P = 0\} \subset \{u \in D^{k+2,p}(R^m); \Delta u \equiv 0\}$. Let now $u \in D^{k+2,p}(R^m)$, $\Delta u \equiv 0$. Fix multiindex α such that $|\alpha| = k + 2$. Since $\partial^\alpha u \in L^p(R^m)$, it is a tempered distribution. (See Sect. 1.20.) Since $u \in H(R^m)$ by Theorem 2.18.2, $\partial^\alpha u \in H(R^m)$ by Corollary 2.2.9. Since $\partial^\alpha u \in L^p(R^m) \cap H(R^m)$, Theorem 2.16.2 forces $\partial^\alpha u \equiv 0$. Since $\partial^\alpha u \equiv 0$ for each multiindex α with $|\alpha| = k + 2$, we deduce that $u \in \tilde{P}_{k+1}(R^m)$.

7.5 The Dirichlet Problem in the Half-Space

In this section, we study the Dirichlet problem for the Poisson equation in $\tilde{W}^{1,p}(R^m_+)$, in $\tilde{W}^{2,p}(R^m_+)$ and in $D^{2,p}(R^m_+)$ for $1 < p < \infty$.

Lemma 7.5.1 *Let* $1 < p < \infty$, $m \in N$, $m \geq 2$. *For* $u \in \mathring{D}^{1,p}(R^m_+)$, *define* $Qu = u$ *in* R^m_+, $Qu(x', x_m) = -u(x', -x_m)$ *for* $x_m < 0$. *Then,* $Qu \in \mathring{D}^{1,p}(R^m)$. *If* $-\Delta u = f$ *in* R^m_+, $f \in L^1_{loc}(\overline{R^m_+})$, *then* $-\Delta Qu = f$ *in* R^m *where* $f(x', x_m) = -f(x', -x_m)$ *for* $x_m < 0$.

Proof Clearly $u = 0$ on $R^m \setminus R^m_+$. Put $\tilde{u}(x', x_m) = u(x', -x_m)$. Then $\tilde{u} \in \mathring{D}^{1,p}(R^m)$. Thus, $Qu = u - \tilde{u} \in \mathring{D}^{1,p}(R^m)$.

Let now $-\Delta u = f$ in R^m_+, $f \in L^1_{loc}(\overline{R^m_+})$, $f(x', x_m) = -f(x', -x_m)$ for $x_m < 0$. If $\varphi \in \mathcal{C}^\infty_c(R^m)$ denote $\varphi_1(x', x_m) = [\varphi(x', x_m) + \varphi(x', -x_m)]/2$, $\varphi_2(x', x_m) = [\varphi(x', x_m) - \varphi(x', -x_m)]/2$. Remark that $\varphi = \varphi_1 + \varphi_2$, $\varphi_1(x', -x_m) = \varphi_1(x', x_m)$, $\Delta\varphi_1(x', -x_m) = \Delta\varphi_1(x', x_m)$, $\varphi_2(x', -x_m) = -\varphi_2(x', x_m)$, $\Delta\varphi_2(x', -x_m) = -\Delta\varphi_2(x', x_m)$. Fix $r > 0$ such that $\varphi = 0$ on $R^m \setminus B(0; r/2)$. Since $\varphi_2(x', 0) = 0$, we have $\varphi \in \mathring{W}^{1,p/(p-1)}(R^m_+)$ by Lemma 1.21.4. By virtue of the Gauss-Green theorem for Sobolev spaces (Proposition 1.24.4) and Theorem 3.19.3,

$$\begin{aligned}
&\int_{R^m} Qu(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m = \int_{R^m} Qu(-\Delta\varphi_1)\, \mathrm{d}\mathcal{H}_m + \int_{R^m} Qu(-\Delta\varphi_2)\, \mathrm{d}\mathcal{H}_m \\
&= \int_{R^m} Qu(-\Delta\varphi_2)\, \mathrm{d}\mathcal{H}_m = 2 \int_{B(0;r)\cap R^m_+} u(-\Delta\varphi_2)\, \mathrm{d}\mathcal{H}_m = 2 \int_{B(0;r)\cap R^m_+} \nabla u \cdot \nabla\varphi_2 \\
&- \int_{\partial[B(0;r)\cap R^m_+]} 2u \frac{\partial\varphi_2}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = 2 \int_{R^m_+} \nabla u \cdot \nabla\varphi_2\, \mathrm{d}\mathcal{H}_m = 2 \int_{R^m_+} f\varphi_2\, \mathrm{d}\mathcal{H}_m \\
&= \int_{R^m} f\varphi_2\, \mathrm{d}\mathcal{H}_m + \int_{R^m} f\varphi_1\, \mathrm{d}\mathcal{H}_m = \int_{R^m} f\varphi\, \mathrm{d}\mathcal{H}_m.
\end{aligned}$$

Hence, $-\Delta Qu = f$ in R^m.

Proposition 7.5.2 *Let* $1 < p, p', q < \infty$, $1/p + 1/p' = 1$. *Then, there exists a positive constant* C *such that for all* $u \in \mathring{D}^{1,q}(R^m_+)$

$$\|\nabla u\|_{L^p(R^m_+)} \le C \sup_{\varphi\in\mathcal{M}} \int\limits_{R^m_+} \frac{\nabla u \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m_+)}}\,\mathrm{d}\mathcal{H}_m$$

where $\mathcal{M} = \{\varphi \in \mathring{D}^{1,p'}(R^m_+) \cap W^{1,\infty}(R^m_+) \setminus \{0\}; \varphi$ *has compact support* $\}$.

Proof Let $u \in \mathring{D}^{1,q}(R^m_+)$. Then, $u = 0$ on $R^m \setminus R^m_+$. For φ defined in R^m, denote $\tilde{\varphi}(x', x_m) = \varphi(x', x_m) - \varphi(x', -x_m)$, $\breve{\varphi}(x', x_m) = \varphi(x', x_m) + \varphi(x', -x_m)$. Then, $\varphi = \tilde{\varphi}/2 + \breve{\varphi}/2$, $\tilde{\varphi}(x', -x_m) = -\tilde{\varphi}(x', x_m)$, $\breve{\varphi}(x', -x_m) = \breve{\varphi}(x', x_m)$. According to Proposition 3.38.2, there exists a positive constant M such that

$$\|\nabla v\|_{L^p(R^m)} \le M \sup_{\varphi\in\mathcal{C}^\infty_c(R^m)\setminus\{0\}} \int\limits_{R^m} \frac{\nabla v \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\,\mathrm{d}\mathcal{H}_m$$

for all $v \in W^{1,1}_{lok}(R^m)$. If for $\varphi \in \mathcal{C}^\infty_c(R^m)$ denote $\hat{\varphi} = \tilde{\varphi}$ on R^m_+, $\hat{\varphi} = 0$ elsewhere then $\hat{\varphi} \in \mathring{D}^{1,p'}(R^m_+)$ (see Proposition 1.24.3). Since $\|\nabla\tilde{\varphi}\|_{L^{p'}(R^m)} \le 2\|\nabla\varphi\|_{L^{p'}(R^m)}$, we have

$$\begin{aligned}
\|\nabla u\|_{L^p(R^m_+)} &= \frac{1}{2}\|\nabla\tilde{u}\|_{L^p(R^m)} \le \frac{M}{2} \sup_{\varphi\in\mathcal{C}^\infty_c(R^m)\setminus\{0\}} \int\limits_{R^m} \frac{\nabla\tilde{u}\cdot\nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\,\mathrm{d}\mathcal{H}_m \\
&= \frac{M}{4} \sup_{\substack{\varphi\,\in\,\mathcal{C}^\infty_c(R^m)\\ \varphi\not\equiv 0}} \int\limits_{R^m} \frac{\nabla\tilde{u}\cdot\nabla\tilde{\varphi} + \nabla\tilde{u}\cdot\nabla\breve{\varphi}}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\,\mathrm{d}\mathcal{H}_m \\
&= \frac{M}{4} \sup_{\substack{\varphi\,\in\,\mathcal{C}^\infty_c(R^m)\\ \varphi\not\equiv 0}} \int\limits_{R^m} \frac{\nabla\tilde{u}\cdot\nabla\tilde{\varphi}}{\|\nabla\varphi\|_{L^{p'}(R^m)}} \le \frac{M}{2} \sup_{\substack{\varphi\,\in\,\mathcal{C}^\infty_c(R^m)\\ \varphi\not\equiv 0}} \int\limits_{R^m} \frac{\nabla\tilde{u}\cdot\nabla\tilde{\varphi}}{\|\nabla\tilde{\varphi}\|_{L^{p'}(R^m)}} \\
&= M \sup_{\substack{\varphi\,\in\,\mathcal{C}^\infty_c(R^m)\\ \varphi\not\equiv 0}} \int\limits_{R^m_+} \frac{\nabla u\cdot\nabla\hat{\varphi}}{\|\nabla\hat{\varphi}\|_{L^{p'}(R^m)}} \le M \sup_{\varphi\in\mathcal{M}} \int\limits_{R^m_+} \frac{\nabla u\cdot\nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m_+)}}\,\mathrm{d}\mathcal{H}_m.
\end{aligned}$$

Theorem 7.5.3 *Let* $1 < p < \infty$, $m \in N$, $m \ge 2$. *Then* Δ *is an isomorphism from* $\mathring{D}^{1,p}(R^m_+)$ *onto* $D^{-1,p}(R^m_+)$.

Proof Δ is a bounded linear operator from $\mathring{D}^{1,p}(R^m_+)$ to $D^{-1,p}(R^m_+)$ by Theorem 3.19.3. If $u \in \mathring{D}^{1,p}(R^m_+)$ then

$$\langle -\Delta u, \varphi\rangle = \int\limits_{R^m_+} \nabla u\cdot\nabla\varphi\,\mathrm{d}\mathcal{H}_m, \qquad \varphi \in \mathring{D}^{1,p'}(R^m_+), \tag{7.10}$$

where $p' = p/(p-1)$ (see Theorem 3.19.3). Since $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm on $\overset{\circ}{D}{}^{1,p}(R^m_+)$ by Lemma 1.25.4, Proposition 7.5.2 forces that Δ is bounded below. Thus, Δ is one-to-one, and its image is the closed subspace of $D^{-1,p}(R^m_+)$ (see Proposition 1.4.13). Suppose now that there exists $f \in D^{-1,p}(R^m_+) \setminus \{\Delta u; u \in \overset{\circ}{D}{}^{1,p}(R^m_+)\}$. Since $\overset{\circ}{D}{}^{1,p'}(R^m_+)$ is reflexive by Lemma 1.25.3, Proposition 1.5.3 gives that there exists $v \in \overset{\circ}{D}{}^{1,p'}(R^m_+)$ such that $\langle f, v\rangle > 0$, $\langle \Delta u, v\rangle = 0$ for all $u \in \overset{\circ}{D}{}^{1,p}(R^m_+)$. By virtue of (7.10), we obtain

$$\int\limits_{R^m_+} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = 0 \qquad \forall u \in \overset{\circ}{D}{}^{1,p}(R^m_+).$$

Proposition 7.5.2 forces that $\|\nabla v\|_{L^{p'}(R^m_+)} = 0$. Since $\|\nabla w\|_{L^{p'}(\Omega)}$ is an equivalent norm on $\overset{\circ}{D}{}^{1,p'}(R^m_+)$ by Lemma 1.25.4, we deduce that $v \equiv 0$. That is a contradiction with the fact that $\langle f, v\rangle > 0$.

Since Δ is one-to-one and onto, it is continuously invertible by Theorem 1.6.3.

Theorem 7.5.4 *Let* $1 < p < \infty$, $m \in N$, $m \geq 2$. *Let* $f \in D^{-1,p}(R^m_+)$, $g \in \tilde{W}^{1,p}(R^m_+)$. *Then there exists a unique solution* $u \in \tilde{W}^{1,p}(R^m_+)$ *of the Dirichlet problem* $-\Delta u = f$ *in* R^m_+, $u = g$ *on* ∂R^m_+. *Moreover,*

$$\|u\|_{\tilde{W}^{1,p}(R^m_+)} \leq M\left[\|g\|_{\tilde{W}^{1,p}(R^m_+)} + \|f\|_{D^{-1,p}(R^m_+)}\right]$$

where M *depends only on* m *and* p.

Proof Theorem 3.19.3 gives that $\Delta g \in D^{-1,p}(R^m_+)$ and

$$\|\Delta g\|_{D^{-1,p}(R^m_+)} \leq C_1 \|g\|_{\tilde{W}^{1,p}(R^m_+)}$$

with C_1 dependent only on m and p. Put $F = f + \Delta g$. According to Theorem 7.5.3, there exists a unique $v \in \overset{\circ}{D}{}^{1,p}(R^m_+)$ such that $-\Delta v = F$. Moreover, there exists a constant C_2 depending only on m and p such that $\|v\|_{\tilde{W}^{1,p}(R^m_+)} \leq C_2 \|F\|_{D^{-1,p}(R^m_+)}$. The function $u = v + g \in \tilde{W}^{1,p}(R^m_+)$ is a solution of the problem $-\Delta u = f$ in R^m_+, $u = g$ on ∂R^m_+. We have $\|u\|_{\tilde{W}^{1,p}(R^m_+)} \leq \|g\|_{\tilde{W}^{1,p}(R^m_+)} + C_2[\|f\|_{D^{-1,p}(R^m_+)} + C_1\|g\|_{\tilde{W}^{1,p}(R^m_+)}]$.

Let now $u \in \tilde{W}^{1,p}(R^m_+)$ be a solution of the problem $-\Delta u = 0$ in R^m_+, $u = 0$ on ∂R^m_+. Then $u \equiv 0$ by Theorems 3.19.3 and 7.5.3.

Proposition 7.5.5 *Let* $m \in N$, $m \geq 2$, $1 < p < \infty$, $f \in L^p(R^m_+)$, $g \in \tilde{W}^{2,p}(R^m_+)$. *Then there exists* $u \in \tilde{W}^{2,p}(R^m_+)$ *such that* $\Delta u = f$ *in* R^m_+, $u = g$ *on* ∂R^m_+ *in the sense of traces. Moreover,* $\{v \in D^{2,p}(R^m_+); \Delta v = f \text{ in } R^m_+,\ v = g \text{ on } \partial R^m_+\} = \{u(x) + a x_m; a \text{ is constant}\}$.

Proof It is satisfactory to suppose that $g \equiv 0$. Define $f(x', x_m) = -f(x', -x_m)$ for $x_m < 0$. According to Proposition 7.4.2, there exists $v \in \overset{\circ}{D}{}^{2,p}(R^m)$ such that $\Delta v = f$. Define $\tilde{v}(x', x_m) = -v(x', -x_m)$. Then, $\tilde{v} \in \overset{\circ}{D}{}^{2,p}(R^m)$ and $\Delta \tilde{v} = f$. According to Proposition 7.4.2, there exists a polynomial P of the first degree such that $\tilde{v} = v + P$, i.e., $-v(x', 0) = v(x', 0) + P(x', 0)$. Clearly, $P \in \overset{\circ}{D}{}^{2,p}(R^m)$. Put $u = v + P/2$. Then $u \in \overset{\circ}{D}{}^{2,p}(R^m)$, $\Delta u = f$ in R^m_+, and $u(x', 0) = v(x', 0) + P(x', 0)/2 = -P(x', 0)/2 + P(x', 0)/2 = 0$.

Suppose now that $u \in D^{2,p}(R^m_+)$, $\Delta u = 0$ in R^m_+, $u = 0$ on ∂R^m_+. Denote $u(x', x_m) = -u(x', -x_m)$ for $x_m < 0$. Fix $r > 0$. Choose $\varphi_r \in \mathcal{C}^\infty_c(R^1)$ such that $\varphi_r(t) = 1$ for $|t| < r$, $\varphi_r(t) = 0$ for $|t| > 2r$. Denote $u_r(x) = \varphi_r(|x|)u(x)$. Then $u_r \in \overset{\circ}{W}{}^{1,p}(R^m_+ \cap B(0; 2r))$. Lemma 7.5.1 gives that $u_r \in \overset{\circ}{W}{}^{1,p}(B(0; 2r)$ and $\Delta u = \Delta u_r = 0$ in $B(0; r)$. Thus, $u \in \mathcal{C}^\infty(B(0; r))$ by Theorem 2.18.2 and Corollary 2.2.8. Hence, $u \in D^{2,p}(R^m) \cap \mathcal{C}^\infty(R^m)$, $\Delta u = 0$ in R^m. Thus, u is a polynomial of the first degree (see Corollary 2.16.4). Since $u = 0$ on ∂R^m_+, we have $u(x) = ax_m$.

Proposition 7.5.6 *Let* $1 < p, q < \infty$, $u \in \overset{\circ}{D}{}^{1,p}(R^m_+)$. *If* $-\Delta u = f \in L^q(R^m_+)$, *then* $u \in W^{2,q}_{loc}(\overline{R^m_+})$.

Proof Put $f(x', x_m) = -f(x', -x_m)$, $u(x', x_m) = -u(x', -x_m)$ for $x_m < 0$. Then $u \in \overset{\circ}{D}{}^{1,p}(R^m)$, $-\Delta u = f$ in R^m by Lemma 7.5.1. Since $f \in L^q(R^m)$, Proposition 3.18.2 gives that $u \in W^{2,q}_{loc}(R^m)$.

7.6 Properties of a Single Layer Potential

In this section, we study properties of single layer potentials with densities from Sobolev and Besov spaces.

Proposition 7.6.1 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary,* $m \geq 2$, $0 < s < 1$, $1 < p, q < \infty$. *Then,* $\mathcal{S}^\Omega$ *is a bounded linear operator from* $B^{p,q}_{-s}(\partial\Omega)$ *to* $B^{p,q}_{1-s+1/p}(\Omega)$ *and from* $W^{-s,p}(\partial\Omega)$ *to* $W^{1-s+1/p,p}(\Omega)$.

Proof Let $f \in B^{p,q}_{-s}(\partial\Omega)$. Denote $p' = p/(p-1)$, $q' = q/(q-1)$. Since the trace onto $\partial\Omega$ is a bounded linear operator from $B^{p',q'}_{s+1/p'}(\Omega)$ to $B^{p',q'}_s(\partial\Omega)$ by Theorem 1.24.7 and $f \in [B^{p',q'}_s(\partial\Omega)]'$, we deduce that $f \in [B^{p',q'}_{s+1/p'}(R^m)]'$. Theorem 1.23.1 gives that $f \in B^{p,q}_{-s-1/p'}(R^m) = B^{p,q}_{-1-s+1/p}(R^m)$. Since f is compactly supported, Proposition 3.18.5 gives that $\mathcal{S}^\Omega f = Vf \in B^{p,q}_{1-s+1/p}(\Omega)$.

Assume $f \in W^{-s,p}(\partial\Omega)$. Theorem 1.23.3 gives $W^{-s,p}(\partial\Omega) = B^{p,p}_{-s}(\partial\Omega)$. Suppose $s \neq 1/p$. Then $\mathcal{S}^\Omega f \in B^{p,p}_{1-s+1/p}(\Omega) = W^{1-s+1/p,p}(\Omega)$ by Proposition 1.23.13. Let now $s = 1/p$. Since the trace is a bounded linear operator from $W^{1,p'}(\Omega)$ to $W^{1-1/p',p'}(\partial\Omega) = W^{1/p,p'}(\partial\Omega)$ (see Theorem 1.24.2) and $f \in W^{-1/p,p}(\partial\Omega) = [W^{1/p,p'}(\partial\Omega)]'$, we infer that $f \in [W^{1,p'}(R^m)]'$. Since $f \in W^{-1,p}(R^m)$ is compactly supported, Proposition 3.18.5 gives $\mathcal{S}^\Omega f = Vf \in W^{1,p}(\Omega)$.

If $x \in \Omega$ and $f_k \to f$ in $B^{p,q}_{-s}(\partial\Omega)$ or in $W^{-s,p}(\partial\Omega)$ then $\mathcal{S}^\Omega f_k(x) \to \mathcal{S}^\Omega f(x)$. So, $\mathcal{S}^\Omega$ is a closed linear operator from $B^{p,q}_{-s}(\partial\Omega)$ to $B^{p,q}_{1-s+1/p}(\Omega)$ and from $W^{-s,p}(\partial\Omega)$ to $W^{1-s+1/p,p}(\Omega)$. The closed-graph theorem (Theorem 1.4.15) gives that this operator is bounded.

Proposition 7.6.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $0 < s < 1$, $1 < p, q < \infty$. If $f \in L^p(\partial\Omega)$, then $\mathcal{S}^\Omega f|\partial\Omega$ is the trace of $\mathcal{S}^\Omega f$ (i.e., $\gamma_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$). If $f \in B^{p,q}_{-s}(\partial\Omega)$, then $\gamma_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$ (the trace for $B^{p,q}_{1-s+1/p}(\Omega)$). If $f \in W^{-s,p}(\partial\Omega)$, then $\gamma_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$ (the trace for $W_{1-s+1/p,p}(\Omega)$).*

Proof If $f \in L^p(\partial\Omega)$, then $\mathcal{S}^\Omega f \in W^{1,p}(\Omega)$ by Propositions 5.2.2 and 5.2.8 and Lemma 1.28.1. The trace of $\mathcal{S}^\Omega f$ coincides with the nontangential limit of $\mathcal{S}^\Omega f$ by the definition of the trace. So, $\gamma_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$ by Proposition 5.2.2.

Let now $f \in B^{p,q}_{-s}(\partial\Omega)$. Since $\mathcal{S}^\Omega$ is a bounded linear operator from $B^{p,q}_{-s}(\partial\Omega)$ to $B^{p,q}_{1-s+1/p}(\Omega)$ by Proposition 7.6.1, the trace γ_Ω is a bounded linear operator from $B^{p,q}_{1-s+1/p}(\Omega)$ to $B^{p,q}_{1-s}(\partial\Omega)$ by Theorem 1.24.7, $\mathcal{S}^\Omega : B^{p,q}_{-s}(\partial\Omega) \to B^{p,q}_{-s}(\partial\Omega)$ is continuous by Corollary 5.4.11 and $\mathcal{C}^0(\partial\Omega)$ is a dense subset of $B^{p,q}_{-s}(\partial\Omega)$ by Theorem 1.23.1; we deduce that $\gamma_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega f$.

If $f \in W^{-s,p}(\partial\Omega)$, we use the fact that $f \in W^{-s,p}(\partial\Omega) = B^{p,p}_{-s}(\partial\Omega)$ (Theorem 1.23.3) and the continuity of the trace operator.

Theorem 7.6.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$, $0 \leq s \leq 1$, $1 < p, q < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Define $Pf = \langle f, 1\rangle / \mathcal{H}_{m-1}(\partial\Omega)$, $T = \mathcal{S}^\Omega(I - P) + P$.*

- *T is an isomorphism from $W^{s-1,p}(\partial\Omega)$ onto $W^{s,p}(\partial\Omega)$. If $0 < s < 1$, then T is an isomorphism from $B^{p,q}_{s-1}(\partial\Omega)$ onto $B^{p,q}_{s}(\partial\Omega)$.*
- *Let $m > 2$ or $\partial\Omega \subset B(0;1)$. Then $\mathcal{S}^\Omega$ is an isomorphism from $W^{s-1,p}(\partial\Omega)$ onto $W^{s,p}(\partial\Omega)$. If $0 < s < 1$, then $\mathcal{S}^\Omega$ is an isomorphism from $B^{p,q}_{s-1}(\partial\Omega)$ onto $B^{p,q}_{s}(\partial\Omega)$.*

Proof Suppose first that $s = 0$ or $s = 1$. According to Proposition 5.18.1, it is enough to prove that T is an isomorphism for $m > 2$. Since $\mathcal{S}^\Omega$ is an isomorphism and $T - \mathcal{S}^\Omega$ is compact, the operator T is Fredholm with index 0. (See Theorem 1.8.3.) Let $f \in W^{s-1,p}(\partial\Omega)$, $Tf = 0$. Since $T : L^2(\partial\Omega) \to W^{1,2}(\partial\Omega)$ is a Fredholm operator with index 0, Lemma 1.8.4 gives that $f \in L^2(\partial\Omega)$. Since $Tf = 0$, $\mathcal{S}^\Omega(I - P)f = -Pf$ and thus

$$\int_{\partial\Omega} (I-P)f\mathcal{S}^\Omega(I-P)f \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} \left(\oint_{\partial\Omega} f\right)\left(f - \oint_{\partial\Omega} f\right) \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Lemma 5.9.4 forces $(I-P)f \equiv 0$. Thus, $Pf = -\mathcal{S}^\Omega(I-P)f = 0$ and $f = Pf + (I-P)f \equiv 0$. Since $T : W^{s-1,p}(\partial\Omega) \to W^{s,p}(\partial\Omega)$ is a Fredholm operator with index 0, it is onto. Theorem 1.6.3 gives that T is an isomorphism.

Let $0 < s < 1$. The proposition is a consequence of the interpolation argument by Proposition 1.23.12, Theorem 1.22.1, and Proposition 1.10.3. ($W^{s-1,p}(\partial\Omega) = B^{p,p}_{s-1}(\partial\Omega)$, $W^{s,p}(\partial\Omega) = B^{p,p}_{s}(\partial\Omega)$ by Theorems 1.23.3 and 1.23.1.)

Proposition 7.6.4 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $1 < p < \infty$, $f \in W^{-1/p,p}(\partial\Omega)$.*

- *If $m > 2$ or $p \geq m$ or $\langle f, 1\rangle = 0$, then $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$ if and only if $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(R^m)$.*
- *If $p > m/(m-1)$, then $\mathcal{S}^\Omega f \in D^{1,p}(R^m)$.*
- *Let $p \leq m/(m-1)$. Then, $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$ if and only if $\langle f, 1\rangle = 0$. If this is true, then $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(R^m)$.*

Proof If $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$, then $\mathcal{S}^\Omega f \in D^{1,p}(R^m)$ by Propositions 7.6.1 and 7.6.2, Lemma 1.21.12, and Proposition 1.24.3.

If $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(R^m)$, then $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$ because $\tilde{W}^{1,p}(R^m) \subset D^{1,p}(\Omega)$. Let now $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$. If $m \leq p$, then $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(R^m)$ because $D^{1,p}(R^m) = \tilde{W}^{1,p}(R^m)$ by Proposition 3.38.4. Let now $p < m$. If $m > 2$ or $\langle f, 1\rangle = 0$, then $\mathcal{S}^\Omega f(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. Thus, $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(R^m)$ by Proposition 3.38.5.

If $r > 0$, then $\mathcal{S}^\Omega f \in W^{1,p}(\Omega \cap B(0; r))$ by Proposition 7.6.1. Since $\Delta\mathcal{S}^\Omega f = 0$ in Ω by Theorem 3.1.5, we have $\mathcal{S}^\Omega f \in \mathcal{C}^\infty(\Omega)$ by Theorem 2.18.2 and Corollary 2.2.8. Since $\nabla\mathcal{S}^\Omega f(x) = O(|x|^{1-m})$ as $|x| \to \infty$ by Proposition 3.1.6, we have $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$ for $p > m/(m-1)$.

Let $p \leq m/(m-1)$. If $\mathcal{S}^\Omega f \in D^{1,p}(\Omega)$, then $f = -\Delta\mathcal{S}^\Omega f \in \mathcal{E}_p(R^m) = \{g \in D^{-1,p}(R^m); \langle g, 1\rangle = 1\}$ by Theorem 3.1.5, Proposition 3.38.3, and Corollary 2.18.6. Let now $\langle f, 1\rangle = 0$. Put $p' = p/(p-1)$. Theorem 1.24.2 gives that $f \in [\overset{\circ}{D}{}^{1,p'}(R^m)]' = D^{-1,p}(R^m)$. Thus, $f \in \mathcal{E}_p(R^m)$. According to Proposition 3.38.3, there exists $\varphi \in D^{1,p}(R^m)$ such that $-\Delta\varphi = f$. Since $-\Delta\mathcal{S}^\Omega f = f$ by Theorem 3.1.5, we have $\Delta(\mathcal{S}^\Omega f - \varphi) = 0$. Thus, $(\mathcal{S}^\Omega f - \varphi) \in H(R^m)$ by Theorem 2.18.2. Proposition 3.1.6 gives $\partial_j\mathcal{S}^\Omega f(x) \to 0$ as $|x| \to \infty$. Since $\Delta\partial_j\varphi = \partial_j\Delta\varphi = 0$ in Ω, we have $\partial_j\varphi \in H(\Omega)$ by Theorem 2.18.2. Since $\partial_j\varphi \in L^p(\Omega)$, Proposition 2.17.4 forces $\partial_j\varphi(x) \to 0$ as $|x| \to \infty$. Thus, $\partial_j(\mathcal{S}^\Omega f(x) - \varphi(x)) \to 0$ as $|x| \to \infty$. Since $\partial_j(\mathcal{S}^\Omega f - \varphi) \in H(R^m)$ by Corollary 2.2.9, Theorem 2.16.1 forces that $\partial_j(\mathcal{S}^\Omega f - \varphi) \equiv 0$ in R^m. Hence, $(\mathcal{S}^\Omega f - \varphi)$ is constant by Lemma 1.20.1. So, $\mathcal{S}^\Omega f \in D^{1,p}(R^m)$.

7.7 Regularity of Double Layer Potentials

In this section, we study properties of $\mathcal{D}^\Omega$ and K_Ω with densities from Sobolev and Besov spaces.

Proposition 7.7.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $0 < s < 1$, $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$. Then K_Ω is a* *bounded linear operator on* $B^{p,q}_s(\partial\Omega)$ *(on* $W^{s,p}(\partial\Omega)$*). Its adjoint* K'_Ω *is a bounded linear operator on* $B^{p',q'}_{-s}(\partial\Omega)$ *(on* $W^{-s,p'}(\partial\Omega)$*).*

Proof K_Ω is a bounded linear operator on $L^p(\partial\Omega)$ by Proposition 5.3.3. and on $W^{1,p}(\partial\Omega)$ by Lemma 5.5.4. By the interpolation argument, we deduce that K_Ω is a bounded linear operator on $B^{p,q}_s(\partial\Omega)$ (see Proposition 1.10.3, Theorem 1.22.1, and Proposition 1.23.12). We have $[B^{p,q}_s(\partial\Omega)]' = B^{p',q'}_{-s}(\partial\Omega)$ by Theorem 1.23.1. The operator K'_Ω is a bounded linear operator on $B^{p',q'}_{-s}(\partial\Omega)$ by Proposition 1.5.11. Theorems 1.23.3 and 1.23.1 give $W^{s,p}(\partial\Omega) = B^{p,p}_s(\partial\Omega)$, $W^{-s,p'}(\partial\Omega) = B^{p',p'}_{-s}(\partial\Omega)$.

Proposition 7.7.2 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary. Let* $1 < p, q < \infty$, $0 < s < 1$. *If* $p \neq 2$, *suppose that* $\partial\Omega$ *is of class* $\mathcal{C}^{0,\alpha}$ *with* $0 < \alpha < 1$. *Then,* $\mathcal{D}^\Omega : W^{s,p}(\partial\Omega) \to W^{s+1/p,p}(\Omega)$, $\mathcal{D}^\Omega : B^{p,q}_s(\partial\Omega) \to B^{p,q}_{s+1/p}(\Omega)$ *are bounded linear operators.*

Proof Let $f \in W^{s,p}(\partial\Omega)$ (or $f \in B^{p,q}_s(\partial\Omega)$). Since $f \in L^p(\partial\Omega)$, $\mathcal{D}^\Omega f$ is an L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f/2 + K_\Omega f$ on $\partial\Omega$ by Theorem 5.3.5. The operator $\frac{1}{2}I + K_\Omega$ is a bounded linear operator on $W^{s,p}(\partial\Omega)$ and on $B^{p,q}_s(\partial\Omega)$ by Proposition 7.7.1. If $m > 2$, define $T\varphi = \mathcal{S}^\Omega\varphi|\partial\Omega$, $\mathcal{S}\varphi = \mathcal{S}^\Omega\varphi|\Omega$. If $m = 2$, define $P\varphi = \langle\varphi, 1\rangle/\mathcal{H}_1(\partial\Omega)$, $T\varphi = [\mathcal{S}^\Omega(I-P)\varphi + P\varphi]|\partial\Omega$, $\mathcal{S}\varphi = [\mathcal{S}^\Omega(I-P)\varphi + P\varphi]|\Omega$. Then, $T : W^{-1,p}(\partial\Omega) \to L^p(\partial\Omega)$, $T : W^{s-1,p}(\partial\Omega) \to W^{s,p}(\partial\Omega)$, $T : B^{p,q}_{s-1}(\partial\Omega) \to B^{p,q}_s(\partial\Omega)$ are isomorphisms by Proposition 5.18.1 and Theorem 7.6.3. Then, $\mathcal{S}T^{-1}(f/2 + K_\Omega f)$ is a unique L^p-solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = f/2 + K_\Omega f$ on $\partial\Omega$ by Propositions 5.18.2 and 5.18.3. Therefore, $\mathcal{D}^\Omega f = \mathcal{S}T^{-1}(f/2 + K_\Omega f)$. Since $\mathcal{S} : W^{s-1,p}(\partial\Omega) \to W^{s+1/p,p}(\Omega)$, $\mathcal{S} : B^{p,q}_{s-1}(\partial\Omega) \to B^{p,q}_{s+1/p}(\Omega)$ are bounded linear operators by Proposition 7.6.1, $\mathcal{D}^\Omega : W^{s,p}(\partial\Omega) \to W^{s+1/p,p}(\Omega)$, $\mathcal{D}^\Omega : B^{p,q}_s(\partial\Omega) \to B^{p,q}_{s+1/p}(\Omega)$ are bounded linear operators.

Proposition 7.7.3 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary,* $1 < p < \infty$, $p' = p/(p-1)$. *Then* $H^\Omega (= n^\Omega \cdot \nabla\mathcal{D}^\Omega)$ *can be extended as a bounded linear operator from* $W^{1-1/p,p}(\partial\Omega)$ *to* $W^{-1/p,p}(\partial\Omega)$. *If* $f \in W^{1-1/p,p}(\partial\Omega)$ *then* $\mathcal{D}^\Omega f \in W^{1,p}(\Omega)$ *and*

$$\int_\Omega \nabla\varphi \cdot \nabla\mathcal{D}^\Omega f \, \mathrm{d}\mathcal{H}_m = \langle H^\Omega f, \varphi\rangle \qquad \forall\varphi \in W^{1,p'}(\Omega).$$

Proof $\mathcal{D}^\Omega : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is a bounded linear operator by Proposition 7.7.2. Denote $Tf \in [W^{1,p'}(\Omega)]'$ by

$$\langle Tf, \varphi\rangle = \int_\Omega \nabla\varphi \cdot \nabla\mathcal{D}^\Omega f \, \mathrm{d}\mathcal{H}_m.$$

Then, $T : W^{1-1/p,p}(\partial\Omega) \to [W^{1,p'}(\Omega)]'$ is a bounded linear operator by Hölder's inequality (Proposition 1.14.1). If $\varphi \in \mathcal{C}^\infty_c(\Omega)$, then $\langle Tf, \varphi\rangle = 0$ by Theorem 2.18.2. Thus, Tf is supported on $\partial\Omega$. Using Theorem 1.24.2, we infer that

$T : W^{1-1/p,p}(\partial\Omega) \to W^{-1/p,p}(\partial\Omega)$ is a bounded linear operator. If $f \in W^{1,p}(\partial\Omega)$, then $\langle Tf, \varphi\rangle = \langle H^\Omega f, \varphi\rangle$ for all $\varphi \in \mathcal{C}_c^\infty(R^m)$ by Theorem 5.5.2 and Lemma 5.8.1. Since $\mathcal{C}_c^\infty(R^m)$ is a dense subset of $W^{1,p'}(\Omega)$ by Lemma 1.21.3, we deduce that $\langle Tf, \varphi\rangle = \langle H^\Omega f, \varphi\rangle$ for all $\varphi \in W^{1,p'}(\Omega)$. Proposition follows from the fact that $W^{1,p}(\partial\Omega)$ is a dense subset of $W^{1-1/p,p}(\partial\Omega)$ (see Theorems 1.23.3 and 1.23.1).

7.8 Dirichlet Problem on Sets with Cracks

In this section, we look for a solution of the Dirichlet problem for the Poisson equation in $W^{1,2}(\Omega \setminus \Gamma)$ where $\Omega \subset R^m$ is a bounded domain with Lipschitz boundary and Γ is a closed subset of a Lipschitz surface.

Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $\omega = \omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\omega}_+ \subset \Omega$. Denote $\omega_- = \Omega \setminus \overline{\omega}_+$. Let $\Gamma \subset \partial\Omega$ be closed. For $u \in W^{1,2}(\Omega \setminus \Gamma)$, denote by $[u]_+$ (by $[u]_-$) the trace of u with respect to ω_+ (with respect to ω_-), respectively. Let $f \in D^{-1,2}(\Omega \setminus \Gamma)$, $g \in H^{1/2}(\partial\Omega)$, $g_+ \in H^{1/2}(\partial\omega)$, $g_- \in H^{1/2}(\partial\omega)$ be given. *We say that $u \in W^{1,2}(\Omega \setminus \Gamma)$ is a solution of the Dirichlet problem*

$$-\Delta u = f \quad \text{in } \Omega \setminus \Gamma, \tag{7.11}$$

$$u = g \quad \text{on } \partial\Omega, \qquad [u]_\pm = g_\pm \quad \text{on } \Gamma \tag{7.12}$$

if the equation (7.11) is fulfilled in the sense of distributions and the boundary conditions (7.12) are fulfilled in the sense of traces.

(Remark that we cannot study (7.11) and (7.12) in $\tilde{W}^{1,2}(\Omega \setminus \Gamma)$ because if $u \in \tilde{W}^{1,2}(\Omega \setminus \Gamma) \subset \tilde{W}^{1,2}(R^m) \subset W^{1,2}(\Omega)$, then $[u]_+ = [u]_-$ on Γ.)

Theorem 7.8.1 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $\omega = \omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\omega}_+ \subset \Omega$. Denote $\omega_- = \Omega \setminus \overline{\omega}_+$. Let $\Gamma \subset \partial\Omega$ be closed. Let $f \in D^{-1,2}(\Omega \setminus \Gamma)$, $g \in H^{1/2}(\partial\Omega)$, $g_+ \in H^{1/2}(\partial\omega)$, $g_- \in H^{1/2}(\partial\omega)$. Define $G = g_+ - g_-$ on Γ, $G = 0$ elsewhere. Then there exists a solution $u \in W^{1,2}(\Omega \setminus \Gamma)$ of the Dirichlet problem (7.11), (7.12) if and only if $G \in H^{1/2}(\partial\omega)$. This solution is unique and*

$$\|u\|_{W^{1,2}(\Omega\setminus\Gamma)} \le C\left(\|g\|_{H^{1/2}(\partial\Omega)} + \|g_+\|_{H^{1/2}(\partial\omega)} + \|g_-\|_{H^{1/2}(\partial\omega)}\right),$$

where C depends only on Ω, ω, and Γ.

Proof If there exists a solution $u \in W^{1,2}(\Omega \setminus \Gamma)$ of the Dirichlet problem (7.11) and (7.12), then $[u]_+, [u]_- \in H^{1/2}(\partial\omega)$ by Theorem 1.24.2. Since $G = [u]_+ - [u]_-$, we infer that $G \in H^{1/2}(\partial\omega)$.

Let now $G \in H^{1/2}(\partial\omega) = W^{1/2,2}(\partial\Omega)$. Define $v = \mathcal{D}^\omega G$. Then $v \in W^{1,2}(\omega_+)$, $v \in W^{1,2}(\omega_-)$ by Proposition 7.7.2. Since $G = 0$ outside Γ, one has $v \in \mathcal{C}^\infty(\Omega \setminus \Gamma)$.

By virtue of characterization of Sobolev spaces in Theorem 1.21.1, we deduce that $v \in W^{1,2}(\Omega \setminus \Gamma)$. We have $[v]_\pm = \pm(1/2)G + K_\omega G$ by Theorem 5.3.5. So, $[v]_+ - [v]_- = G$.

Define $\tilde{g} = g - v$ on $\partial\Omega$, $\tilde{g} = g_- - [v]_-$ on $\partial\omega$. Then $\tilde{g} \in H^{1/2}(\partial\omega_-)$. According to Theorem 1.24.2, there exists $\tilde{g} \in W^{1,2}(\omega_\pm)$ such that $[\tilde{g}]_\pm = \tilde{g}$. Characterization of Sobolev spaces in Theorem 1.21.1 gives $\tilde{g} \in W^{1,2}(\Omega)$. But $W^{1,2}(\Omega) = \tilde{W}^{1,2}(\Omega)$ by Proposition 1.25.2. According to Theorem 7.1.2, there exists a solution $w \in \tilde{W}^{1,2}(\Omega \setminus \Gamma) \subset W^{1,2}(\Omega \setminus \Gamma)$ of the problem

$$-\Delta w = f \quad \text{in } \Omega \setminus \Gamma, \qquad w = \tilde{g} \quad \text{on } \partial(\Omega \setminus \Gamma).$$

Since $w \in \tilde{W}^{1,2}(\Omega \setminus \Gamma) = W^{1,2}(\Omega)$, we deduce that $w = \tilde{g}$ on $\partial(\Omega \setminus \Gamma)$ in the sense of traces.

Put $u = v + w$. Then, $u \in W^{1,2}(\Omega \setminus \Gamma)$. Since $G = 0$ outside Γ, Theorem 1.13.7 gives that $\Delta v = 0$ in $R^m \setminus \Gamma$. Thus, $-\Delta u = f$ in $\Omega \setminus \Gamma$. Clearly, $u = g$ on $\partial\Omega$, $[u]_- = g_-$ on Γ. Since $[v]_+ - [v]_- = g_+ - g_-$, we obtain

$$[u]_+ = [v]_+ + [w]_+ = ([v]_+ - [v]_-) + ([v]_- + [w]_-) = (g_+ - g_-) + g_- = g_+$$

on Γ.

Let now $u \in W^{1,2}(\Omega \setminus \Gamma)$ be a solution of the Dirichlet problem (7.11), (7.12) with $f \equiv 0$, $g \equiv 0$, $g_+ = g_- \equiv 0$. Since $[u]_+ = [u]_-$ on $\partial\Omega$, Theorem 1.21.1 gives that $u \in W^{1,2}(\Omega)$. Define $F \in [W^{1,2}(\Omega)]'$ by

$$\langle F, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m.$$

Since $\Delta u = 0$ in $\Omega \setminus \Gamma$, we have $\langle F, \varphi \rangle = 0$ for $\varphi \in \mathcal{C}_c^\infty(\Omega \setminus \Gamma)$. (See Theorem 3.19.3.) Therefore, F is supported on $\partial(\Omega \setminus \Gamma)$. Since $u = 0$ on $\partial(\Omega \setminus \Gamma)$, we obtain

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m = \langle F, u \rangle = 0.$$

So, u is constant by Lemma 1.20.1. Since $u = 0$ on $\partial\Omega$, we infer that $u \equiv 0$.

Denote $X = \{h|\Gamma; h \in H^{1/2}(\partial\Omega)\}$ equipped with the norm

$$\|h\|_X = \inf\{\|\tilde{h}\|_{H^{1/2}(\partial\omega)}; \tilde{h}|\Gamma = h\}.$$

The operator $Tu = [-\Delta u, u|\partial\Omega, [u]_+|\Gamma, [u]_-|\Gamma]$ is a bounded linear operator from $W^{1,2}(\Omega \setminus \Gamma)$ to $D^{-1,2}(\Omega) \times H^{1/2}(\partial\Omega) \times X \times X$ by Theorems 3.19.3 and 1.24.2. We have proved that T is one-to-one and onto. So, T is an isomorphism by Theorem 1.6.3.

7.9 Integral Representation

If we discuss the uniqueness of boundary value problems for the Poisson equation, then it is very useful to express a solution of the Laplace equation as the double layer potential corresponding to the trace of the function plus the single layer potential corresponding to the Neumann condition. In this section, we generalize this integral representation for functions from Sobolev spaces, from Besov spaces, and from homogeneous Sobolev spaces for domains with compact Lipschitz boundaries.

If $u \in \mathcal{C}^\infty(R^m)$, $\Omega \subset R^m$ is a bounded domain with smooth boundary and u is a solution of the problem

$$-\Delta u = f \text{ in } \Omega, \quad \partial u/\partial n = g \text{ on } \partial\Omega, \tag{7.13}$$

then the Green identity (Theorem 2.2.1) gives

$$\int_\Omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \int_\Omega f\varphi\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} g\varphi\,\mathrm{d}\mathcal{H}_{m-1}$$

for all $\varphi \in \mathcal{C}_c^\infty(R^m)$. If $F = f\mathcal{H}_m|\Omega + g\mathcal{H}_{m-1}|\partial\Omega$, then F is a distribution supported on $\overline{\Omega}$ and

$$\int_\Omega u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi\rangle, \quad \varphi \in \mathcal{C}_c^\infty(R^m). \tag{7.14}$$

This motivates our definition of a very weak solution of the Neumann problem.

Suppose that $1 < p, q < \infty$, $1/p < s < \infty$ and Ω is an open set with compact Lipschitz boundary. If $u \in B_s^{p,q}(\Omega)$ or $u \in W^{s,p}(\Omega)$ or $u \in D^{k,p}(\Omega)$ with $k \in N$, then there exists the trace $\gamma_\Omega u \in L^p(\partial\Omega)$ by Theorem 1.24.7. If F is a distribution supported in $\overline{\Omega}$, then we say that u is a very weak solution of the Neumann problem $-\Delta u = F|\Omega$ in Ω, $\partial u/\partial n = F|\partial\Omega$ in $\partial\Omega$, if (7.14) holds true.

The following lemma is a generalization of the theorem on three potentials (Theorem 2.2.4).

Lemma 7.9.1 *Suppose that* $1 < p, q < \infty$, $1/p < s < \infty$ *and* $\Omega \subset R^m$ *is a bounded open set with Lipschitz boundary. Let* F *be a distribution supported in* $\overline{\Omega}$. *Suppose that* $u \in B_s^{p,q}(\Omega)$ *or* $u \in W^{s,p}(\Omega)$. *If* u *is a very weak solution of the Neumann problem for the Poisson equation* $-\Delta u = F|\Omega$ *in* Ω, $\partial u/\partial n = F|\partial\Omega$ *on* $\partial\Omega$, *then*

$$u = VF + \mathcal{D}^\Omega u = h_m * F + \mathcal{D}^\Omega u.$$

Proof Fix $z \in \Omega$ and $r > 0$ such that $B(z; 4r) \subset \Omega$. Choose $\psi \in \mathcal{C}_c^\infty(B(z; 4r)$ such that $\psi = 1$ on $B(z; 3r)$. Denote $u_1 = u\psi$, $u_2 = u - u_1$. Then $u_1 \in B_s^{p,q}(\Omega)$ or $u_1 \in W^{s,p}(\Omega)$, u_1 has a compact support in Ω (see Lemma 1.23.9). Denote $F_1 = -\Delta u_1$ in the sense of distribution. Since $u_1 = 0$ on a neighborhood of $\partial\Omega$, we see that u_1 is a very weak solution of the Neumann problem $-\Delta u_1 = F_1|\Omega$ in Ω, $\partial u_1/\partial n = F_1|\partial\Omega$ on $\partial\Omega$, i.e., the problem $-\Delta u_1 = F_1$ in Ω, $\partial u_1/\partial n = 0$ on $\partial\Omega$. Since $u_1 = 0$ on $\partial\Omega$ and u_1 has compact support, we have

$$u_1 = \delta_0 * u_1 = (-\Delta h_m) * u_1 = h_m * (-\Delta u_1) = h_m * F_1 + \mathcal{D}^\Omega u_1.$$

Clearly, u_2 is a very weak solution of the Neumann problem $-\Delta u_2 = F_2|\Omega$ in Ω, $\partial u_2/\partial n = F_2|\partial\Omega$ on $\partial\Omega$, where $F_2 = F - F_1$. Remark that $-\Delta u_2 = F_2$ in Ω in the sense of distributions. Since $u_2 = 0$ in $B(z; 3r)$, we have $F_2 = 0$ in $B(z; 3r)$. Fix $x \in B(z; r)$. Choose $\Phi_1 \in \mathcal{C}_c^\infty(B(x; 2r))$ and $\Phi_2 \in \mathcal{C}_c^\infty(R^m)$ such that $\Phi_1 = 1$ on $B(x; r)$ and $\Phi_2 = 1$ on a neighborhood of $\overline{\Omega}$. Set $\phi(y) = h_m(y-x)[1 - \Phi_1(y)]\Phi_2(y)$. Since $u_2 = 0$ in $B(x; 2r)$ and $\Delta\varphi(y) = \Delta h_m(y - x) = 0$ in $\Omega \setminus B(x; 2r)$, we have by the definition of a very weak solution

$$\langle F_2, \varphi\rangle = \int\limits_\Omega u_2(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} u_2 \frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1} = -\mathcal{D}^\Omega u_2.$$

Since $\varphi(y) = h_m(y - x)$ on $\operatorname{spt} F$, we deduce

$$\begin{aligned} u_2(x) = 0 = \mathcal{D}^\Omega u_2(x) + \langle F_2, \varphi\rangle &= \mathcal{D}^\Omega u_2(x) + \langle F_2, h_m(\cdot - x)\rangle \\ &= \mathcal{D}^\Omega u_2(x) + (h_m * F_2)(x). \end{aligned}$$

We have proved that $u = u_1 + u_2 = h_m * F + \mathcal{D}^\Omega u$ in $B(z; r)$.

Proposition 7.9.2 *Suppose that* $1 < p < \infty$, $k \in N$ *and* $\Omega \subset R^m$ *is an unbounded open set with compact Lipschitz boundary. Let* F *be a distribution supported on* $\partial\Omega$. *If* $u \in D^{k,p}(\Omega)$ *is a very weak solution of the Neumann problem for the Poisson equation* $-\Delta u = 0$ *in* Ω, $\partial u/\partial n = F$ *on* $\partial\Omega$, *then there exists a polynomial* P *of degree at most* $k - 1$ *such that*

$$u = \mathcal{S}^\Omega F + \mathcal{D}^\Omega u + P.$$

Proof Choose $r \in (0, \infty)$ such that $\partial\Omega \subset B(0; r)$. Define $\omega = \Omega \cap B(0; r)$, $f = F$ on $\partial\Omega$, $f = \partial u/\partial n$ on $\partial B(0; r)$. Choose an open set G with smooth boundary such that $\partial\Omega \subset G \subset \overline{G} \subset B(0; r)$, and $\psi \in \mathcal{C}_c^\infty(B(0; r))$ such that $\psi = 1$ on $\overline{G}$. If

$\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\int_{\omega} u(-\Delta\varphi)\, d\mathcal{H}_m + \int_{\partial\omega} u\frac{\partial\varphi}{\partial n}\, d\mathcal{H}_{m-1} = \int_{\Omega} u[-\Delta(\psi\varphi)]\, d\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial(\psi\varphi)}{\partial n}$$

$$+ \int_{B(0;r)\setminus\overline{G}} u\{-\Delta[(1-\psi)\varphi]\}\, d\mathcal{H}_m + \int_{\partial[B(0;r)\setminus\overline{G}]} u\frac{\partial[(1-\psi)\varphi]}{\partial n}\, d\mathcal{H}_{m-1}$$

$$= \langle F, \varphi\rangle + \int_{\partial B(0;r)} \varphi\frac{\partial u}{\partial n}\, d\mathcal{H}_{m-1} = \langle f, \varphi\rangle$$

by Theorem 2.2.1. So, u is a very weak solution of the problem $\Delta u = 0$ in ω, $\partial u/\partial n = f$ on $\partial\omega$. Lemma 7.9.1 gives

$$u = \mathcal{S}^\omega f + \mathcal{D}^\omega u \qquad \text{in } \omega. \tag{7.15}$$

Define

$$P = \begin{cases} u - \mathcal{S}^\Omega F - \mathcal{D}^\Omega u & \text{on } \Omega, \\ \mathcal{S}^{B(0;r)}(\partial u/\partial n) + \mathcal{D}^{B(0;r)} u & \text{on } B(0;r). \end{cases}$$

The function P is well defined by (7.15) and harmonic in R^m. Let α be a multiindex with $|\alpha| = k$. Then $|\partial^\alpha \mathcal{S}^\Omega F(x)| + |\partial^\alpha \mathcal{D}^\Omega u(x)| \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. Since $\partial^\alpha u \in L^p(\Omega)$ and $\partial^\alpha u \in H(\Omega)$ by Corollary 2.2.9, Proposition 2.17.4 forces $\partial^\alpha u(x) \to 0$ as $|x| \to \infty$. So, $\partial^\alpha P(x) \to 0$ as $|x| \to \infty$. Since $\partial^\alpha P \in H(R^m)$ by Corollary 2.2.9, we obtain by the Liouville theorem (Theorem 2.16.1) that $\partial^\alpha P \equiv 0$. Since a multiindex α with $|\alpha| = k$ is arbitrary, we deduce that P is a polynomial of degree at most $k - 1$.

7.10 The Dirichlet Problem in $W^{s,p}(\Omega)$, $B_s^{p,q}(\Omega)$

In this section, we prove that there exists a unique solution of the Dirichlet problem for the Poisson equation in the Sobolev space $W^{s,p}(\Omega)$ or in the Besov space $B_s^{p,q}(\Omega)$, where Ω is a bounded domain with Ljapunov boundary and $1 < p, q < \infty$, $1/p < s < 1 + 1/p$.

Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p, q < \infty$, $1/p < s < \infty$. We say that $u \in W^{s,p}(\Omega)$ (or $u \in B_s^{p,q}(\Omega)$) is a solution of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{7.16}$$

if $-\Delta u = f$ in Ω in the sense of distributions and the trace of u is equal to g.

Let now $u \in W^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$. If u is a solution of the problem (7.16), then $f \in W^{-1,p}(\Omega)$ by Theorem 3.19.3 and $g \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2. Fix $f \in W^{-1,p}(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$. According to Theorem 1.24.2, we can suppose that $g \in W^{1,p}(\Omega)$ and g on the $\partial\Omega$ is the trace. Then, u is a solution of the Dirichlet problem (7.16) if and only if $u - g \in \mathring{W}^{1,p}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m = \langle f, \varphi\rangle \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega)$$

(or equivalently for all $\varphi \in \mathring{W}^{1,p'}(\Omega) = \mathring{D}^{1,p'}(\Omega)$ with $p' = p/(p-1)$). (See Proposition 1.24.3 and Theorem 3.19.3.)

Theorem 7.10.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

- *There exists a solution $u \in B_s^{p,q}(\Omega)$ of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$ if and only if $g \in B_{s-1/p}^{p,q}(\partial\Omega), f \in B_{s-2}^{p,q}(\Omega)$. This solution is unique and*

  $$\|u\|_{B_s^{p,q}(\Omega)} \leq C[\|f\|_{B_{s-2}^{p,q}(\Omega)} + \|g\|_{B_{s-1/p}^{p,q}(\partial\Omega)}]$$

  with a constant C depending only on Ω, s, p, and q.
- *If $f \in B_{s-2}^{p,q}(\Omega)$, $u_0 \in B_s^{p,q}(\Omega)$, then there exists a unique $u \in B_s^{p,q}(\Omega)$ such that $-\Delta u = f$ in Ω, $u - u_0 \in \mathring{B}_s^{p,q}(\Omega)$. Moreover,*

  $$\|u\|_{B_s^{p,q}(\Omega)} \leq C[\|f\|_{B_{s-2}^{p,q}(\Omega)} + \|u_0\|_{B_s^{p,q}(\Omega)}]$$

  with a constant C depending only on Ω, s, p, and q.

Proof We can suppose that $\overline{\Omega} \subset B(0;1)$.

Suppose that there exists a solution $u \in B_s^{p,q}(\Omega)$ of the Dirichlet problem (7.16). We have $g \in B_{s-1/p}^{p,q}(\partial\Omega)$ by Theorems 1.24.7. Theorem 1.23.8 gives that $f \in B_{s-2}^{p,q}(\Omega)$.

Let now $g \in B_{s-1/p}^{p,q}(\partial\Omega), f \in B_{s-2}^{p,q}(\Omega)$ be fixed. We can suppose $f \in B_{s-2}^{p,q}(R^m)$. We can even suppose that f has a compact support. Then $Vf \in B_s^{p,q}(G)$ for all bounded open G (see Proposition 3.18.5). Denote by h the trace of Vf. We have $h \in B_{s-1/p}^{p,q}(\partial\Omega)$ by Theorem 1.24.7. According to Theorem 7.6.3, there exists $\Psi \in B_{s-1-1/p}^{p,q}(\partial\Omega)$ such that $\mathcal{S}^\Omega\Psi = g - h$ on $\partial\Omega$. Put $u = Vf + \mathcal{S}^\Omega\Psi$. Then, $u \in B_s^{p,q}(\Omega)$ by Proposition 7.6.1. The trace of u is equal to g by Proposition 7.6.2. Moreover, $-\Delta u = f$ in Ω by Theorem 3.1.5.

Let now u be a solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. Define the distribution F by

$$\langle F, \varphi\rangle = \int_\Omega u(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\, \mathrm{d}\mathcal{H}_{m-1}.$$

Then F is a distribution supported on $\overline{\Omega}$. Since $u = 0$ on $\partial\Omega$, Theorem 1.24.9 gives that $u \in \mathring{B}^{p,q}_s(\Omega)$. If we define $u = 0$ on $R^m \setminus \Omega$, then $u \in B^{p,q}_s(R^m)$. If $\varphi \in \mathcal{C}^\infty_c(R^m)$ then

$$\langle F, \varphi \rangle = \int\limits_{R^m} u(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m.$$

Thus, $F = -\Delta u$ in R^m in the sense of distributions. Theorems 1.23.8 and 1.23.1 give $F \in B^{p,q}_{s-2}(R^m) = [B^{p',q'}_{2-s}(R^m)]'$ with $p' = p/(p-1)$, $q' = q/(q-1)$. Since $\Delta u = 0$ in $R^m \setminus \partial\Omega$, the distribution F is supported on $\partial\Omega$. Thus, F is a bounded linear operator on the space of traces of $B^{p',q'}_{2-s}(\Omega)$, i.e., on $B^{p',q'}_{2-s-1/p'}(\partial\Omega) = B^{p',q'}_{1+1/p-s}(\partial\Omega)$ (compare Theorems 1.24.7 and 1.24.9). So, $F \in B^{p,q}_{-1+s-1/p}(\partial\Omega)$ by Theorem 1.23.1. Lemma 7.9.1 gives that $u = \mathcal{S}^\Omega F$ in Ω. According to Proposition 7.6.2, we have $0 = u = \mathcal{S}^\Omega F$ on $\partial\Omega$. Since $\mathcal{S}^\Omega$ is an isomorphism on $B^{p,q}_{-1+s-1/p}(\partial\Omega)$ by Theorem 7.6.3, we deduce that $F = 0$. Thus, $u \equiv 0$.

Since the mapping $u \mapsto [-\Delta u, \gamma_\Omega u]$ is a one-to-one bounded linear operator $B^{p,q}_s(\Omega)$ onto $B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$, it is an isomorphism by Theorem 1.6.3.

Let now $f \in B^{p,q}_{s-2}(\Omega)$, $u_0 \in B^{p,q}_s(\Omega)$. The trace γ_Ω is a bounded linear operator from $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s-1/p}(\Omega)$ by Theorem 1.24.7. So, $g = \gamma_\Omega u \in B^{p,q}_{s-1/p}(\partial\Omega)$. Theorem 1.24.9 gives that $u \in B^{p,q}_s(\Omega)$ is a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$ if and only if $-\Delta u = f$ in Ω, $u - u_0 \in \mathring{B}^{p,q}_s(\Omega)$.

Theorem 7.10.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

- *Then, there exists a solution $u \in W^{s,p}(\Omega)$ of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$ if and only if $g \in W^{s-1/p,p}(\partial\Omega)$, $f \in W^{s-2,p}(\Omega)$. This solution is unique and*

  $$\|u\|_{W^{s,p}(\Omega)} \leq C[\|f\|_{W^{s-2,p}(\Omega)} + \|g\|_{W^{s-1/p,p}(\partial\Omega)}] \tag{7.17}$$

  with a constant C depending only on Ω, s, and p.
- *If $u_0 \in W^{s,p}(\partial\Omega)$, $f \in W^{s-2,p}(\Omega)$, then there exists a unique $u \in W^{s,p}(\Omega)$ such that $-\Delta u = f$ in Ω, $u - u_0 \in \mathring{W}^{s,p}(\Omega)$. Moreover,*

  $$\|u\|_{W^{s,p}(\Omega)} \leq C[\|f\|_{W^{s-2,p}(\Omega)} + \|u_0\|_{W^{s,p}(\Omega)}] \tag{7.18}$$

  with a constant C depending only on Ω, s, and p. If $s = 1$, then u is a unique solution of the problem $u - u_0 \in \mathring{W}^{1,p}(\Omega)$,

  $$\int\limits_\Omega \nabla u \cdot \nabla\varphi = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{C}^\infty_c(\Omega) \tag{7.19}$$

  (or equivalently for all $\varphi \in \mathring{W}^{1,p'}(\Omega)$ with $p' = p/(p-1)$).

Proof We can suppose that $\overline{\Omega} \subset B(0;1)$. If $s \neq 1$, we can use Theorem 7.10.1 and the fact that $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$, $W^{s-2,p}(\Omega) = B_{s-2}^{p,p}(\Omega)$, $W^{s-1/p,p}(\partial\Omega) = B_{s-1/p}^{p,p}(\partial\Omega)$ by Proposition 1.23.13 and Theorem 1.23.3. Let $s = 1$. We can suppose that $\overline{\Omega} \subset B(0;1)$. The uniqueness follows from the fact that $W^{s,p}(\Omega) \subset W^{t,p}(\Omega)$ for $t < s$.(See Lemma 1.21.16.)

If $u \in W^{1,p}(\Omega)$ is a solution of the problem (7.16), then $f \in W^{-1,p}(\Omega)$ by Theorem 3.19.3 and $g \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2.

Fix $f \in W^{-1,p}(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$. According to Theorem 1.24.2, we can suppose that $g \in W^{1,p}(\Omega)$ and g on the $\partial\Omega$ is the trace. Then, u is a solution of the Dirichlet problem (7.16) if and only if $u - g \in \overset{\circ}{W}{}^{1,p}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = \langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

(See Proposition 1.24.3 and Theorem 3.19.3.) According to Hahn-Banach's theorem (Theorem 1.5.2), we can extend f as an element $\tilde{f} \in W^{-1,p}(R^m)$, such that $\|\tilde{f}\|_{W^{-1,p}(R^m)} \leq \|f\|_{W^{-1,p}(\Omega)}$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on a neighborhood of $\overline{\Omega}$. Then, $\varphi\tilde{f} \in W^{-1,p}(R^m)$ and $\|\varphi\tilde{f}\|_{W^{-1,p}(R^m)} \leq C_1\|f\|_{W^{-1,p}(\Omega)}$ with C_1 depending only on φ and p. Then, $V(\varphi\tilde{f}) \in W^{1,p}(\Omega)$ by Proposition 3.18.5 and $\|V(\varphi\tilde{f})\|_{W^{1,p}(\Omega)} \leq C_2\|f\|_{W^{-1,p}(\Omega)}$ with C_2 depending only on φ and p. Denote by h the trace of $V(\varphi\tilde{f})$. Then, $h \in W^{1-1/p,p}(\partial\Omega)$ and $\|h\|_{W^{1-1/p,p}(\partial\Omega)} \leq C_3\|f\|_{W^{-1,p}(\Omega)}$ with C_3 depending only on Ω, φ, and p (see Theorem 1.24.2). According to Theorem 7.6.3, there exists $\Psi \in W^{-1/p,p}(\partial\Omega)$ such that $\mathcal{S}^\Omega\Psi = g - h$ on $\partial\Omega$ and

$$\|\Psi\|_{W^{-1/p,p}(\partial\Omega)} \leq C_4\|g - h\|_{W^{1-1/p,p}(\partial\Omega)}.$$

Put $u = V(\varphi\tilde{f}) + \mathcal{S}^\Omega\Psi$. Then, $u \in W^{1,p}(\Omega)$ and (7.17) holds by Proposition 7.6.1. The trace of u is equal to g by Proposition 7.6.2. Moreover, $-\Delta u = f$ in Ω by Theorem 3.1.5. Remember that $u = g$ on $\partial\Omega$ if and only if $u - g \in \overset{\circ}{W}{}^{1,p}(\Omega)$. (See Proposition 1.24.3.)

Theorem 7.10.3 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary and $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$ suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $c \in L^\infty(\Omega)$, $c \geq 0$.*

- *There exists a solution $u \in B_s^{p,q}(\Omega)$ of the Dirichlet problem*

$$-\Delta u + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{7.20}$$

if and only if $g \in B_{s-1/p}^{p,q}(\partial\Omega)$, $f \in B_{s-2}^{p,q}(\Omega)$. This solution is unique and

$$\|u\|_{B_s^{p,q}(\Omega)} \leq C[\|f\|_{B_{s-2}^{p,q}(\Omega)} + \|g\|_{B_{s-1/p}^{p,q}(\partial\Omega)}]$$

with a constant C depending only on Ω, c, s, p, and q.

- *If* $f \in B^{p,q}_{s-2}(\Omega)$, $u_0 \in B^{p,q}_s(\Omega)$, *then there exists a unique* $u \in B^{p,q}_s(\Omega)$ *such that* $-\Delta u + cu = f$ *in* Ω, $u - u_0 \in \mathring{B}^{p,q}_s(\Omega)$. *Moreover,*

$$\|u\|_{B^{p,q}_s(\Omega)} \le C[\|f\|_{B^{p,q}_{s-2}(\Omega)} + \|u_0\|_{B^{p,q}_s(\Omega)}]$$

with a constant C *depending only on* Ω, c, s, p, *and* q.

Proof The mapping $u \mapsto [-\Delta u, \gamma_\Omega u]$ is an isomorphism from $B^{p,q}_s(\Omega)$ onto $B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$ by Theorem 7.10.1. The mapping $u \mapsto cu$ is a compact linear mapping from $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s-2}(\Omega)$ by Theorem 1.23.14, Propositions 1.23.15 and 1.23.13, and Theorem 1.7.4. The operator $T : u \mapsto [-\Delta u + cu, \gamma_\Omega u]$ is a Fredholm operator with index 0 from $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$ (see Theorem 1.8.3).

We have proved that $T : B^{2,2}_{3/4} \to B^{2,2}_{-5/4}(\Omega) \times B^{2,2}_{1/4}(\partial\Omega)$ is a Fredholm operator with index 0. We have $B^{2,2}_{3/4}(\Omega) = W^{3/4,2}(\Omega)$ by Proposition 1.23.13, $B^{2,2}_{-5/4}(\Omega) = W^{-5/4,2}(\Omega)$ by Proposition 1.23.13, and $B^{2,2}_{1/4}(\partial\Omega) = W^{1/4,2}(\partial\Omega)$ by Theorem 1.23.3. Since $T : W^{1,2}(\Omega) \to W^{-1,2}(\Omega) \times W^{1/2,2}(\Omega)$ is an isomorphism by Theorem 7.1.2, Lemma 1.8.4 gives that $T : B^{2,2}_1(\Omega) \to B^{2,2}_{-1}(\Omega) \times B^{2,2}_{1/2}(\partial\Omega)$, $T : B^{p,q}_1(\Omega) \to B^{p,q}_{-1}(\Omega) \times B^{p,q}_{1-1/p}(\partial\Omega)$, $T : B^{p,q}_s(\Omega) \to B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$ are isomorphisms.

Let now $f \in B^{p,q}_{s-2}(\Omega)$, $u_0 \in B^{p,q}_s(\Omega)$. The trace γ_Ω is a bounded linear operator from $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s-1/p}(\Omega)$ by Theorem 1.24.7. So, $g = \gamma_\Omega u \in B^{p,q}_{s-1/p}(\partial\Omega)$. Theorem 1.24.9 gives that $u \in B^{p,q}_s(\Omega)$ is a solution of the Dirichlet problem $-\Delta u + cu = f$ in Ω, $u = g$ on $\partial\Omega$ if and only if $-\Delta u + cu = f$ in Ω, $u - u_0 \in \mathring{B}^{p,q}_s(\Omega)$.

Theorem 7.10.4 *Let* $\Omega \subset R^m$ *be a bounded open set with Lipschitz boundary and* $1 < p < \infty$, $1/p < s < 1 + 1/p$. *If* $p \neq 2$, *suppose that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $c \in L^\infty(\Omega)$, $c \ge 0$.

- *Then, there exists a solution* $u \in W^{s,p}(\Omega)$ *of the Dirichlet problem (7.20) if and only if* $f \in W^{s-2,p}(\Omega)$, $g \in W^{s-1/p,p}(\partial\Omega)$. *This solution is unique and*

$$\|u\|_{W^{s,p}(\Omega)} \le C[\|f\|_{W^{s-2,p}(\Omega)} + \|g\|_{W^{s-1/p,p}(\partial\Omega)}] \tag{7.21}$$

with a constant C *depending only on* Ω, c, s, *and* p.

- *If* $f \in W^{s-2,p}(\Omega)$, $u_0 \in W^{s,p}(\Omega)$, *then there exists a unique* $u \in W^{s,p}(\Omega)$ *such that* $-\Delta u + cu = f$ *in* Ω, $u - u_0 \in \mathring{W}^{s,p}(\Omega)$. *Moreover,*

$$\|u\|_{W^{s,p}(\Omega)} \le C[\|f\|_{W^{s-2,p}(\Omega)} + \|u_0\|_{W^{s,p}(\Omega)}] \tag{7.22}$$

with a constant C *depending only on* Ω, c, s, *and* p.

Proof If $s \neq 1$, we can use Theorem 7.10.3 and the fact that $W^{s,p}(\Omega) = B^{p,p}_s(\Omega)$, $W^{s-2,p}(\Omega) = B^{p,p}_{s-2}(\Omega)$, $W^{s-1/p,p}(\partial\Omega) = B^{p,p}_{s-1/p}(\partial\Omega)$. (See Proposition 1.23.13 and Theorem 1.23.3.) Let $s = 1$. The uniqueness follows from the fact that $W^{s,p}(\Omega) \subset W^{t,p}(\Omega)$ for $t < s$. (See Lemma 1.21.16.)

Define $T_1 u = [-\Delta u, \gamma_\Omega u]$, $T_2 u = [-\Delta u + cu, \gamma_\Omega u]$. The operator $T_1 : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ is an isomorphism by Theorem 7.10.2. Since $T_2 - T_1 : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ is a compact operator by Lemma 1.21.8, Proposition 1.21.9, and Theorem 1.7.4, the operator $T_2 : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ is Fredholm with index 0 by Theorem 1.8.3. Since it is one-to-one, it is onto. Theorem 1.6.3 gives that $T_2 : W^{1,p}(\Omega) \to W^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ is an isomorphism.

Fix $u_0 \in W^{1,p}(\partial\Omega)$, $f \in W^{-1,p}(\Omega)$. Denote $F = f + \Delta u_0 - cu_0$. Then $F \in W^{-1,p}(\Omega)$ and

$$\| F\|_{W^{-1,p}(\Omega)} \le \|f\|_{W^{-1,p}(\Omega)} + C_1 \|u_0\|_{W^{1,p}(\partial\Omega)}$$

where C_1 depends only on Ω and p (see Proposition 1.21.11). We have proved that there exists $v \in W^{1,p}(\Omega)$ such that $-\Delta v + cv = F$ in Ω, $v = 0$ on $\partial\Omega$ and $\|v\|_{W^{1,p}(\Omega)} \le C_3 \| F\|_{W^{-1,p}(\Omega)}$, where C_3 depends only on Ω and p. Proposition 1.24.3 gives $v \in \overset{\circ}{W}{}^{1,p}(\Omega)$. Define $u = u_0 + v$. Then, $u - u_0 \in \overset{\circ}{W}{}^{1,p}(\Omega)$, $-\Delta u + cu = f$, and the estimate (7.22) holds.

Example 7.10.5 Fix $2 < p < \infty$. We show that there exists a bounded domain Ω with Lipschitz boundary and $g \in W^{1,1-1/p}(\partial\Omega)$ such that the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ does not have a solution $u \in W^{1,p}(\Omega)$. Choose $\alpha \in (\pi p/(p-1), 2\pi)$. Denote $G_\alpha = \{x = [x_1, x_2]; x_1 = r\cos\varphi, x_2 = r\sin\varphi, r > 0, 0 < \varphi < \alpha\}$. Define $u(x) = [u_1(x), u_2(x)] = [r^{\pi/\alpha}\cos(\varphi\pi/\alpha), r^{\pi/\alpha}\sin(\varphi\pi/\alpha)]$ for $x = [r\cos\varphi, r\sin\varphi]$. Easy calculation yields that $u_1 + iu_2$ is a holomorphic function in G_α. If $z \in \partial G_\alpha$, then there exists a finite limit $u(z)$ of u and $u_2(z) = 0$. Choose a bounded open set Ω such that $\Omega \cap B(0;\rho) = G_\alpha \cap B(0;\rho)$ for some $\rho > 0$ and $\partial\Omega$ is of class $\mathcal{C}^\infty$ away from 0. Put $v = u_2$ on $\overline{\Omega}$, $g = v$ on $\partial\Omega$. Then $v \in H(\Omega)$ by Proposition 2.20.2. Clearly, $g \in \mathcal{C}^\infty(\partial\Omega)$. Easy calculation yields that $c_1|x|^{\pi/\alpha-1} \le |\nabla v(x)| \le c_2|x|^{\pi/\alpha-1}$ for $x \in \Omega$, where c_1 and c_2 are positive constants. Since $\beta = 2\pi/\alpha - 2 > -1$ and $|\nabla v(x)|^2 \le c_2^2|x|^\beta$, we infer $v \in W^{1,2}(\Omega)$. If there exists a solution $u \in W^{1,p}(\Omega)$ of the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, then $u \in W^{1,2}(\Omega)$ by Hölder's inequality (Proposition 1.14.1). The uniqueness of a solution of the Dirichlet problem (Theorem 7.1.2) gives that $u = v$. Since $\gamma = p(\pi/\alpha - 1) < \pi p(p-1)/(p\pi) - p = -1$ and $|\nabla v(x)|^p \ge c_1^p|x|^\gamma$, we deduce $v \notin W^{1,p}(\Omega)$.

Corollary 7.10.6 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $1 < p < m/(m-1)$, $p' = p/(p-1)$ and μ be a finite real measure in R^m. Then, $\mu \in [W^{1,p'}(\Omega)]' \subset W^{-1,p}(\Omega)$. If $g \in W^{1-1/p,p}(\Omega)$, $c \in L^\infty(\Omega)$, $c \ge 0$, and $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, then there exists a unique $u \in W^{1,p}(\Omega)$ such that $-\Delta u + cu = \mu$ in Ω, $u = g$ on $\partial\Omega$. Moreover,*

$$\|u\|_{W^{1,p}(\Omega)} \le C[|\mu|(R^m) + \|g\|_{W^{1-1/p,p}(\partial\Omega)}]$$

where C depends only on Ω and p.

Proof $W^{1,p'}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$ by Theorem 1.21.13. Thus, there exists C_1 such that

$$\sup_{x\in\Omega} |\varphi(x)| \le C_1 \|\varphi\|_{W^{1,p'}(\Omega)}$$

for $\varphi \in W^{1,p'}(\Omega)$. So,

$$\left| \int_\Omega \varphi \, \mathrm{d}\mu \right| \le |\mu|(R^m) \sup_{x\in\Omega} |\varphi(x)| \le C_1 |\mu|(R^m) \|\varphi\|_{W^{1,p'}(\Omega)}.$$

Thus, $\mu \in [W^{1,p'}(\Omega)]'$ and $\|\mu\| \le C_1|\mu|(R^m)$. The rest is a consequence of Theorem 7.10.4.

Corollary 7.10.7 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary and $1 < p < \infty$, $0 < s < 2$. If $s = 1$ and $p \ne 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $\Delta u = 0$ in Ω in the sense of distributions. Then, $u \in W^{s,p}(\Omega)$ if and only if $u \in B_s^{p,p}(\Omega)$. Moreover,*

$$C^{-1}\|u\|_{W^{s,p}(\Omega)} \le \|u\|_{B_s^{p,p}(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)}$$

where a positive constant C depends only on Ω, s, and p.

Proof $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$ for $s \ne 1$ or $p = 2$ and the corresponding norms are equivalent, but $W^{1,p}(\Omega) \ne B_1^{p,p}(\Omega)$ for $p \ne 2$. (See Proposition 1.23.13 and Lemma 1.23.6.)

Suppose now that $s = 1$ and $p \ne 2$. Let $u \in W^{1,p}(\Omega) \cup B_1^{p,p}(\Omega)$. Then, the trace $u \in W^{1-1/p,p}(\partial\Omega) = B_{1-1/p}^{p,p}(\partial\Omega)$ by Theorems 1.24.2, 1.24.7, and 1.23.3. According to Theorems 7.10.1 and 7.10.2, there exist $v \in B_1^{p,p}(\Omega)$ and $w \in W^{1,p}(\Omega)$ such that $\Delta v = \Delta w = 0$ in Ω, $v = w = u$ on $\partial\Omega$. Choose $t \in (1/p, 1)$. Then, $u, v, w \in W^{t,p}(\Omega) = B_t^{p,p}(\Omega)$ by Lemma 1.21.16 and Proposition 1.23.7. Uniqueness of the Dirichlet problem in $B_t^{p,p}(\Omega)$ (Theorem 7.10.1) gives that $u = v = w$.

$W^{s,p}(\Omega) \cap H(\Omega)$ is a closed subspace of $W^{s,p}(\Omega)$; $B_s^{p,p}(\Omega) \cap H(\Omega)$ is a closed subspace of $B_s^{p,p}(\Omega)$. (See Proposition 2.11.4.) Since $W^{s,p}(\Omega) \cap H(\Omega) = B_s^{p,p}(\Omega) \cap H(\Omega)$, the corresponding norms are equivalent by Proposition 1.3.6.

7.11 Solutions of the Dirichlet Problem in Different Senses

In this section, we compare solutions of the Dirichlet problem for the Laplace equation in different senses: PWB-solutions, L^p-solutions, and solutions in Sobolev and Besov spaces.

Theorem 7.11.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p < \infty$. If u is an L^p-solution of the Dirichlet problem*

$$\Delta u = 0 \quad in\ \Omega, \qquad u = g \quad on\ \partial\Omega \tag{7.23}$$

such that $M_a(\nabla u) \in L^p(\partial\Omega)$, then u is a solution of the problem (7.23) in $W^{1,p}(\Omega)$.

Proof $u \in W^{1,p}(\Omega)$ by Lemma 1.28.1. So, there exists the trace. By the definition of the nontangential limit, we see that the trace is equal to the nontangential limit.

Theorem 7.11.2 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$.*

1. *If u is an L^p-solution of the problem (7.23) such that $M_a(\nabla u) \in L^p(\partial\Omega)$, then u is a solution of the problem (7.23) in $W^{s,p}(\Omega)$ and in $B_s^{p,q}(\Omega)$.*
2. *If u is a solution of the problem (7.23) in $W^{s,p}(\Omega)$ or in $B_s^{p,q}(\Omega)$, then u is an L^p-solution of the problem (7.23).*
3. *If u is an L^p-solution of the problem (7.23), then u is a PWB-solution of the problem (7.23).*

Proof We can suppose that $\overline{\Omega} \subset B(0; 1)$.

(1) According to Proposition 5.18.2, there exists $\psi \in L^p(\partial\Omega)$ such that $u = \mathcal{S}^\Omega \psi$. Propositions 7.6.1 and 7.6.2 give that u is a solution of the problem (7.23) in $W^{s,p}(\Omega)$ and in $B_s^{p,q}(\Omega)$.

(2) We can suppose that $s < 1$. Since $W^{s,p}(\Omega) = B_s^{p,q}(\Omega)$ by Proposition 1.23.13, we can suppose that $u \in B_s^{p,q}(\Omega)$. According to Theorems 7.10.1 and 7.6.3 and Propositions 7.6.1 and 7.6.2, there exists a unique $\varphi \in B_{s-1/p-1}^{p,q}(\Omega)$ such that $u = \mathcal{S}^\Omega \varphi$ and $g = \mathcal{S}^\Omega \varphi$. Since $s > 1/p$, Proposition 1.23.7 and Theorems 1.23.5 and 1.22.1 give $\varphi \in B_{s-1-1/p}^{p,q}(\partial\Omega) \subset F_{-1}^{p,2}(\partial\Omega) = W^{-1,p}(\partial\Omega)$. Corollary 5.18.4 gives that u is an L^p-solution of the problem.

(3) See Theorem 5.16.2.

Corollary 7.11.3 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p, q < \infty$, $0 < s < 1$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $g \in L^p(\partial\Omega)$. Then, there exists a unique PWB-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. If $g \in W^{s,p}(\partial\Omega)$, then $u \in W^{s+1/p,p}(\Omega)$. If $g \in B_s^{p,q}(\partial\Omega)$, then $u \in B_{s+1/p}^{p,q}(\Omega)$.*

Proof According to Theorem 5.16.2, there is a unique *PWB*-solution u of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$, and u is an L^p-solution of the problem. If $g \in B_s^{p,q}(\partial\Omega)$, then there exists a solution $v \in B_{s+1/p}^{p,q}(\Omega)$ of the problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$ (see Theorem 7.10.1). If $g \in W^{s,p}(\partial\Omega)$, then there exists a solution $v \in W^{s+1/p,p}(\Omega)$ of the problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$ (see Theorem 7.10.2). Theorem 7.11.2 gives that v is a PWB-solution of the problem. Thus, $u = v$.

Theorem 7.11.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$, $g \in \tilde{W}^{1,2}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$. If u is a solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ in $\tilde{W}^{1,2}(\Omega)$, then u is a PWB-solution of this problem.*

Proof Since $g \in \mathcal{C}^0(\partial\Omega)$, there exists a PWB-solution v of the Dirichlet problem $\Delta v = 0$ in Ω, $v = g$ on $\partial\Omega$ by Theorem 4.2.7. Fix an open set ω with Lipschitz boundary such that $\overline{\omega} \subset \Omega$. Choose an increasing sequence of open sets Ω_n with Lipschitz boundary such that $\omega \subset \Omega_1$ and $\Omega = \cup\Omega_n$. Denote by v_n the PWB-solution of the Dirichlet problem $\Delta v_n = 0$ in Ω_n, $v_n = g$ on $\partial\Omega_n$. Then $v_n \to v$ in ω by Theorem 4.3.1. According to Theorem 7.1.2, there exists a solution u_n of the Dirichlet problem $\Delta u_n = 0$ in Ω_n, $u_n = g$ on $\partial\Omega_n$ in $\tilde{W}^{1,2}(\Omega_n)$. Theorem 7.2.1 forces that $u_n \to u$ in $W^{1,2}(\omega)$. According to Proposition 1.14.8, we can suppose that $u_n \to u$ almost everywhere. Since $u_n = v_n$ by Theorem 7.11.2, we have $u(x) = \lim u_n(x) = \lim v_n(x) = v(x)$ for almost all $x \in \omega$.

Corollary 7.11.5 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, $m \geq 2$, $1 < p < \infty$. Let $u \in W^{1,p}(\Omega)$ be a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. If $g \in \mathcal{C}^0(\partial\Omega)$ and $f \in L^q(\Omega)$ with $q > m/2$, then $u \in \mathcal{C}^0(\overline{\Omega})$.*

Proof Put $r = \min(p, q)$. Define $f = 0$ on $R^m \setminus \Omega$. Then, $Vf \in \mathcal{C}^0(\overline{\Omega}) \cap W^{1,r}(\Omega)$ by Theorem 3.13.3, Proposition 3.13.2, and Theorems 3.14.5 and 1.21.13. Since $-\Delta Vf = f$ in the sense of distributions by Theorem 3.1.5, the function $v = u - Vf \in W^{1,r}(\Omega)$ is a solution of the problem $\Delta v = 0$ in Ω, $v = G := g - Vf$ on $\partial\Omega$. So, v is a PWB-solution of this problem by Theorem 7.11.2. Since Ω is a regular set (see Theorems 4.5.7 and 4.5.8) and $G \in \mathcal{C}^0(\partial\Omega)$, we have $v \in \mathcal{C}^0(\overline{\Omega})$.

Corollary 7.11.6 *Let $\Omega \subset R^m$ be a regular bounded open set, $m \geq 2$. Let $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. If $g \in \mathcal{C}^0(\overline{\Omega})$ and $f \in L^q(\Omega)$ with $q > m/2$, then $u \in \mathcal{C}^0(\overline{\Omega})$.*

Proof Define $f = 0$ on $R^m \setminus \Omega$. Then, $Vf \in \mathcal{C}^0(\overline{\Omega}) \cap D^{2,q}(R^m)$ by Theorems 3.13.3, 3.14.5, and 1.21.13. Fix a ball B such that $\overline{\Omega} \subset B$. By virtue of Lemma 1.21.12, Theorem 1.21.13, and Proposition 1.14.1, we get $Vf \in W^{2,m/2}(B) \subset W^{1,m}(B) \subset W^{1,2}(B)$. Since $-\Delta Vf = f$ in the sense of distributions by Theorem 3.1.5, the function $v = u - Vf \in \tilde{W}^{1,2}(\Omega)$ is a solution of the problem $\Delta v = 0$ in Ω, $v = G \equiv g - Vf$ on $\partial\Omega$. So, v is a PWB-solution of this problem by Theorem 7.11.4. Since $G \in \mathcal{C}^0(\partial\Omega)$, we have $v \in \mathcal{C}^0(\overline{\Omega})$.

Corollary 7.11.7 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary. Let $u \in W^{1,2}(\Omega)$ be a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. If $g \in \mathcal{C}^0(\partial\Omega)$ and $f \in L^q(\Omega)$ with $q > m/2$, then $u \in \mathcal{C}^0(\overline{\Omega})$.*

Proof According to Theorem 7.1.2, there exists a solution $h \in W^{1,2}(\Omega)$ of the problem $\Delta h = 0$ in Ω, $h = g$ on $\partial\Omega$. Theorem 7.11.2 gives that h is a PWB-solution of the problem. The set Ω is a regular set by Theorems 4.5.7 and 4.5.8. So, $h \in \mathcal{C}^0(\overline{\Omega})$. Since u is a solution of the problem $-\Delta u = f$ in Ω, $u = h$ on $\partial\Omega$, Corollary 7.11.6 gives $u \in \mathcal{C}^0(\overline{\Omega})$.

7.12 The Maximum Principle in $B_s^{p,q}(\Omega)$

Proposition 7.12.1 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^{1,\alpha}$*,* $0 < \alpha < 1 < p, q < \infty$*,* $1/p < s < 1 + 1/p$*,* M *be a constant. Suppose that* $u \in W^{s,p}(\Omega)$ *or* $u \in B_s^{p,q}(\Omega)$*. If* $\Delta u = 0$ *in* Ω*,* $u \le M$ *on* $\partial\Omega$ *(in the sense of a trace), then* $u \le M$*.*

Proof Put $g = u$ on $\partial\Omega$. Then $g \le M$ almost everywhere on $\partial\Omega$. Theorem 7.11.2 gives that u is a PWB-solution of the Dirichlet problem with the boundary condition $g = u$. If $x \in \Omega$ then

$$u(x) = \int_{\partial\Omega} g \, \mathrm{d}\mu_x^{\Omega}, \tag{7.24}$$

where μ_x^{Ω} is the harmonic measure relative to Ω and x (see Corollary 4.4.6). Since $\mu_x^{\Omega}(\partial\Omega) = 1$ by Theorem 4.4.1 and μ_x^{Ω} is absolutely continuous with respect to the surface measure on $\partial\Omega$ (see Theorem 5.16.2), (7.24) gives $u(x) \le M$.

7.13 The Dirichlet Problem in $D^{1,p}(\Omega)$ on Exterior Domains

In this section, we prove the existence of a solution of the Dirichlet problem for the Poisson equation in $D^{1,p}(\Omega)$ for an exterior domain with Ljapunov boundary and characterize the set of all solutions.

Let $\Omega \subset R^m$ be an unbounded domain with compact nonempty boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $1 < p < \infty$. Recall that $D^{1,p}(\Omega)$ consists of those functions in $L^p_{loc}(\Omega)$ for which all generalized derivatives of the order 1 are in $L^p(\Omega)$. Denote by $\mathring{D}^{1,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $D^{1,p}(\Omega)$. Denote by $D^{-1,p}(\Omega)$ the dual space of $\mathring{D}^{1,p'}(\Omega)$ where $p' = p/(p-1)$. Denote $\tilde{W}^{1,p}(\Omega) = \{u|\Omega; u \in \mathring{D}^{1,p}(R^m)\}$. Remark that $u \in \mathring{D}^{1,p}(\Omega)$ if and only if $u \in \tilde{W}^{1,p}(\Omega)$ and the trace $\gamma_\Omega u = 0$. If $p \ge m$, then $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$. If $p < m$, then $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega) \oplus \{c; c \in R^1\}$. (See Proposition 3.38.4.)

We say that $u \in D^{1,p}(\Omega)$ is a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$ if $-\Delta u = f$ in Ω in the sense of distributions and the trace $\gamma_\Omega u = g$.

Lemma 7.13.1 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact nonempty boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*, and* $1 < p < \infty$*. If* $u \in D^{1,p}(\Omega)$ *is a solution of the Dirichlet problem* $-\Delta u = f$ *in* Ω*,* $u = g$ *on* $\partial\Omega$*, then* $f \in D^{-1,p}(\Omega)$*,* $g \in W^{1-1/p,p}(\partial\Omega)$*.*

Proof $f \in D^{-1,p}(\Omega)$ by Theorem 3.19.3. If $\partial\Omega \subset B(0; r)$, then $u \in W^{1,p}(\Omega \cap B(0; r))$ by Proposition 1.25.2. Thus, $\gamma_\Omega u \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2.

Lemma 7.13.2 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact nonempty boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*,* $m > 2$*, and* $1 < p < \infty$*. If* $u \in D^{1,p}(\Omega)$

is a solution of the Dirichlet problem $-\Delta u = 0$ *in* Ω, $u = g$ *on* $\partial\Omega$ *then* u *is an* L^p*-solution of this problem.*

Proof Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $g = u$ on $\partial B(0;r)$. Then $u \in W^{1,p}(\Omega \cap B(0;r))$ is a solution of the problem $-\Delta u = 0$ in $\Omega \cap B(0;r)$, $u = g$ on $\partial[\Omega \cap B(0;r)]$. Theorem 7.11.2 gives that u is an L^p-solution of this problem.

Since $|\nabla u| \in L^p(\Omega)$, Proposition 2.17.5 gives that u is bounded in a neighborhood of infinity. Thus, u is an L^p-solution of the problem $-\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$.

Theorem 7.13.3 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact nonempty boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$, $m > 2$, *and* $1 < p < \infty$. *Denote by* u_0 *the* L^p*-solution of the Dirichlet problem* $\Delta u_0 = 0$ *in* Ω, $u_0 = 0$ *on* $\partial\Omega$, $u_0(x) \to 1$ *as* $|x| \to \infty$. *Fix* $f \in D^{-1,p}(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$.

- *If* $p \le m/(m-1)$, *then there exists a unique solution* $u \in D^{1,p}(\Omega)$ *of the problem*

  $$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{7.25}$$

  Moreover,

  $$\|u\|_{D^{1,p}(\Omega)} \le C[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{D^{-1,p}(\Omega)}] \tag{7.26}$$

  where C *depends only on* p *and* Ω. *The problem (7.25) with* $f = 0$, $g = 1$ *does not have a solution in* $\tilde{W}^{1,p}(\Omega)$.
- *If* $m/(m-1) < p < m$, *then there exists a unique solution* $u \in \tilde{W}^{1,p}(\Omega)$ *of the problem (7.25). Moreover,*

  $$\|u\|_{\tilde{W}^{1,p}(\Omega)} \le C[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{D^{-1,p}(\Omega)}] \tag{7.27}$$

  where C *depends only on* p *and* Ω. *The set of all solutions of the problem (7.25) in* $D^{1,p}(\Omega)$ *is* $\{u + cu_0; c \in R^1\}$. *If* $v \in D^{1,p}(\Omega)$ *is a solution of the problem (7.25), then there exists* $c \in R^1$ *such that*

  $$\lim_{r\to\infty} \oint_{\partial B(0;r)} v \, d\mathcal{H}_{m-1} = c. \tag{7.28}$$

  On the other hand, if $c \in R^1$ *is given, then* $v = u + cu_0$ *is a unique solution* $v \in D^{1,p}(\Omega)$ *of the problem (7.25) such that (7.28) holds. Moreover,*

  $$\|v\|_{D^{1,p}(\Omega)} \le C[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{D^{-1,p}(\Omega)} + |c|] \tag{7.29}$$

  where C *depends only on* p *and* Ω.
- *If* $p \ge m$, *then there exists a solution* $u \in \tilde{W}^{1,p}(\Omega) = D^{1,p}(\Omega)$ *of the problem (7.25). The set of all solutions of the problem (7.25) in* $D^{1,p}(\Omega)$ *is* $\{u + cu_0; c \in R^1\}$.

Proof According to Lemma 3.38.12, there exists $w \in \tilde{W}^{1,p}(\Omega)$ such that $-\Delta w = f$ in Ω. Denote $\tilde{g} = g - w$ on $\partial\Omega$. Then, $\tilde{g} \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2. If $u = v + w$, then u is a solution of the problem (7.25) in $D^{1,p}(\Omega)$ (or in $\tilde{W}^{1,p}(\Omega)$) if and only if v is a solution of the problem

$$-\Delta v = 0 \quad \text{in } \Omega, \qquad v = \tilde{g} \quad \text{on } \partial\Omega \tag{7.30}$$

in $D^{1,p}(\Omega)$ (or in $\tilde{W}^{1,p}(\Omega)$).

According to Theorem 7.6.3, there exists $\varphi \in W^{-1/p,p}(\partial\Omega)$ with $\langle\varphi, 1\rangle = 0$ and a constant c such that $\mathcal{S}^{\Omega}\varphi + c = \tilde{g}$ on $\partial\Omega$. Denote $v = \mathcal{S}^{\Omega}\varphi + c$ in Ω. Then, $v \in W^{1,p}(\Omega \cap B(0;r))$ for every $r > 0$ by Proposition 7.6.1, and $\tilde{g}$ is the trace of v by Proposition 7.6.2. Since $\langle\varphi, 1\rangle = 0$, we have $|\nabla v(x)| = O(|x|^{-m})$ as $|x| \to \infty$. Thus, $v \in D^{1,p}(\Omega)$. Since $\Delta v = 0$ in Ω in the sense of distributions (see Theorem 3.1.5), v is a solution of (7.30) in $D^{1,p}(\Omega)$.

According to Theorem 7.6.3, there exists $\varphi_0 \in L^p(\partial\Omega) \cap L^2(\partial\Omega)$ such that $\mathcal{S}^{\Omega}\varphi_0 = 1$ on $\partial\Omega$. Put $u_0 = 1 - \mathcal{S}^{\Omega}\varphi_0$. Then, u_0 is a unique L^p-solution of the problem $\Delta u_0 = 0$ in Ω, $u_0 = 0$ on $\partial\Omega$, $u_0(x) \to 1$ as $|x| \to \infty$. (See Proposition 5.2.3 and Theorem 5.15.4.) Since $M_a(\nabla u_0) \in L^p(\partial\Omega)$ (see Proposition 5.2.8), Lemma 1.28.1 gives that $u_0 \in W^{1,p}(B(0;r))$ for all $r > 0$. Since $|\nabla u_0(x)| = O(|x|^{1-m})$ as $|x| \to \infty$, we have $u_0 \in D^{1,p}(\Omega)$ for $p > m/(m-1)$.

Let now $w \in D^{1,p}(\Omega)$ be a solution of the problem (7.30). Then $v - w$ is an L^p-solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ by Lemma 7.13.2. According to Theorem 5.15.4, there exists $c \in R^1$ such that $v - w = cu_0$. If $p > m/(m-1)$, then $\{cu_0; c \in R^1\}$ is the set of all solutions of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ in $D^{1,p}(\Omega)$.

Let now $p \le m/(m-1)$. We show that $u_0 \notin D^{1,p}(\Omega)$. Clearly, $u_0 \in D^{1,p}(\Omega)$ if and only if $\mathcal{S}^{\Omega}\varphi_0 \in D^{1,p}(R^m)$. Since $\mathcal{S}^{\Omega}\varphi_0(x) \to 0$ as $|x| \to \infty$, Theorem 3.38.5 gives that $\mathcal{S}^{\Omega}\varphi_0 \in D^{1,p}(R^m)$ if and only if $\mathcal{S}^{\Omega}\varphi_0 \in \mathring{D}^{1,p}(R^m)$. If $\int_{\partial\Omega} \varphi_0 \, \mathrm{d}\mathcal{H}_{m-1} = 0$, then

$$0 = \int_{\partial\Omega} \varphi_0 \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\partial\Omega} \varphi_0 \mathcal{S}^{\Omega}\varphi_0 \, \mathrm{d}\mathcal{H}_{m-1}.$$

Lemma 5.9.3 gives $\varphi_0 \equiv 0$, what is a contradiction. Thus, $\int_{\partial\Omega} \varphi_0 \, \mathrm{d}\mathcal{H}_{m-1} \neq 0$. Hence, there exist positive constants a and r such that $|\mathcal{S}^{\Omega}\varphi_0(x)| \ge a|x|^{2-m}$ for $|x| > r$. Fix $z \in R^m \setminus \Omega$. According to Proposition 2.13.1, there exist $u_1 \in H(R^m)$, $u_2 \in H(\Omega)$ and a constant b such that $\mathcal{S}^{\Omega}\varphi_0(x) = u_1(x) + u_2(x) + bh_m(x-z)$ in Ω and $u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$. Since $u_1(x) = \mathcal{S}^{\Omega}\varphi_0(x) - u_2(x) - bh_m(x-z) \to 0$ as $|x| \to \infty$, Theorem 2.16.1 gives that $u_1 \equiv 0$. We have $b \neq 0$ because $|\mathcal{S}^{\Omega}\varphi_0(x)| \ge a|x|^{2-m}$ for $|x| > r$. Since $u_2(x) = O(|x|^{1-m})$ as $|x| \to \infty$, we have $|\nabla u_2(x)| = O(|x|^{-m})$ as $|x| \to \infty$ by Proposition 2.12.1. Therefore, there exist positive constants c_1 and c_2 such that $|\nabla \mathcal{S}^{\Omega}\varphi_0(x)| \ge c_1|\nabla h_m(x-z)| \ge c_2|x|^{1-m}$ for $|x| > r$. So, $\mathcal{S}^{\Omega}\varphi_0 \notin D^{1,p}(\Omega)$ because $p \le m/(m-1)$. Since $u_0 \notin D^{1,p}(\Omega)$, there is a unique solution $v \in D^{1,p}(\Omega)$ of the problem (7.30). Thus, $u \mapsto [-\Delta u, \gamma_{\Omega} u]$ is a one-to-one linear mapping $D^{1,p}(\Omega)$ onto $D^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$. It is bounded by

Theorems 3.19.3 and 1.24.2. So, it is an isomorphism by Theorem 1.6.3. Let now $v \in \tilde{W}^{1,p}(\Omega)$ be a solution of the problem $\Delta v = 0$ in Ω, $v = 1$ on $\partial\Omega$. Then v is an L^p-solution of the problem by Lemma 7.13.2. According to Theorem 5.15.4, there exists a constant v_∞ such that $v(x) \to v_\infty$ as $|x| \to \infty$. Since $v \in \tilde{W}^{1,p}(\Omega)$, we have $v_\infty = 0$ (see Proposition 3.38.5). From the uniqueness of an L^p-solution of the Dirichlet problem, we infer that $v = \mathcal{S}^\Omega \varphi_0$ (see Theorem 5.15.4). But $\mathcal{S}^\Omega \varphi_0 \notin D^{1,p}(\Omega)$, what is a contradiction.

Let now $m/(m-1) < p < m$. If $v \in D^{1,p}(\Omega)$ is a solution of the problem (7.25), then there exists $c \in R^1$ such that (7.28) holds by Corollary 3.38.6. Let now c_∞ be fixed. Then, there exists a solution $v \in D^{1,p}(\Omega)$ of the problem (7.25). Let $c \in R^1$ be such that (7.28) holds. Then $u = v + (c_\infty - c)u_0$ is a unique solution of the problem (7.25) such that

$$\lim_{r\to\infty} \oint\limits_{\partial B(0;r)} u \, \mathrm{d}\mathcal{H}_{m-1} = c_\infty .$$

Moreover, $u \in \tilde{W}^{1,p}(\Omega)$ if and only if $c_\infty = 0$ (see Proposition 3.38.5). Theorems 3.19.3 and 1.24.2 give that $u \mapsto [-\Delta u, \gamma_\Omega u]$ is a bounded one-to-one linear mapping $\tilde{W}^{1,p}(\Omega)$ onto $D^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$. So, it is an isomorphism by Theorem 1.6.3. Since $\tilde{W}^{1,p}(\Omega)$ is a closed subspace of $D^{1,p}(\Omega)$ of codimension 1 by Lemma 3.38.11 and $u_0 \in D^{1,p}(\Omega) \setminus \tilde{W}^{1,p}(\Omega)$, we have $D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega) \oplus \{cu_0; c \in R^1\}$. Thus,

$$u \mapsto \left[-\Delta u, \gamma_\Omega u, \lim_{r\to\infty} \oint\limits_{\partial B(0;r)} v \, \mathrm{d}\mathcal{H}_{m-1} \right]$$

is a bounded one-to-one mapping $D^{1,p}(\Omega)$ onto $D^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega) \times R^1$. So, it is an isomorphism by Theorem 1.6.3.

Lemma 7.13.4 *Let* $\Omega \subset R^2$ *be an unbounded domain with nonempty compact boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$, *and* $1 < p < \infty$. *If* $g \in W^{1-1/p,p}(\partial\Omega)$, *then there exists a unique* L^p*-solution* u *of the problem* $\Delta u = 0$ *in* Ω, $u = g$ *on* $\partial\Omega$. *Moreover,* u *is a solution of this problem in* $D^{1,p}(\Omega)$.

Proof According to Theorem 7.6.3, there exist $\varphi \in W^{-1/p,p}(\partial\Omega)$ with $\langle \varphi, 1 \rangle = 0$ and a constant c such that $\mathcal{S}^\Omega \varphi + c = g$ on $\partial\Omega$. Denote $u = \mathcal{S}^\Omega \varphi + c$ in Ω. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Then $u \in W^{1,p}(\Omega \cap B(0;r))$ by Proposition 7.6.1, and g is the trace of u by Proposition 7.6.2. Define $g = u$ on $\partial(\Omega \cap B(0;r))$. Then $u \in W^{1,p}(\Omega \cap B(0;r))$ is a solution of $\Delta u = 0$ in $\Omega \cap B(0;r)$, $u = g$ on $\partial(\Omega \cap B(0;r))$. Theorem 7.11.2 gives that u is an L^p-solution of this problem. Since u is bounded on $R^2 \setminus B(0;r)$, the function u is an L^p-solution of the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$. This solution is unique by Theorem 5.15.3. Since $\langle \varphi, 1 \rangle = 0$, we have $|\nabla u(x)| = O(|x|^{-2})$ as $|x| \to \infty$. Thus, $u \in D^{1,p}(\Omega)$.

Theorem 7.13.5 *Let* $\Omega \subset R^2$ *be an unbounded domain with nonempty compact boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*, and* $1 < p < \infty$*. Fix* $z \in R^2 \setminus \overline{\Omega}$*. Denote by* u_0 *the* L^p*-solution of the Dirichlet problem* $\Delta u = 0$ *in* Ω*,* $u(x) = h_2(x-z)$ *on* $\partial\Omega$*. Put* $u_1(x) = h_2(x-z) - u_0$*. Let* $f \in D^{-1,p}(\Omega)$*,* $g \in W^{1-1/p,p}(\partial\Omega)$*.*

- *If* $1 < p \leq 2$*, then there exists a unique solution* $u \in D^{1,p}(\Omega)$ *of the problem (7.25). Moreover, the estimate (7.26) holds with C depending only on p and* Ω*.*
- *Let* $2 < p < \infty$*. Then, there exists a solution* $u \in D^{1,p}(\Omega)$ *of the problem (7.25). The set of all solutions of the problem (7.25) in* $D^{1,p}(\Omega)$ *is* $\{u + au_1; a \in R^1\}$*.*

Proof According to Lemma 3.38.12, there exists $w \in D^{1,p}(\Omega)$ such that $-\Delta w = f$ in Ω. Denote $\tilde{g} = g - w$ on $\partial\Omega$. Then, $\tilde{g} \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2. Lemma 7.13.4 gives that there exists a solution $v \in D^{1,p}(\Omega)$ of the problem $\Delta v = 0$ in Ω, $v = \tilde{g}$ on $\partial\Omega$. Put $u = v + w$. Then u is a solution of the problem (7.25) in $D^{1,p}(\Omega)$.

According to Lemma 7.13.4, there exists a unique L^p-solution u_0 of the problem $\Delta u_0 = 0$ in Ω, $u_0(x) = h_2(x-z)$ on $\partial\Omega$, and $u_0 \in D^{1,p}(\Omega)$. Since $c_1|x-z|^{-1} \leq |\nabla h_2(x-z)| \leq c_2|x-z|^{-1}$ for some positive constants c_1 and c_2, the function $u_1 \in D^{1,p}(\Omega)$ if and only if $p > 2$. Moreover, $\Delta u_1 = 0$ in Ω, $u_1 = 0$ on $\partial\Omega$.

Let now $u \in D^{1,p}(\Omega)$ be a solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. Since $|\nabla u| \in L^p(\Omega)$, Proposition 2.17.6 gives that there exists $a \in R^1$ such that the function $v(x) = u(x) - ah_2(x-z)$ is bounded in a neighborhood of infinity. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $g = v$ on $\partial B(0;r)$, $g(x) = -ah_2(x-z)$ on $\partial\Omega$. Since $v(x) = u(x) - ah_2(x-z) = -ah_2(x-z)$ on $\partial\Omega$, $v \in W^{1,p}(\Omega \cap B(0;r))$ is a solution of the problem $\Delta v = 0$ in $\Omega \cap B(0;r)$, $v = g$ on $\partial[\Omega \cap B(0;r)]$. Theorem 7.11.2 gives that v is an L^p-solution of this problem. Since v is bounded in a neighborhood of infinity, v is an L^p-solution of the problem $\Delta v = 0$ in Ω, $v(x) = -ah_2(x-z)$ on $\partial\Omega$. Thus, $v = -au_0 \in D^{1,p}(\Omega)$ by Lemma 7.13.4. Hence, $u = au_1$. If $p > 2$, then $\{au_1; a \in R^1\}$ is the set of all solutions of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$ in $D^{1,p}(\Omega)$. Let now $p \leq 2$. Since $u = au_1 \in D^{1,p}(\Omega)$ but $u_1 \notin D^{1,p}(\Omega)$, we infer that $a = 0$. Therefore, $u = 0$.

Let $p \leq 2$. Then, $u \mapsto [-\Delta u, \gamma_\Omega u]$ is a one-to-one linear mapping $D^{1,p}(\Omega)$ onto $D^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$. It is bounded by Theorems 3.19.3 and 1.24.2. So, it is an isomorphism by Theorem 1.6.3.

7.14 The Dirichlet Problem in $W^{2+k,p}(\Omega)$

In this section, we shall study a solution of the Dirichlet problem in the space $W^{2+k,p}(\Omega) = D^{2+k,p}(\Omega)$ where $k \in N_0$ and Ω is a bounded domain with boundary of class $\mathcal{C}^{k+2}$.

Lemma 7.14.1 *Let* $1 < p < \infty$, $v \in L^p(R^m_+)$. *Fix* $k \in \{1, \dots, m-1\}$. *For a nonzero real number* δ, *denote*

$$P^\delta_k v(x) = v(x_1, x_2, \dots, x_{k-1}, x_k + \delta, x_{k+1}, \dots, x_m), \quad D^\delta_k v = \frac{P^\delta_k v - v}{\delta}.$$

If there exist positive constants C *and* r *such that*

$$\|D^\delta_k v\|_{L^p(R^m_+)} \leq C, \quad 0 < \delta < r,$$

then $\partial_k v \in L^p(R^m_+)$.

Proof According to Theorem 1.5.9, there exists a sequence $\delta(n) \downarrow 0$ and $w \in L^p(R^m_+)$ such that $D^{\delta(n)}_k v$ converges weakly to w in $L^p(R^m_+)$. Fix now $\varphi \in \mathcal{C}^\infty_c(R^m_+)$. By the substitution,

$$\int\limits_{R^m_+} \varphi D^{\delta(n)}_k v \, \mathrm{d}\mathcal{H}_m = - \int\limits_{R^m_+} (D^{-\delta(n)}_k \varphi) v \, \mathrm{d}\mathcal{H}_m.$$

Clearly, $D^{-\delta(n)}_k \varphi \to \partial_k \varphi$ as $n \to \infty$. Fix c such that $|\nabla\varphi| \leq c$. Then for each x, there exists z such that $D^{-\delta(n)}_k \varphi(x) = \partial_k \varphi(z)$. Hence, $|D^{-\delta(n)}_k \varphi| \leq c$. Since $D^{-\delta(n)}_k \varphi$ are supported in the same compact we obtain by virtue of Lebesgue lemma (Lemma 1.13.2),

$$\int\limits_{R^m_+} w\varphi \, \mathrm{d}\mathcal{H}_m = - \lim_{n\to\infty} \int\limits_{R^m_+} (D^{-\delta(n)}_k \varphi) v \, \mathrm{d}\mathcal{H}_m = - \int\limits_{R^m_+} (\partial_k \varphi) v \, \mathrm{d}\mathcal{H}_m.$$

This means that $\partial_k v = w \in L^p(R^m_+)$.

Proposition 7.14.2 *Let* $\Omega \subset R^m$ *be a bounded open set with boundary of class* $\mathcal{C}^{2+k}$, $m \geq 2$, $k \in N_0$, $1 < p < \infty$. *If* $u \in \overset{\circ}{W}{}^{1,p}(\Omega)$, $\Delta u = f$ *in* Ω, $f \in W^{k,p}(\Omega)$, *then* $u \in W^{2+k,p}(\Omega)$.

Proof We prove the proposition by the induction argument. Suppose that $k = 0$ or that the proposition is true for $k - 1$. Proposition 3.18.2 gives $u \in W^{2+k,p}_{loc}(\Omega)$. So, it is enough to prove that u is of class $W^{2+k,p}$ on a neighborhood of each point of the boundary. Fix $z \in \partial\Omega$. Then, there exist a coordinate system, $\Phi \in \mathcal{C}^{2+k,0}(R^{m-1})$ and $\rho > 0$ such that $z = 0$ and $\Omega \cap B(0; \rho) = B(0; \rho) \cap \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x')\}$. We can moreover suppose that $\nabla\Phi(0) = (0, \dots, 0)$.

Set $p' = p/(p - 1)$. Denote

$$\mathcal{A} = \left\{ \varphi \in \mathcal{C}^\infty_c(R^m_+); \|\nabla\varphi\|_{L^{p'}(R^m_+)} \leq 1 \right\}.$$

According to Proposition 7.5.2, there exists a constant $C_1 \geq 1$ such that

$$\|\nabla v\|_{L^p(R^m_+)} \leq C_1 \sup_{\varphi \in \mathcal{A}} \int\limits_{R^m_+} \nabla v \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m \qquad \forall v \in \mathring{W}^{1,p}(R^m_+). \tag{7.31}$$

Let $C_2 \geq 1$ be a constant such that $|\nabla \Phi| + |\nabla^2 \Phi| \leq C_2$. Fix $r \in (0; \rho/4)$ such that $|\Phi| \leq \rho$ and $|\nabla \Phi| \leq (4mC_1)^{-1}$ for $|x'| < 4r$. Choose $\psi \in \mathcal{C}^\infty_c(B(0; 2r))$ such that $\psi = 1$ on $B(0; r)$. Put $\tilde{u} = \psi u$. Then, $\tilde{u} \in \mathring{W}^{1,p}(B(0; 2r) \cap \Omega)$, $\operatorname{spt} \tilde{u} \subset B(0; \rho/2)$, $g = \Delta \tilde{u} = \psi \Delta u + 2\nabla \psi \cdot \nabla u + u \Delta \psi \in W^{k,p}(\Omega)$. Denote $w(x', x_m) = \tilde{u}(x', x_m + \Phi(x'))$. Then, $w(x', x_m) = 0$ for $x_m \leq 0$. If $0 \neq w(x) = \tilde{u}(x', x_m - \Phi(x'))$, then $|[x', x_m - \Phi(x')]| < \rho/2$ and $|\Phi(x')| < \rho$. Thus, w is supported in $B(0; 2\rho)$. Proposition 1.21.14 gives that $w \in W^{1,p}(R^m)$. According to Proposition 1.21.15, it is enough to prove that $w \in W^{2+k,p}(R^m)$. Since $\tilde{u} \in W^{2+k,p}_{loc}(B(0; 2r) \cap \Omega)$, we have $w \in W^{2+k,p}_{loc}(R^m_+)$. Since $\tilde{u}(x', x_m) = w(x', x_m - \Phi(x'))$, easy calculation yields

$$g(x) = \Delta w(\tilde{x}) + |\nabla \Phi(x')|^2 \frac{\partial^2 w(\tilde{x})}{\partial x_m^2} - \sum_{j=1}^{m-1} 2(\partial_j \Phi(x')) \frac{\partial^2 w(\tilde{x})}{\partial x_j \partial x_m} - \partial_m w(\tilde{x}) \Delta \Phi(x')$$

where $\tilde{x} = [x', x_m - \Phi(x')]$. Define

$$h := -\Delta w - |\nabla \Phi|^2 \frac{\partial^2 w}{\partial x_m^2} + \sum_{j=1}^{m-1} 2(\partial_j \Phi) \frac{\partial^2 w}{\partial x_m \partial x_j}. \tag{7.32}$$

Then,

$$h(\tilde{x}) = -g(x) - \partial_m w(\tilde{x}) \Delta \Phi(x'). \tag{7.33}$$

Since $\partial_m w \in L^p(R^m)$, we have $h \in L^p(R^m_+)$. We now show that $h \in W^{k,p}(R^m_+)$ forces $w \in W^{2+k,p}(R^m)$. Suppose first that $k = 0$.

We show

$$\int\limits_{R^m_+} \{\nabla w \cdot \nabla \varphi + (\partial_m \varphi)[|\nabla \Phi|^2 (\partial_m w) - 2\nabla \Phi \cdot \nabla w]\} = \int\limits_{R^m_+} h\varphi \tag{7.34}$$

for $\varphi \in \mathcal{C}^\infty_c(R^m_+)$. Theorem 3.19.3 gives

$$\int\limits_{R^m_+} \nabla w \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m_+} (-\Delta w) \varphi \, \mathrm{d}\mathcal{H}_m.$$

Since Φ does not depend on x_m, we obtain by integration per partes with respect to x_m

$$\int\limits_{R^m_+} (\partial_m\varphi)[|\nabla\Phi|^2(\partial_m w) - 2\nabla\Phi\cdot\nabla w]\,\mathrm{d}\mathcal{H}_m$$

$$= \int\limits_{R^m_+} \varphi\Big[-|\nabla\Phi|^2\frac{\partial^2 w}{\partial x_m^2} + 2\sum_{j=1}^{m-1}(\partial_j\Phi)\frac{\partial^2 w}{\partial x_m\partial x_j}\Big]\,\mathrm{d}\mathcal{H}_m.$$

Adding, we get (7.34).

Fix $n\in\{1,\dots,m-1\}$. For a nonzero real number δ, denote

$$P_n^\delta\varphi(x) = \varphi(x_1,x_2,\dots,x_{n-1},x_n+\delta,x_{n+1},\dots,x_m),\quad D_n^\delta\varphi = \frac{P_n^\delta\varphi-\varphi}{\delta}.$$

According to Theorem 1.21.2,

$$\|D_n^\delta\varphi\|_{L^{p'}(R^m)} \le \|\nabla\varphi\|_{L^{p'}(R^m)}$$

for all n, δ and $\varphi\in\mathcal{C}_c^\infty(R^m)$. If $\varphi\in\mathcal{C}_c^\infty(R^m_+)$, then $D_n^{-\delta}\varphi\in\mathcal{C}_c^\infty(R^m_+)$, and according to Hölder's inequality (Proposition 1.14.1),

$$-\int\limits_{R^m_+}\{\nabla w\cdot\nabla D_n^{-\delta}\varphi + (\partial_m D_n^{-\delta}\varphi)[|\nabla\Phi|^2\partial_m w - 2\nabla\Phi\cdot\nabla w]\}\,\mathrm{d}\mathcal{H}_m$$

$$\overset{(7.34)}{=} -\int\limits_{R^m_+} hD_n^{-\delta}\varphi\,\mathrm{d}\mathcal{H}_m \le \|h\|_{L^p(R^m_+)}\|D_n^{-\delta}\varphi\|_{L^{p'}(R^m_+)} \le \|h\|_{L^p}\|\nabla\varphi\|_{L^{p'}}.$$

So, if $\varphi\in\mathcal{A}$, we get by the substitution

$$\|h\|_{L^p(R^m_+)} \ge \int\limits_{R^m_+}\{\nabla D_n^\delta w\cdot\nabla\varphi + |(\partial_m\varphi)D_n^\delta[|\nabla\Phi|^2\partial_m w - 2\nabla w\cdot\nabla\Phi]\}.$$

Using this relation, (7.31) and Hölder's inequality (Proposition 1.14.1), we get

$$\|\nabla D_n^\delta w\|_{L^p} \le C_1\sup_{\varphi\in\mathcal{A}}\int\limits_{R^m_+}\nabla D_n^\delta w\cdot\nabla\varphi\,\mathrm{d}\mathcal{H}_m$$

$$\le C_1\left\{\|h\|_{L^p} + \|D_n^\delta[|\nabla\Phi|^2\partial_m w]\|_{L^p} + \|2D_n^\delta[\nabla w\cdot\nabla\Phi]\|_{L^p}\right\}.$$

If α and β are functions, then $D_n^\delta(\alpha\beta) = (D_n^\delta\alpha)P_n^\delta\beta + \alpha D_n^\delta\beta$. Since for each x and j there exists z such that $D_n^\delta\partial_j\Phi(x) = \partial_n\partial_j\Phi(z)$, we have $|D_n^\delta\nabla\Phi| \le C_2$. Similarly, $D_n^\delta(\partial_j\Phi(x'))^2 = [P_n^\delta\partial_j\Phi(x)]\partial_n\partial_j\Phi(z') + \partial_j\Phi(x)\partial_n\partial_j\Phi(z')$ and thus $|D_n^\delta|\nabla\Phi(x')|^2| \le 2mC_2^2$. Therefore for $0 < |\delta| < r$, we have $|D_n^\delta[\nabla w \cdot \nabla\Phi]| \le \sum_j [|D_n^\delta\partial_j w|\,|P_n^\delta\nabla\Phi| + |\partial_j w||D_n^\delta\Phi|] \le (4C_1)^{-1}|\nabla D_n^\delta w| + mC_2|\nabla w|$; and $|D_n^\delta[|\nabla\Phi|^2\partial_m w]| \le |D_n^\delta\partial_m w||P_n^\delta|\nabla\Phi|^2| + |\partial_m w||D_n^\delta|\nabla\Phi|^2| \le (4C_1)^{-2}|\nabla D_n^\delta w| + 2mC_2^2|\partial_m w|$. Hence,

$$\|\nabla D_n^\delta w\|_{L^p(R_+^m)} \le C_1\|h\|_{L^p(R_+^m)} + \frac{1}{4}\|\nabla D_n^\delta w\|_{L^p(R_+^m)} + 2mC_1C_2^2\|\partial_m w\|_{L^p(R_+^m)}$$
$$+\frac{2}{4}\|\nabla D_n^\delta w\|_{L^p(R_+^m)} + 2mC_1C_2\|\nabla w\|_{L^p(R_+^m)}.$$

Subtracting $(3/4)\|\nabla D_n^\delta w\|_{L^p}$ from both sides we obtain

$$\frac{1}{4}\|D_n^\delta\nabla w\|_{L^p(R_+^m)} \le C_1\|h\|_{L^p(R_+^m)} + 4mC_1C_2^2\|\nabla w\|_{L^p(R_+^m)} < \infty.$$

If $j \in \{1, \dots, m\}$, then $\{D_n^\delta\partial_j w; 0 < \delta < r\}$ is a bounded subset of $L^p(R_+^m)$. So, $\partial_n\partial_j w \in L^p(R_+^m)$ by Lemma 7.14.1. We have $\partial_j\partial_n w = \partial_n\partial_j w \in L^p(R_+^m)$. To finish, we must prove that $\partial^2 w/\partial x_m^2 \in L^p(R_+^m)$. According to (7.32), we have

$$\frac{\partial^2 w}{\partial x_m^2} = (1 + |\nabla\Phi|^2)^{-1}\left\{\sum_{j=1}^{m-1}\left[2(\partial_j\Phi)\frac{\partial^2 w}{\partial x_m\partial x_j} - \frac{\partial^2 w}{\partial x_j^2}\right] - h\right\} \in L^p(R_+^m).$$

Let now $k = l + 1$, $l \in N_0$, and $h \in W^{l,p}(R_+^m)$ force $w \in W^{2+l,p}(R^m)$. Suppose that $h \in W^{k,p}(R_+^m)$. Fix $n \in \{1, \dots, m-1\}$. We know that $w \in W^{2+l,p}(R_+^m)$, $w = 0$ on ∂R_+^m. Thus, $\partial_n w \in W^{1+l,p}(R_+^m)$ and $\partial_n w = 0$ on ∂R_+^m. Define

$$H_n := -\Delta(\partial_n w) - |\nabla\Phi|^2\frac{\partial^2}{\partial x_m^2}(\partial_n w) + \sum_{j=1}^{m-1} 2(\partial_j\Phi)\frac{\partial^2}{\partial x_m\partial x_j}(\partial_n w)$$

for $\partial_n w$ by the same way like h for w. Using ∂_n onto (7.32), we get

$$\partial_n h = -\Delta(\partial_n w) - |\nabla\Phi|^2\frac{\partial^2}{\partial x_m^2}(\partial_n w) - \frac{\partial^2 w}{\partial x_m^2}\partial_n|\nabla\Phi|^2$$
$$+\sum_{j=1}^{m-1} 2(\partial_j\Phi)\frac{\partial^2}{\partial x_m\partial x_j}(\partial_n w) + \sum_{j=1}^{m-1} 2(\partial_n\partial_j\Phi)\frac{\partial^2 w}{\partial x_m\partial x_j}.$$

Hence,

$$H_n = \partial_n h + \frac{\partial^2 w}{\partial x_m^2}\partial_n|\nabla\Phi|^2 - \sum_{j=1}^{m-1} 2(\partial_n\partial_j\Phi)\frac{\partial^2 w}{\partial x_m \partial x_j} \in W^{l,p}(R^m_+).$$

Since $H_n \in W^{l,p}(R^m_+)$, the induction assumption forces $\partial_n w \in W^{2+l,p}(R^m_+)$. Therefore, if $|\alpha| = k+2$ and $\alpha \neq (0,\dots,0,k+2)$, then $\partial^\alpha w \in L^p(R^m_+)$. Since

$$\frac{\partial^2 w}{\partial x_m^2} = (1+|\nabla\Phi|^2)^{-1}\left\{\sum_{j=1}^{m-1}\left[2(\partial_j\Phi)\frac{\partial^2 w}{\partial x_m\partial x_j} - \frac{\partial^2 w}{\partial x_j^2}\right] - h\right\} \in W^{k,p}(R^m_+)$$

we deduce that $w \in W^{2+k,p}(R^m_+)$.

We are now able to finish the proof. We have proved the proposition for $k = 0$. Suppose that the proposition holds for $k = l \in N_0$. Let $k = l+1$. By the induction assumption $w \in W^{k+1,p}(R^m_+)$. Then (7.32) gives that $h \in W^{k,p}(R^m_+)$. We have proved that $w \in W^{2+k,p}(R^m_+)$.

Theorem 7.14.3 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{2+k}$, $m \geq 2$, $k \in N_0$, $1 < p < \infty$, $c \in \mathcal{C}^{k,0}(\Omega)$, $c \geq 0$. Then there exists a solution $u \in W^{k+2,p}(\Omega)$ of the Dirichlet problem $-\Delta u + cu = f$ in Ω, $u = g$ on $\partial\Omega$ if and only if $f \in W^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$. This solution is unique and*

$$\|u\|_{W^{k+2,p}(\Omega)} \leq C\left[\|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+2-1/p,p}(\partial\Omega)}\right]$$

where C depends only on Ω, k, p, and c.

Proof If u is a solution of the problem, then $f \in W^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$ by Theorem 1.24.5.

The uniqueness is given by Theorem 7.10.4.

Suppose first that $c \equiv 0$. According to Theorem 1.24.6, there exists $v \in W^{k+2,p}(\Omega)$ such that $v = g$ on $\partial\Omega$. Denote $\tilde{f} = f + \Delta v$. Then, $\tilde{f} \in W^{k,p}(\Omega)$. According to Theorem 7.10.2, there exists $w \in \overset{\circ}{W}{}^{1,p}(\Omega)$ such that $-\Delta w = \tilde{f}$ in Ω. Proposition 7.14.2 gives that $w \in W^{2+k,p}(\Omega)$. The function $u = v + w$ is a solution of the problem.

Let now c be arbitrary. We have proved that $Tu = [-\Delta u, \gamma_\Omega u]$ is a Fredholm operator with index 0 from $W^{2+k,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{2+k-1/p,p}(\partial\Omega)$. Since $u \mapsto cu$ is a compact linear operator from $W^{2+k,p}(\Omega)$ to $W^{k,p}(\Omega)$ by Lemma 1.21.8 and Theorem 1.7.4, the operator $T_c : u \mapsto [-\Delta u + cu, \gamma_\Omega u]$ is a Fredholm operator with index 0 from $W^{2+k,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{2+k-1/p,p}(\partial\Omega)$ (see Theorem 1.8.3). Since T_c is one-to-one, it is onto. Theorem 1.6.3 forces that T_c is isomorphism.

Theorem 7.14.4 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^{2+k}$, $m \geq 2$, $k \in N_0$, $1 < p, q < \infty$, $c \in \mathcal{C}^{k,0}(\Omega)$, $c \geq 0$, $1 < s < k+2$. Then there exists a solution $u \in B^{p,q}_s(\Omega)$ of the Dirichlet problem $-\Delta u + cu = f$ in Ω, $u = g$ on $\partial\Omega$*

if and only if $f \in B^{p,q}_{s-2}(\Omega)$, $g \in B^{p,q}_{s-1/p}(\partial\Omega)$. *This solution is unique and*

$$\|u\|_{B^{p,q}_s(\Omega)} \le C\left[\|f\|_{B^{p,q}_{s-2}(\Omega)} + \|g\|_{B^{p,q}_{s-1/p}(\partial\Omega)}\right]$$

where C *depends only on* Ω, k, p, q, s, *and* c.

Proof The operator

$$T : u \mapsto [-\Delta u + cu, \gamma_\Omega u]$$

is an isomorphism from $W^{2+k,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{2+k-1/p,p}(\partial\Omega)$ (see Theorem 7.14.3) and from $W^{1,p}(\Omega)$ to $W^{-1,p}(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ (see Theorems 7.10.4 and 1.24.2). Put $\theta = (s-1)/(k+1)$. Then

$$B^{p,q}_{s-1/p}(\partial\Omega) = [W^{1-1/p,p}(\partial\Omega), W^{2+k-1/p,p}(\partial\Omega)]_{\theta,q}$$

by Theorems 1.23.3 and 1.23.11. If l is integer, then $W^{l,p}(\Omega) = F^{p,2}_l(\Omega)$ by Proposition 1.23.13. Theorem 1.23.16 gives

$$[W^{1,p}(\Omega), W^{k+2,p}(\Omega)]_{\theta,q} = B^{p,q}_s(\Omega),$$
$$[W^{-1,p}(\Omega), W^{k,p}(\Omega)]_{\theta,q} = B^{p,q}_{s-2}(\Omega).$$

The interpolation theorem (Proposition 1.10.3) gives that T is a bounded linear operator from $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$. Using the interpolation theorem onto T^{-1}, we obtain that T is an isomorphism from $B^{p,q}_s(\Omega)$ onto $B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1/p}(\partial\Omega)$.

Corollary 7.14.5 *Let* $U, \Omega \subset R^m$ *be open sets,* $m \ge 2$. *Suppose that* $L = U \cap \partial\Omega$ *is a surface of class* $\mathcal{C}^{k+2}$ *with* $k \in N_0$. *Let* $1 < p < \infty$, $u \in W^{1,p}(\Omega)$, $f \in W^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(L)$, $\Delta u = f$ *in* Ω, $u = g$ *on* L. *Then* $u \in W^{k+2,p}_{loc}(\Omega \cup L)$.

Proof $u \in W^{k+2,p}_{loc}(\Omega)$ by Proposition 3.18.2. Let now $z \in L$. Fix $r > 0$ such that $B(z; 3r) \subset U$. Choose $\varphi \in \mathcal{C}^\infty_c(B(z; 2r))$ such that $\varphi = 1$ on $B(z; r)$. We can choose a bounded open set $\omega \subset B(0; 3r) \cap \Omega$ with boundary of class $\mathcal{C}^{k+2}$ such that $B(0; 2r) \cap \partial\omega = B(0; 2r) \cap \partial\Omega \subset L$. We have $\varphi u \in W^{1,p}(\omega)$, $\varphi u \in W^{k+2-1/p,p}(\partial\omega)$, $\Delta(\varphi u) = \varphi f + 2\nabla u \cdot \nabla\varphi + u\Delta\varphi$. We prove by the induction that $\varphi u \in W^{l+2,p}(\omega)$ for $l = 0, 1, \dots, k$. Suppose first that $l = 0$. Since $u \in W^{1,p}(\omega)$, we infer that $\Delta(\varphi u) \in L^p(\omega)$. So, $\varphi u \in W^{2,p}(\omega)$ by Theorems 7.14.3 and 7.10.2. Let now $u \in W^{l+2,p}(\omega)$ and $l + 1 \le k$. Then $\Delta(\varphi u) = \varphi f + 2\nabla u \cdot \nabla\varphi + u\Delta\varphi \in W^{l+1,p}(\omega)$. Hence, $\varphi u \in W^{l+3,p}(\omega)$ by Theorems 7.14.3 and 7.10.2.

Corollary 7.14.6 *Let* $\Omega \subset R^m$ *be a bounded domain with boundary of class* $\mathcal{C}^\infty$. *If* $g \in \mathcal{C}^\infty(\partial\Omega)$, $c, f \in \mathcal{C}^\infty(\overline{\Omega})$, $c \ge 0$, *then there exists a unique classical solution* $u \in \mathcal{C}^\infty(\overline{\Omega})$ *of the Dirichlet problem* $-\Delta u + cu = f$ *in* Ω, $u = g$ *on* $\partial\Omega$.

Proof According to Theorem 7.10.4, there exists a unique solution of this problem in $W^{1,2}(\Omega)$. Theorem 7.14.3 gives that $u \in W^{k,2}(\Omega)$ for all $k \in N$. Therefore, $u \in \mathcal{C}^\infty(\overline{\Omega})$ by Theorem 1.21.13 and Lemma 1.21.12.

Proposition 7.14.7 *Let* $\Omega \subset R^m$ *be a bounded open set with boundary of class* $\mathcal{C}^{2+k}$, $m \geq 2$, $k \in N_0$, $c \in \mathcal{C}^{k,0}(\Omega)$, $c \geq 0$, $1 < p < \infty$, $p' = p/(p-1)$, $f \in W^{k,p}(\Omega)$. *If* $u \in L^1(\Omega)$, *then* u *is a solution of the Dirichlet problem* $-\Delta u + cu = f$ *in* Ω, $u = 0$ *on* $\partial\Omega$ *in* $W^{k+2,p}(\Omega)$ *if and only if*

$$\int_\Omega u[-\Delta\varphi + c\varphi]\, \mathrm{d}\mathcal{H}_m = \int_\Omega f\varphi\, \mathrm{d}\mathcal{H}_m \tag{7.35}$$

for all $\varphi \in \overset{\circ}{W}{}^{1,p'}(\Omega) \cap W^{2+k,p'}(\Omega)$ *with* $[-\Delta\varphi + c\varphi] \in L^\infty(\Omega)$.

Proof Suppose first that u is a solution of the Dirichlet problem $-\Delta u + cu = f$ in Ω, $u = 0$ on $\partial\Omega$ in $W^{k+2,p}(\Omega)$. If $\varphi \in \overset{\circ}{W}{}^{1,p'}(\Omega) \cap W^{2+k,p'}(\Omega)$, then the Gauss-Green theorem for Sobolev spaces (Proposition 1.24.4) yields

$$\int_\Omega u[-\Delta\varphi + c\varphi]\, \mathrm{d}\mathcal{H}_m - \int_\Omega f\varphi\, \mathrm{d}\mathcal{H}_m = \int_\Omega u[-\Delta\varphi + c\varphi]\, \mathrm{d}\mathcal{H}_m$$
$$-\int_\Omega \varphi[-\Delta u + cu]\, \mathrm{d}\mathcal{H}_m = \int_{\partial\Omega} \left[\varphi\frac{\partial u}{\partial n} - u\frac{\partial\varphi}{\partial n}\right] \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Let now (7.35) hold for all $\varphi \in \overset{\circ}{W}{}^{1,p'}(\Omega) \cap W^{2+k,p'}(\Omega)$ with $[-\Delta\varphi + c\varphi] \in L^\infty(\Omega)$. According to Theorem 7.14.3, there exists a solution $v \in W^{k+2,p}(\Omega)$ of the Dirichlet problem $-\Delta v + cv = f$ in Ω, $v = 0$ on $\partial\Omega$. Subtracting (7.35) for u and v, we obtain

$$\int_\Omega (u-v)[-\Delta\varphi + c\varphi]\, \mathrm{d}\mathcal{H}_m = 0$$

for all $\varphi \in \overset{\circ}{W}{}^{1,p'}(\Omega) \cap W^{2+k,p'}(\Omega)$ with $-\Delta\varphi + c\varphi \in L^\infty(\Omega)$. Fix $\psi \in \mathcal{C}_c^\infty(\Omega)$. According to Theorem 7.14.3, there exists a solution $\varphi \in W^{k+2,p'}(\Omega)$ of the Dirichlet problem $-\Delta\varphi + c\varphi = \psi$ in Ω, $\varphi = 0$ on $\partial\Omega$. Since $\varphi \in \overset{\circ}{W}{}^{1,p'}(\Omega)$ by Proposition 1.24.3, we have

$$\int_\Omega (u-v)\psi\, \mathrm{d}\mathcal{H}_m = \int_\Omega (u-v)[-\Delta\varphi + c\varphi]\, \mathrm{d}\mathcal{H}_m = 0.$$

Thus, $u - v = 0$ in Ω in the sense of distribution. Hence, $u = v$.

7.15 Exterior Dirichlet Problem in $D^{k+2,p}(\Omega)$

In this section, we prove the existence of a solution of the Dirichlet problem for the Poisson equation in $D^{k,p}(\Omega)$ for an exterior domain with boundary of class $\mathcal{C}^k$, where $k \in N$, $k \geq 2$, $1 < p < \infty$.

Theorem 7.15.1 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$, $m \geq 2$, $1 < p < \infty$. If $f \in L^p(\Omega)$, $g \in W^{2-1/p,p}(\partial\Omega)$, then there exists $u \in D^{2,p}(\Omega)$ such that $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. The dimension of the space $\{u \in D^{2,p}(\Omega); \Delta u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega\}$ is $m + 1$.*

Proof Define $f = 0$ outside Ω. According to Theorem 7.4.2, there exists $v \in \overset{\circ}{D}{}^{2,p}(R^m)$ such that $-\Delta v = f$ in R^m. Put $\tilde{g} = g - v$. Then $\tilde{g} \in W^{2-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. According to Theorems 5.15.3 and 5.15.4 and Proposition 2.17.3, there exists an L^p-solution w of the Dirichlet problem $\Delta w = 0$ in Ω, $w = \tilde{g}$ on $\partial\Omega$ such that $w(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Put $\tilde{g} = w$ on $\partial B(0; r)$. According to Theorem 7.14.3, there exists a solution $\tilde{w} \in D^{2,p}(\Omega \cap B(0; r))$ of the problem $\Delta\tilde{w} = 0$ in $\Omega \cap B(0; r)$, $\tilde{w} = \tilde{g}$ on $\partial[\Omega \cap B(0; r)]$. Theorem 7.11.2 gives that $\tilde{w}$ is an L^p-solution w of that problem. The uniqueness of an L^p-solution of the Dirichlet problem gives that $w = \tilde{w} \in D^{2,p}(B(0; r) \cap \Omega)$ (see Proposition 5.15.1). Since $w(x) = O(|x|^{2-m})$, Proposition 2.12.1 gives $|\nabla^2 w(x)| = O(|x|^{-m})$ as $|x| \to \infty$. Hence, $w \in D^{2,p}(\Omega)$. The function $u = v + w$ is a solution of the problem.

Denote $v_j(x) = x_j$ for $j = 1, \dots, m$. If $m > 2$, put $v_{m+1}(x) = 1$. If $m = 2$, fix $z \in R^2 \setminus \overline{\Omega}$ and set $v_{m+1}(x) = \ln|x - z|$. We have proved that there exist $w_j \in D^{2,p}(\Omega)$ such that $\Delta w_j = 0$ in Ω, $w_j = v_j$ on $\partial\Omega$, and $w_j(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Set $u_j = v_j - w_j$. Then, $u_j \in D^{2,p}(\Omega)$ are linearly independent, $\Delta u_j = 0$ in Ω and $u_j = 0$ on $\partial\Omega$.

Let now $u \in D^{2,p}(\Omega)$, $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. Since $|\nabla^2 u| \in L^p(\Omega)$, there exist constants $a_1, \dots, a_{m+1}$ such that $v(x) - P(x) = O(|x|^{2-m})$ as $|x| \to \infty$ for $P(x) = a_1 v_1(x) + \cdots + a_{m+1} v_{m+1}(x)$ (see Proposition 2.17.5). Put $w = u - a_1 u_j - \cdots - a_{m+1} u_{m+1}$. Then, $w \in D^{2,p}(\Omega)$, $w = 0$ on $\partial\Omega$, $w(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Put $h = 0$ on $\partial\Omega$, $h = w$ on $\partial[\Omega \cap B(0; r)]$. Then w is an L^p-solution of the Dirichlet problem $\Delta w = 0$ in $\Omega \cap B(0; r)$, $w = h$ on $\partial[\Omega \cap B(0; r)]$ (see Theorem 7.11.2). So, w is an L^p-solution of the Dirichlet problem $\Delta w = 0$ in Ω, $w = 0$ on $\partial\Omega$. Since $w(x) = O(|x|^{2-m})$ as $|x| \to \infty$, Theorems 5.15.3 and 5.15.4 give $w \equiv 0$.

Proposition 7.15.2 *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{k+2}$, and $k, l \in N_0$, $l \leq k$, $1 < p < \infty$. Let $f \in D^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$. Suppose that $f = 0$ outside of a bounded set. If $u \in D^{l+1,p}(\Omega)$ is a solution of the Dirichlet problem $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$, then $u \in D^{k+2,p}(\Omega)$.*

Proof For Ω bounded, see Corollary 7.14.5. Suppose that Ω is unbounded. Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$ and $f = 0$ on $\Omega \setminus B(0; r)$. Then, $u \in D^{k+2,p}(\Omega \cap$

$B(0; 2r))$ by Corollary 7.14.5. Since $|\nabla^{l+1}u| \in L^p(\Omega \setminus B(0; 2r))$, Proposition 2.17.7 gives $|\nabla^{k+2}u| \in L^p(\Omega \setminus B(0; 2r))$. Hence, $u \in D^{k+2,p}(\Omega)$.

Theorem 7.15.3 *Let $k \in N$, $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{k+2}$, $m \geq 2$, $1 < p < \infty$. Then, there exists a solution $u \in D^{k+2,p}(\Omega)$ of the Dirichlet problem*

$$-\Delta u = f \textit{ in } \Omega, \quad u = g \textit{ on } \partial\Omega$$

if and only if $f \in D^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$. If $f \in \tilde{W}^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$, then there exists a solution $u \in \tilde{W}^{k+2,p}(\Omega)$ of the problem. The dimension of the space $\{u \in D^{k+2,p}(\Omega); \Delta u = 0 \textit{ in } \Omega, u = 0 \textit{ on } \partial\Omega\}$ is equal to $\dim[H(R^m) \cap \tilde{P}_{k+1}(R^m)]$, *where $H(R^m)$ is the space of all harmonic function in R^m and $\tilde{P}_{k+1}(R^m)$ is the space of all polynomials of degree at most* $k + 1$.

Proof Let $u \in D^{k+2,p}(\Omega)$ be a solution of the problem $\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$. Then $f \in D^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$ by Theorem 1.24.5.

Let now $f \in D^{k,p}(\Omega)$, $g \in W^{k+2-1/p,p}(\partial\Omega)$ be fixed. We can suppose that $f \in D^{k,p}(R^m)$. If $f \in \tilde{W}^{k,p}(\Omega)$, we can suppose that $f \in \tilde{W}^{k,p}(R^m)$. (See Lemma 1.21.12.) According to Theorem 3.42.1, there exists $v \in D^{k+2,p}(R^m)$ such that $-\Delta v = f$. If $f \in \tilde{W}^{k,p}(R^m)$, we can find $v \in \tilde{W}^{k+2,p}(R^m)$ by Theorem 3.17.2. Define $\tilde{g} = g - v$. Then, $\tilde{g} \in W^{k+2-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. If w is a solution of the Dirichlet problem

$$\Delta w = 0 \text{ in } \Omega, \quad w = \tilde{g} \text{ on } \partial\Omega \tag{7.36}$$

then $u = v + w$ is a solution of the original problem. So, it is enough to prove that there exists a solution $w \in \tilde{W}^{2+k,p}(\Omega)$ of the problem (7.36).

We show that there exists a solution $w \in \tilde{W}^{2+k,p}(\Omega)$ of the problem (7.36). According to Theorems 5.15.3 and 5.15.4 and Proposition 2.17.3, there exists an L^p-solution w of the Dirichlet problem $\Delta w = 0$ in Ω, $w = \tilde{g}$ on $\partial\Omega$ such that $w(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Put $\tilde{g} = w$ on $\partial B(0; r)$. According to Theorem 7.14.3, there exists a solution $\tilde{w} \in W^{k+2,p}(\Omega \cap B(0; r))$ of the problem $\Delta\tilde{w} = 0$ in $\Omega \cap B(0; r)$, $\tilde{w} = \tilde{g}$ on $\partial[\Omega \cap B(0; r)]$. Theorem 7.11.2 gives that $\tilde{w}$ is an L^p-solution w of that problem. The uniqueness of an L^p-solution of the Dirichlet problem gives that $w = \tilde{w} \in D^{2+k,p}(B(0; r) \cap \Omega)$ (see Proposition 5.15.1). If α is a multiindex, then $\partial^\alpha w(x) = O(|x|^{2-m-|\alpha|})$ as $|x| \to \infty$ by Proposition 2.12.1. Hence, $w \in D^{k+2,p}(\Omega)$. According to Lemma 1.21.12, we can extend w as a function from $D^{k+2,p}(R^m)$. Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on $B(0; 1)$, $\varphi = 0$ on $R^m \setminus B(0; 2)$. Then, there exists a constant C_1 such that $|\partial^\alpha\varphi| \leq C_1$ for $|\alpha| \leq k + 2$. Denote $\varphi_n(x) = \varphi(x/n)$, $w_n = w\varphi_n$. Then, $w_n \in W^{k+2,p}(R^m) = \overset{\circ}{W}{}^{k+2,p}(R^m) \subset \tilde{W}^{k+2,p}(R^m)$ (compare Lemma 1.21.4). If β is a multiindex with $|\beta| \leq k + 2$, then $|\partial^\beta\varphi_n| \leq C_1 n^{-|\beta|}$. So, there exists a constant C_2 such that

$$|\nabla^{k+2}[(1 - \varphi_n(x))w(x)]| \leq C_2 n^{-m-k}, \qquad n < |x| < 2n$$

for sufficiently large n. For sufficiently large n, we have

$$\|w - w_n\|^p_{D^{k+2,p}(R^m)} = \int\limits_{B(0;2n)\setminus B(0;n)} |\nabla^{k+2}(w - w_n)|^p \, d\mathcal{H}_m$$
$$\le \frac{C_2^p \mathcal{H}_m(B(0;2n))}{n^{mp+pk}} \to 0.$$

Therefore, $w \in \tilde{W}^{k+2,p}(R^m)$.

Let now $u \in D^{k+2,p}(\Omega)$, $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. Put $h = \partial u/\partial n$. Proposition 7.9.2 gives

$$u = \mathcal{S}^{\Omega} h + P,$$

where $P \in \tilde{P}_{k+1}(R^m)$. Denote $X = \{Q \in \tilde{P}_{k+1}(R^m); Q(0) = 0, \nabla Q(0) = 0\}$. There exist $Q_1 \in \tilde{P}_1(R^m)$ and $Q_2 \in X$ such that $P = Q_1 + Q_2$. Put $v = \mathcal{S}^{\Omega} h + Q_1$. Since $u, v \in H(\Omega)$, we infer that $Q_2 = u - v \in H(\Omega)$. Moreover, $v = u - Q_2 \in D^{k+2,p}(\Omega)$ is a solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad v = -Q_2 \text{ on } \partial\Omega.$$

Let now $Q_2 \in X \cap H(\Omega)$ be given. According to Theorem 7.15.1, there exists a solution $v \in D^{2,p}(\Omega)$ of this problem. Proposition 7.15.2 gives that $v \in D^{k+2,p}(\Omega)$. So, $u = v + Q_2 \in D^{k+2,p}(\Omega)$ is a solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. By virtue of Theorem 7.15.1, we deduce that the space $\{u \in D^{k+2,p}(\Omega); \Delta u = 0$ in $\Omega, u = 0$ on $\partial\Omega\}$ has dimension equal to $m + 1 + \dim(X \cap H(\Omega)) = \dim(\tilde{P}_{k+1}(R^m) \cap H(\Omega))$.

7.16 The Perturbation Problem

Proposition 7.16.1 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$, $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$. For $\epsilon \ge 0$, denote by u_ϵ a solution of the problem*

$$-\Delta u_\epsilon + \epsilon u_\epsilon = f \quad \text{in } \Omega, \quad u_\epsilon = g \quad \text{on } \partial\Omega \tag{7.37}$$

in $\tilde{W}^{1,2}(\Omega)$. Then, $u_\epsilon \to u_0$ in $\tilde{W}^{1,2}(\Omega)$ as $\epsilon \downarrow 0$.

Proof According to Lemma 1.21.10, we can suppose that $\overset{\circ}{W}{}^{1,2}(\Omega)$ is equipped with the norm $\|\nabla v\|_{L^2(\Omega)}$. Hence, there exists a constant $C \ge 1$ such that $\|v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)}$ for all $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

Put $v_\epsilon = u_\epsilon - g$, $F = f + \Delta g$. Then, $F \in D^{-1,2}(\Omega)$ by Theorem 3.19.3, and v_ϵ is a solution of the Dirichlet problem $-\Delta v_\epsilon + \epsilon v_\epsilon = F - \epsilon g$ in Ω, $v_\epsilon = 0$ on $\partial\Omega$. We have

$$\int\limits_\Omega [\nabla v_\epsilon \cdot \nabla w + \epsilon v_\epsilon w] \,\mathrm{d}\mathcal{H}_m = \langle F - \epsilon g, w\rangle \quad \forall w \in \mathring{W}^{1,2}(\Omega). \tag{7.38}$$

Since $v_\epsilon \in \mathring{W}^{1,2}(\Omega)$, we obtain for $\epsilon < 1$

$$\int\limits_\Omega [|\nabla v_\epsilon|^2 + \epsilon v_\epsilon^2] \,\mathrm{d}\mathcal{H}_m = \langle F - \epsilon g, v_\epsilon\rangle \le [\| F\|_{D^{-1,2}(\Omega)} + \|g\|_{D^{-1,2}(\Omega)}]\|\nabla v_\epsilon\|_{L^2(\Omega)}.$$

Since

$$\|\nabla v_\epsilon\|^2_{L^2(\Omega)} \le [\| F\|_{D^{-1,2}(\Omega)} + \|g\|_{D^{-1,2}(\Omega)}]\|\nabla v_\epsilon\|_{L^2(\Omega)}$$

we infer

$$\|\nabla v_\epsilon\|_{L^2(\Omega)} \le [\| F\|_{D^{-1,2}(\Omega)} + \|g\|_{D^{-1,2}(\Omega)}]. \tag{7.39}$$

Since $v_\epsilon - v_0 \in \mathring{W}^{1,2}(\Omega)$, (7.38) gives

$$\int\limits_\Omega [\nabla v_\epsilon \cdot \nabla(v_\epsilon - v_0) + \epsilon v_\epsilon(v_\epsilon - v_0)] \,\mathrm{d}\mathcal{H}_m - \int\limits_\Omega \nabla v_0 \cdot \nabla(v_\epsilon - v_0) \,\mathrm{d}\mathcal{H}_m$$
$$= -\int\limits_\Omega \epsilon g(v_\epsilon - v_0) \,\mathrm{d}\mathcal{H}_m.$$

Using this relation, Hölder's inequality (Proposition 1.14.1), and (7.39), we get

$$\int\limits_\Omega |\nabla(v_\epsilon - v_0)|^2 = -\int\limits_\Omega \epsilon(v_\epsilon + g)(v_\epsilon - v_0) \le \epsilon\|v_\epsilon - v_0\|_{L^2(\Omega)}\Big(\|v_\epsilon\|_{L^2(\Omega)}$$
$$+\|g\|_{L^2(\Omega)}\Big) \le C^2\epsilon\Big[\|\nabla v_\epsilon\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\Big]\|\nabla(v_\epsilon - v_0)\|_{L^2(\Omega)}$$
$$\le 2\epsilon C^2\Big[\| F\|_{D^{-1,2}(\Omega)} + \|g\|_{D^{-1,2}(\Omega)} + \|g\|_{L^2(\Omega)}\Big]^2 \to 0.$$

7.17 The Singular Perturbation Problem

Theorem 7.17.1 *Let* $\Omega \subset R^m$ *be a bounded open set,* $m \geq 2$, $f \in H^{-1}(\Omega)$. *For* $\epsilon > 0$, *denote by* u_ϵ *a solution of the problem*

$$-\epsilon \Delta u_\epsilon + u_\epsilon = f \quad \text{in } \Omega\,, \quad u_\epsilon = 0 \quad \text{on } \partial\Omega \tag{7.40}$$

in $\tilde{W}^{1,2}(\Omega)$. *Then,* $u_\epsilon \to f$ *in* $H^{-1}(\Omega)$ *as* $\epsilon \searrow 0$.

Proof According to Lemma 1.21.10, we can suppose that $\mathring{W}^{1,2}(\Omega)$ is equipped with the norm $\|\nabla u\|_{L^2(\Omega)}$.

Since u_ϵ is a solution of the problem (7.40), we have

$$\int_\Omega [\epsilon \nabla u_\epsilon \cdot \nabla w + u_\epsilon w]\, \mathrm{d}\mathcal{H}_m = \langle f, w\rangle \quad \forall w \in \mathring{W}^{1,2}(\Omega). \tag{7.41}$$

Therefore,

$$\langle f - u_\epsilon, w\rangle = \int_\Omega \epsilon \nabla u_\epsilon \cdot \nabla w \, \mathrm{d}\mathcal{H}_m$$

for all $w \in \mathring{W}^{1,2}(\Omega)$. Hence,

$$\|f - u_\epsilon\|_{H^{-1}(\Omega)} = \|\nabla \epsilon u_\epsilon\|_{L^2(\Omega)}. \tag{7.42}$$

Suppose that u_ϵ does not converge to f in $H^{-1}(\Omega)$ as $\epsilon \to 0_+$. Then there exist $\delta > 0$ and a decreasing sequence of positive numbers $\epsilon(n)$ such that $\epsilon(n) \to 0$ as $n \to \infty$ and $\|\nabla \epsilon(n) u_{\epsilon(n)}\|_{L^2(\Omega)} > \delta$.

Putting $w = \epsilon u_\epsilon$, we get from (7.41)

$$\int_\Omega [|\nabla \epsilon u_\epsilon|^2 + \epsilon u_\epsilon^2]\, \mathrm{d}\mathcal{H}_m = \langle f, \epsilon u_\epsilon\rangle \leq \|f\|_{H^{-1}(\Omega)} \|\nabla \epsilon u_\epsilon\|_{L^2(\Omega)}. \tag{7.43}$$

Therefore,

$$\|\nabla \epsilon u_\epsilon\|^2_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|\nabla \epsilon u_\epsilon\|_{L^2(\Omega)}.$$

Dividing this inequality by $\|\nabla \epsilon u_\epsilon\|_{L^2(\Omega)}$, we obtain

$$\|\nabla \epsilon u_\epsilon\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}. \tag{7.44}$$

Since $\epsilon(n)u_{\epsilon(n)}$ is a bounded sequence in $\overset{\circ}{W}^{1,2}(\Omega)$, there are $v \in \overset{\circ}{W}^{1,2}(\Omega)$ and a subsequence of $\epsilon(n)u_{\epsilon(n)}$ which converges weakly to v (see Theorem 1.5.10). So, we can suppose that $\epsilon(n)u_{\epsilon(n)}$ converges weakly to v in $\overset{\circ}{W}^{1,2}(\Omega)$. Since $\epsilon(n)u_{\epsilon(n)}$ converges weakly to v in $\overset{\circ}{W}^{1,2}(\Omega)$ and the injection is a compact linear operator from $\overset{\circ}{W}^{1,2}(\Omega)$ to $L^2(\Omega)$ (see Lemma 1.21.7), Proposition 1.7.8 gives that $\epsilon(n)u_{\epsilon(n)} \to v$ in $L^2(\Omega)$. According to (7.43) and (7.44),

$$\|\epsilon(n)u_{\epsilon(n)}\|^2_{L^2(\Omega)} \le \epsilon(n)\left\{\int\limits_{\Omega}[|\nabla\epsilon(n)u_{\epsilon(n)}|^2 + \epsilon(n)u^2_{\epsilon(n)}]\,\mathrm{d}\mathcal{H}_m\right\}$$

$$\le \epsilon(n)\|f\|_{H^{-1}(\Omega)}\|\nabla\epsilon(n)u_{\epsilon(n)}\|_{L^2(\Omega)} \le \epsilon(n)\|f\|^2_{H^{-1}(\Omega)} \to 0.$$

Thus, $v \equiv 0$. Since $\epsilon(n)u_{\epsilon(n)}$ converges weakly to $v \equiv 0$ in $\overset{\circ}{W}^{1,2}(\Omega)$, we obtain by (7.43)

$$0 < \delta^2 \le \lim_{n\to\infty} \|\nabla\epsilon(n)u_{\epsilon(n)}\|^2_{L^2(\Omega)}$$

$$\le \lim_{n\to\infty}\int\limits_{\Omega}[|\nabla\epsilon(n)u_{\epsilon}(n)|^2 + \epsilon(n)u^2_{\epsilon(n)}]\,\mathrm{d}\mathcal{H}_m = \lim_{n\to\infty}\langle f, \epsilon(n)u_{\epsilon(n)}\rangle = 0$$

what is a contradiction.

Proposition 7.17.2 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$, $f \in L^2(\Omega)$. For $\epsilon > 0$, denote by u_ϵ a solution of the problem (7.40) in $\tilde{W}^{1,2}(\Omega)$. Then $u_\epsilon \to f$ in $L^2(\Omega)$ as $\epsilon \searrow 0$.*

Proof Since u_ϵ is a solution of the problem (7.40), we have

$$\int\limits_{\Omega}[\epsilon\nabla u_\epsilon \cdot \nabla w + u_\epsilon w]\,\mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} fw\,\mathrm{d}\mathcal{H}_m \quad \forall w \in \overset{\circ}{W}^{1,2}(\Omega).$$

For $w = u_\epsilon$, we obtain using Hölder's inequality (Proposition 1.14.1)

$$\|u_\epsilon\|^2_{L^2(\Omega)} \le \epsilon\|\nabla u_\epsilon\|^2_{L^2(\Omega)} + \|u_\epsilon\|^2_{L^2(\Omega)} = \int\limits_{\Omega} fu_\epsilon\,\mathrm{d}\mathcal{H}_m \le \|u_\epsilon\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}.$$

Dividing by $\|u_\epsilon\|_{L^2(\Omega)}$, we get

$$\|u_\epsilon\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}. \tag{7.45}$$

Since u_ϵ is bounded in $L^2(\Omega)$, we can choose from each sequence $\epsilon(n) \searrow 0$ a subsequence $\epsilon(n(j))$ such that $u_{\epsilon(n(j))}$ converges weakly to some v in $L^2(\Omega)$. (See Theorem 1.5.10.) Since $u_{\epsilon(n(j))}$ converges weakly to v in $H^{-1}(\Omega)$ and $u_{\epsilon(n(j))} \to f$ in

$H^{-1}(\Omega)$ by Theorem 7.17.1, we infer that $u_{\epsilon(n(j))}$ converges weakly to f in $L^2(\Omega)$. Since u_ϵ converges weakly to f in $L^2(\Omega)$ as $\epsilon \searrow 0$, we get by virtue of (7.45)

$$\begin{aligned}\|u_\epsilon - f\|^2_{L^2(\Omega)} &= \|u_\epsilon\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} - 2\int_\Omega f u_\epsilon \, \mathrm{d}\mathcal{H}_m \\ &\le 2\|f\|^2_{L^2(\Omega)} - 2\int_\Omega f u_\epsilon \, \mathrm{d}\mathcal{H}_m \to 0.\end{aligned}$$

7.18 The Green Function and Sobolev Spaces

In this section, we study the Green function $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$. We prove that $g_\Omega(\cdot,y)$ is a solution of the Dirichlet problem $\Delta u = 0$ in Ω, $u = h_m(\cdot,y)$ on $\partial\Omega$ in $\tilde{W}^{1,2}(\Omega)$, in $W^{s,p}(\Omega)$, and in $B^{p,q}_s(\Omega)$.

Proposition 7.18.1 *Let $\Omega \subset R^m$ be a nonempty domain, $m \ge 2$. If $m = 2$, suppose moreover that Ω is bounded. Let $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. Fix $y \in \Omega$. Then $g_\Omega(\cdot,y)$ is a solution of the Dirichlet problem*

$$-\Delta u = 0 \quad in\ \Omega, \qquad u(x) = h_m(x-y) \quad on\ \partial\Omega$$

in $\tilde{W}^{1,2}(\Omega)$.

Proof Choose $r > 0$ with $\overline{B(y;3r)} \subset \Omega$ and $\varphi \in \mathcal{C}^\infty(R^m)$ such that $\varphi = 0$ in $B(y;r)$ and $\varphi = 1$ in $R^m \setminus B(y;2r)$. If $m > 2$, set $f(x) = \varphi(x)h_m(x-y)$. Then, $f \in \mathcal{C}^\infty(R^m)$. Since $|\nabla f(x)| = O(|x|^{1-m})$ as $|x| \to \infty$, we have $f \in D^{1,2}(R^m)$. Since $f(x) \to 0$ as $|x| \to \infty$, Proposition 3.38.5 gives that $f \in \overset{\circ}{D}{}^{1,2}(R^m)$. If $m = 2$, choose $\psi \in \mathcal{C}^\infty_c(R^m)$ such that $\psi = 1$ on a neighborhood of $\overline{\Omega}$ and put $f(x) = \varphi(x)\psi(x)h_2(x-y)$. Then, $f \in \mathcal{C}^\infty_c(R^2)$.

According to Theorem 7.1.2, there exists a unique $u \in \tilde{W}^{1,2}(\Omega)$ such that $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$. Theorem 2.18.2 gives that $u \in H(\Omega)$. Choose an increasing sequence of bounded domains Ω_k with boundary of class $\mathcal{C}^\infty$ such that $\cup\Omega_k = \Omega$ and $B(y;3r) \subset \Omega_1$. According to Corollary 3.9.6 and Proposition 3.9.8, there exist Green's functions $G_k(x,y) = h_m(x-y) - g_k(x,y)$ of Ω_k. Lemma 3.9.9 and Theorem 6.4.2 give that $g_k(\cdot,y)$ is a classical solution of the Dirichlet problem

$$-\Delta u_k = 0 \quad \text{in } \Omega_k, \qquad u_k = f \quad \text{on } \partial\Omega_k. \tag{7.46}$$

Proposition 6.12.1 gives that $g_k(\cdot - y) \in \mathcal{C}^1(\overline{\Omega_k})$, so $g_k(\cdot - y)$ is a solution of the problem (7.46) in $\tilde{W}^{1,2}(\Omega_k)$. If ω is a bounded open set such that $\overline{\omega} \subset \Omega$, then $g_k(\cdot,y) \to u$ in $L^2(\omega)$ by Theorem 7.2.1. Proposition 2.11.4 gives that $g_k(\cdot,y) \to u$ locally uniformly in Ω. Since $h_m(x-y) - g_k(x,y) = G_k(x,y) \to G(x,y) = h_m(x-y) - g_\Omega(x,y)$ by Theorem 3.9.13, we infer that $g_\Omega(x,y) = u(x)$.

Proposition 7.18.2 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$. Let $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. If $y \in \Omega$, then $g_\Omega(\cdot,y) \in \tilde{W}^{1,2}(\Omega)$ and*

$$\|g_\Omega(\cdot,y)\|_{\tilde{W}^{1,2}(\Omega)} \leq C \operatorname{dist}(y,\partial\Omega)^{1-m}.$$

with a constant C dependent only on Ω.

Proof Choose $\varphi \in \mathcal{C}^\infty(R^m))$ such that $\varphi(x) = 0$ for $|x| < 1/4$, and $\varphi(x) = 1$ for $|x| > 1/2$. Define $f_y(x) = h_m(x-y)\varphi((x-y)/\operatorname{dist}(y,\partial\Omega))$ for $y \in \Omega$ and $x \in R^m$. Then $f_y(x) = h_m(x-y)$ for $|x-y| \geq \operatorname{dist}(y,\partial\Omega)/2$ and $f_y(x) = 0$ for $|x-y| < \operatorname{dist}(y,\partial\Omega)/4$. Fix $r > 0$ such that $\overline{\Omega} \subset B(0;r)$. Easy calculation yields that there exists a constant C_1 such that $|f_y(x)| + |\nabla f_y(x)| \leq C_1 \operatorname{dist}(y,\partial\Omega)^{1-m}$ for $x \in B(0;r)$, $y \in \Omega$. Thus, there exists a constant C_2 such that

$$\|f_y\|_{W^{1,2}(B(0;r))} \leq C_2 \operatorname{dist}(y,\partial\Omega)^{1-m}, \quad \forall y \in \Omega.$$

Proposition 7.18.1 gives that $g_\Omega(\cdot,y)$ is a solution of the problem

$$\Delta u = 0 \quad \text{in}\,\Omega, \quad u = f_y \quad \text{on}\,\partial\Omega$$

in $\tilde{W}^{1,2}(\Omega)$. According to Theorem 7.1.2, there exists a constant C_3 dependent only on Ω such that

$$\|g_\Omega(\cdot,y)\|_{\tilde{W}^{1,2}(\Omega)} \leq C_3 \|f_y\|_{\tilde{W}^{1,2}(\Omega)}.$$

According to Proposition 1.25.2, there exists a constant C_4 such that

$$\|f\|_{\tilde{W}^{1,2}(B(0;r))} \leq C_4 \|f\|_{W^{1,2}(B(0;r))} \quad \forall f \in \tilde{W}^{1,2}(B(0;r)) = W^{1,2}(B(0;r)).$$

Hence,

$$\|g_\Omega(\cdot,y)\|_{\tilde{W}^{1,2}(\Omega)} \leq C_3 \|f_y\|_{\tilde{W}^{1,2}(B(0;r))} \leq C_2 C_3 C_4 \operatorname{dist}(y,\partial\Omega)^{1-m}, \quad \forall y \in \Omega.$$

Proposition 7.18.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$. Let $G_\Omega(x,y) = h_m(x-y) - g_\Omega(x,y)$ be the Green function of Ω. Let $1 < p,q < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If $y \in \Omega$, then $g_\Omega(\cdot,y) \in W^{s,p}(\Omega) \cap B_s^{p,q}(\Omega)$ and*

$$\|g_\Omega(\cdot,y)\|_{W^{s,p}(\Omega)} + \|g_\Omega(\cdot,y)\|_{B_s^{p,q}(\Omega)} \leq C \operatorname{dist}(y,\partial\Omega)^{1-m} \tag{7.47}$$

with a constant C that does not depend on y. If $p < m/(m-1)$, then $G_\Omega(\cdot,y) \in W^{1,p}(\Omega)$.

Proof Put $f_y(x) = h_m(x-y)$. According to Propositions 7.18.1 and 1.25.2, the function $g_\Omega(\cdot, y)$ is a solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = f_y \quad \text{on } \partial\Omega$$

in $W^{1,2}(\Omega)$. According to Theorems 7.10.1 and 7.10.2, we have $g_\Omega(\cdot, y) \in W^{s,p}(\Omega) \cap B_s^{p,q}(\Omega)$ and

$$\|g_\Omega(\cdot, y)\|_{B_s^{p,q}(\Omega)} \le C_1 \|f_y\|_{B_{s-1/p}^{p,q}(\partial\Omega)},$$

$$\|g_\Omega(\cdot, y)\|_{W^{s,p}(\Omega)} \le C_1 \|f_y\|_{W^{s-1/p,p}(\partial\Omega)},$$

where C_1 does not depend on y. Easy calculation yields that there exists a constant C_2 such that

$$|h_m(x-y)| + |\nabla h_m(x-y)| \le C_2 \operatorname{dist}(y, \partial\Omega)^{1-m},$$

for $x \in \partial\Omega$, $y \in \Omega$. (Compare Corollary 2.9.6.) Thus,

$$\|f_y\|_{W^{1,p}(\partial\Omega)} \le C_3 \operatorname{dist}(y, \partial\Omega)^{1-m},$$

where a constant C_3 does not depend on y. According to Proposition 1.23.7 and Theorems 1.23.3, 1.23.5, and 1.22.1, there exists a constant C_4 that does not depend on y, such that

$$\|f_y\|_{W^{s-1/p,p}(\partial\Omega)} + \|f_y\|_{B_{s-1/p}^{p,q}(\partial\Omega)} \le C_4 \|f_y\|_{W^{1,p}(\partial\Omega)}.$$

This gives (7.47) with $C = C_1 C_3 C_4$.

Let now $p < m/(m-1)$. Clearly there exists a constant C_5 such that $|h_m(x-y)| + |\nabla h_m(x-y)| \le C_5 |x-y|^{1-m}$ for $x, y \in \Omega$. Thus, $h_m(\cdot - y) \in W^{1,p}(\Omega)$. Since $g_\Omega(\cdot, y) \in W^{1,p}(\Omega)$, we obtain $G_\Omega(\cdot, y) \in W^{1,p}(\Omega)$.

Corollary 7.18.4 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{2+k}$, $m \ge 2$, $k \in N_0$. Let $G_\Omega(x, y) = h_m(x-y) - g_\Omega(x, y)$ be the Green function of Ω. Let $1 < p, q < \infty$, $1 < s < k+2$. If $y \in \Omega$, then $g_\Omega(\cdot, y) \in W^{k+2,p}(\Omega) \cap B_s^{p,q}(\Omega)$.*

Proof Put $f_y(x) = h_m(x-y)$. According to Propositions 7.18.1 and 1.25.2 the function $g_\Omega(\cdot, y)$ is a solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = f_y \quad \text{on } \partial\Omega$$

in $W^{1,2}(\Omega)$. We have $g_\Omega(\cdot, y) \in W^{1,p}(\Omega)$ by Proposition 7.18.3. According to Theorem 7.14.3, one has $g_\Omega(\cdot, y) \in W^{2+k,p}(\Omega)$. Theorem 7.14.4 forces $g_\Omega(\cdot, y) \in B_s^{p,q}(\Omega)$.

7.19 Strong, Weak, and Very Weak Solution of the Robin Problem

In next sections, we shall study the Neumann problem and the Robin problem for the Poisson equation in Sobolev and Besov spaces. First we define a strong solution, a weak solution, and a very weak solution of these problems. We show that a strong solution is also a weak solution, and a weak solution is also a very weak solution.

We have shown that a classical solution of the Robin problem for the Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega$$

is an L^p-solution of the problem. An L^p-solution of the problem is a sort of a strong solution of the problem. (The equation $\Delta u = 0$ in Ω is satisfied in a classical sense, and the boundary condition is satisfied in the sense of a nontangential limit.)

In this chapter, we study a strong solution of the Robin problem for the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega \tag{7.48}$$

in another sense. Suppose that $\Omega \subset R^m$ is a domain with locally Lipschitz boundary and $u \in D^{2,p}(\Omega)$ with $1 < p < \infty$. Then, $\Delta u \in L^p(\Omega)$ and the traces $\gamma_\Omega u$ and $\gamma_\Omega \partial_j u$ are from $L^p_{loc}(\partial\Omega)$ (see Theorem 1.24.1). *If $h \in L^\infty(\partial\Omega)$, $f \in L^p(\Omega)$, and $g \in L^p_{loc}(\partial\Omega)$, we say that $u \in D^{2,p}(\Omega)$ is a strong solution of the problem (7.48) if $\Delta u = f$ in Ω in the sense of distributions and the boundary condition $\partial u/\partial n + hu = g$ on $\partial\Omega$ is fulfilled in the sense of traces.* (If $h \equiv 0$, we call it a strong solution of the Neumann problem for the Poisson equation.)

Now we define a weak solution of the Robin problem for the Poisson equation in $D^{1,p}(\Omega)$ (and therefore in $\tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$). If $u \in \mathcal{C}^\infty(R^m)$, $\Omega \subset R^m$ is a bounded domain with smooth boundary and u is a solution of the Robin problem (7.48), then the Gauss-Green theorem (Theorem 1.19.1) gives

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} hu\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g\varphi, \qquad \forall \varphi \in \mathcal{C}_c^\infty(R^m).$$

If $F = f\mathcal{H}_m|\Omega + g\mathcal{H}_{m-1}|\partial\Omega$, then

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m + \int_{\partial\Omega} hu\varphi \, d\mathcal{H}_{m-1} = \langle F, \varphi \rangle \qquad \forall \varphi \in \mathcal{C}_c^\infty(R^m). \tag{7.49}$$

This motivates our definition of the Robin problem for the Poisson equation in $D^{1,p}(\Omega)$.

Let $\Omega \subset R^m$ be an open set with locally Lipschitz boundary, $m \geq 2$, $1 < p < \infty$, $p' = p/(p-1)$, $h \in L^\infty(\partial\Omega)$, $F \in [\tilde{W}^{1,p'}(\Omega)]'$. Then $u \in D^{1,p}(\Omega)$ is a weak solution of the Robin problem

$$-\Delta u = F|\Omega \quad \text{in}\ \Omega, \qquad \frac{\partial u}{\partial n} + hu = F|\partial\Omega \quad \text{on}\ \partial\Omega \tag{7.50}$$

if (7.49) holds. If h has compact support, then $u \in D^{1,p}(\Omega)$ is a weak solution of the Robin problem (7.50) if and only if (7.49) holds for all $\varphi \in \tilde{W}^{1,p'}(\Omega)$.

We cannot study the problem for $F \in [D^{1,p'}(\Omega)]'$ because for $D^{1,p'}(\Omega) \neq \tilde{W}^{1,p'}(\Omega)$, there exists $F \in [D^{1,p'}(\Omega)]'$ which is not a distribution.

We define a very weak solution of the Robin problem for the Poisson equation in Besov and Lizorkin-Triebel spaces. If $u \in \mathcal{C}^\infty(R^m)$, $\Omega \subset R^m$ is a bounded domain with smooth boundary, $h \in L^\infty(\partial\Omega)$ and u is a solution of the Robin problem (7.48) then the Green identity (Theorem 2.2.1) gives

$$\int\limits_\Omega u(-\Delta\varphi)\, d\mathcal{H}_m + \int\limits_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\, d\mathcal{H}_{m-1} = \int\limits_\Omega f\varphi\, d\mathcal{H}_m + \int\limits_{\partial\Omega} g\varphi\, d\mathcal{H}_{m-1}$$

for all $\varphi \in \mathcal{C}_c^\infty(R^m)$. If $F = f\mathcal{H}_m|\Omega + g\mathcal{H}_{m-1}|\partial\Omega$, then F is a distribution supported on $\overline{\Omega}$ and

$$\int\limits_\Omega u(-\Delta\varphi) + \int\limits_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right) = \langle F, \varphi\rangle, \quad \varphi \in \mathcal{C}_c^\infty(R^m). \tag{7.51}$$

This motivates our definition of a very weak solution of the Robin problem.

Suppose that $1 < p, q < \infty$, $1/p < s < \infty$ and Ω is an open set with compact Lipschitz boundary. If $u \in B_s^{p,q}(\Omega)$ or $u \in F_s^{p,q}(\Omega)$ or $u \in D^{1,p}(\Omega)$, then there exists the trace $\gamma_\Omega u \in L^p(\partial\Omega)$ by Theorem 1.24.7. *If F is a distribution supported in $\overline{\Omega}$, $h \in L^\infty(\partial\Omega)$, then we say that u is a very weak solution of the Robin problem (7.50) if (7.51) holds true.*

Proposition 7.19.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $m \geq 2$, $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega)$. Denote $F = f\mathcal{H}_m|\Omega + g\mathcal{H}_{m-1}|\partial\Omega$. Then $F \in [\tilde{W}^{1,p'}(\Omega)]'$, where $p' = p/(p-1)$. If $u \in D^{2,p}(\Omega) = W^{2,p}(\Omega)$ is a strong solution of the Robin problem for the Poisson equation (7.48), then u is a weak solution of the Robin problem for the Poisson equation (7.50) in $\tilde{W}^{1,p}(\Omega) = W^{1,p}(\Omega)$.*

Proof The trace is a bounded linear operator from $W^{1,p'}(\Omega)$ to $L^{p'}(\partial\Omega)$ by Theorem 1.24.1. Theorem 1.14.3 gives that $F \in [W^{1,p'}(\Omega)]'$.

Let $\varphi \in W^{1,p'}(\Omega)$. According to the Gauss-Green theorem for Sobolev spaces (Proposition 1.24.4)

$$\langle F, \varphi\rangle = \int\limits_\Omega (-\Delta u)\varphi + \int\limits_{\partial\Omega} \varphi\left[\frac{\partial u}{\partial n} + hu\right] = \int\limits_\Omega \nabla u \cdot \nabla\varphi + \int\limits_{\partial\Omega} hu\varphi.$$

Proposition 7.19.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m \geq 2$. Suppose $1 < p < \infty$, $p' = p/(p-1)$, $h \in L^\infty(\partial\Omega)$.*

- *Let F be a distribution supported on $\overline{\Omega}$. If there exists $u \in D^{1,p}(\Omega)$ that is a very weak solution of the Robin problem for the Poisson equation (7.50), then $F \in [\tilde{W}^{1,p'}(\Omega)]'$.*
- *Let $F \in [\tilde{W}^{1,p'}(\Omega)]'$. Then $u \in D^{1,p}(\Omega)$ is a very weak solution of the Robin problem for the Poisson equation (7.50) if and only if u is a weak solution of (7.50).*

Proof Suppose first that $F \in [\tilde{W}^{1,p'}(\Omega)]'$ and $u \in D^{1,p}(\Omega)$ is a weak solution of (7.50). Let $\varphi \in \mathcal{C}_c^\infty(R^m)$. Fix $r > 0$ such that $\operatorname{spt}\varphi \in B(0;r)$. According to the Gauss-Green theorem for Sobolev spaces (Proposition 1.24.4)

$$\int_\Omega u(-\Delta\varphi) + \int_{\partial\Omega} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right) = \int_{\Omega\cap B(0;r)} u(-\Delta\varphi) + \int_{\partial[\Omega\cap B(0;r)]} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)$$

$$= \int_{\Omega\cap B(0;r)} \nabla u\cdot\nabla\varphi + \int_{\partial[\Omega\cap B(0;r)]} hu\varphi = \int_\Omega \nabla u\cdot\nabla\varphi + \int_{\partial\Omega} hu\varphi = \langle F,\varphi\rangle.$$

So, u is a very weak solution of (7.50).

Let now F be a distribution supported on $\overline{\Omega}$ and $u \in D^{1,p}(\Omega)$ be a very weak solution of (7.50). Define a distribution G by

$$\langle G,\varphi\rangle = \int_\Omega \nabla u\cdot\nabla\varphi + \int_{\partial\Omega} hu\varphi.$$

Then $G \in [\tilde{W}^{1,p'}(\Omega)]'$ by Theorems 1.24.1 and 1.14.3. Since u is a weak solution of the problem $-\Delta u = G|\Omega$ in Ω, $\partial u/\partial n + hu = G|\partial\Omega$ on $\partial\Omega$, it is a very weak solution of the problem. Thus, $G = F$. Hence, $F \in [\tilde{W}^{1,p'}(\Omega)]'$ and u is a weak solution of (7.50).

7.20 The Neumann Problem in $\tilde{W}^{1,2}(\Omega)$

In this section, we study the Neumann problem for the Poisson equation in $D^{1,2}(\Omega)$ for a general domain and in $\tilde{W}^{1,2}(\Omega)$ for a domain with compact Lipschitz boundary. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness of a solution.

Proposition 7.20.1 *Let* $\Omega \subset R^m$ *be a domain,* $m \geq 2$*,* $F \in [D^{1,2}(\Omega)]'$*. Then there exists* $u \in D^{1,2}(\Omega)$ *such that*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle F, \varphi \rangle \qquad \forall \varphi \in D^{1,2}(\Omega) \tag{7.52}$$

if and only if $\langle F, 1 \rangle = 0$*. This solution is unique up to an additive constant.*

Proof Put $\varphi \equiv 1$. If u is a solution of the problem (7.52), then

$$\langle F, 1 \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = 0.$$

Suppose now that $\langle F, 1 \rangle = 0$. Fix a nonempty bounded open set ω such that $\overline{\omega} \subset \Omega$. Put

$$\|u\| = \left[\left(\int_\omega u \, \mathrm{d}\mathcal{H}_m \right)^2 + \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m \right]^{1/2}.$$

Then $\| \ \|$ is a norm on $D^{1,2}(\Omega)$. Put

$$V = \{u \in D^{1,2}(\Omega); \int_\omega u \, \mathrm{d}\mathcal{H}_m = 0\}.$$

If we define the inner product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m,$$

then V endowed with this inner product is a Hilbert space and the corresponding norm is $\| \ \|$. Riesz representation theorem (Theorem 1.9.10) gives that there exists $u \in V$ such that (7.52) holds for all $\varphi \in V$. Since $D^{1,2}(\Omega)$ is the direct sum of V and the space of constant functions, we infer that u is a solution of the Neumann problem (7.52).

Let u and v be two solutions of the problem (7.52). Then,

$$\int_\Omega |\nabla(u - v)|^2 \, \mathrm{d}\mathcal{H}_m = \int_\Omega \nabla u \cdot \nabla(u - v) \, \mathrm{d}\mathcal{H}_m - \int_\Omega \nabla v \cdot \nabla(u - v) \, \mathrm{d}\mathcal{H}_m$$

$$= \langle F, u - v \rangle - \langle F, u - v \rangle = 0.$$

Since $\nabla(u - v) = 0$ in Ω, $u - v$ is constant (see Lemma 1.20.1).

Theorem 7.20.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m \geq 2$, $F \in [\tilde{W}^{1,2}(\Omega)]'$.*

- *If Ω is unbounded and $m > 2$, then there exists a unique weak solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem $-\Delta u = F|\Omega$ in Ω, $\partial u/\partial n = F|\partial\Omega$ on $\partial\Omega$.*
- *If Ω is bounded or $m = 2$, then there exists a weak solution $u \in \tilde{W}^{1,2}(\Omega)$ of the problem $-\Delta u = F|\Omega$ in Ω, $\partial u/\partial n = F|\partial\Omega$ on $\partial\Omega$ if and only if $\langle F, 1\rangle = 0$. This solution is unique up to an additive constant.*

Proof If Ω is bounded or $m = 2$, then $D^{1,2}(\Omega) = \tilde{W}^{1,2}(\Omega)$ by Proposition 1.25.2 and Lemma 3.38.11. Proposition 7.20.1 gives that there exists a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the problem $-\Delta u = F|\Omega$ in Ω, $\partial u/\partial n = F|\partial\Omega$ on $\partial\Omega$ if and only if $\langle F, 1\rangle = 0$ and this solution is unique up to an additive constant.

Let now Ω is unbounded and $m > 2$. Then $D^{1,2}(\Omega) = \tilde{W}^{1,2}(\Omega) \oplus \{c; c \in R\}$ (see Lemma 3.38.11). We extend F onto $D^{1,2}(\Omega)$ by $\langle F, \varphi + c\rangle = \langle F, \varphi\rangle$ for $\varphi \in \tilde{W}^{1,2}(\Omega)$, $c \in R$. According to Proposition 7.20.1, there exist $w \in D^{1,2}(\Omega)$ such that (7.52) holds. This w is unique up to an additive constant. So, there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the problem $-\Delta u = F|\Omega$ in Ω, $\partial u/\partial n = F|\partial\Omega$ on $\partial\Omega$.

7.21 The Robin Problem in $\tilde{W}^{1,2}(\Omega)$ for the Helmholtz Equation

In this section, we study a solution of the Robin problem for the Helmholtz equation in $W^{1,2}(\Omega)$ and in $\tilde{W}^{1,2}(\Omega)$. We prove the monotony principle and the existence of a unique solution of the problem.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $1 < p < \infty$, $p' = p/(p-1)$. Suppose that $c \in L^\infty(R^m)$ has compact support. If $F \in [\tilde{W}^{1,p'}(\Omega)]'$, then $u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the problem

$$-\Delta u + cu = F|\Omega \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = F|\partial\Omega \quad \text{on } \partial\Omega \tag{7.53}$$

if u is a weak solution of the problem $-\Delta u = -cu\mathcal{H}_m|\Omega + F|\Omega$ on Ω, $\frac{\partial u}{\partial n} = -hu\mathcal{H}_{m-1}|\partial\Omega + F|\partial\Omega$ on $\partial\Omega$.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < p < \infty$, $p' = p/(p-1)$, $c \in L^\infty(\Omega)$, $h \in L^\infty(\partial\Omega)$. If $F \in [W^{1,p'}(\Omega)]'$, then $u \in W^{1,p}(\Omega)$ is a weak solution of the Robin problem (7.53) if

$$\int_\Omega [\nabla u \cdot \nabla\varphi + cu\varphi]\, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} hu\varphi = \langle F, \varphi\rangle \qquad \forall\varphi \in \mathcal{C}_c^\infty(R^m)$$

or equivalently for all $\varphi \in W^{1,p'}(\Omega)$.

Proposition 7.21.1 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $c \in L^\infty(R^m)$, $h \in L^\infty(\partial\Omega)$, and $h, c \geq 0$. Suppose that*

$$\int_\Omega c(x)\, \mathrm{d}\mathcal{H}_m(x) + \int_{\partial\Omega} h(x)\, \mathrm{d}\mathcal{H}_{m-1}(x) > 0. \tag{7.54}$$

Let $X = W^{1,2}(\Omega)$ or $X = \tilde{W}^{1,2}(\Omega)$. If $X = \tilde{W}^{1,2}(\Omega)$, suppose that c has compact support. Let $u \in X$ be a weak solution of the Robin problem $-\Delta u + cu = F|\Omega$ in Ω, $\partial u/\partial n + hu = F|\partial\Omega$ on $\partial\Omega$, where $F \in X'$. If $F \geq 0$ (i.e., F is a nonnegative measure), then $u \geq 0$.

Proof $u^- = \max(-u, 0) \in W^{1,2}_{loc}(\Omega)$, $\partial_j u^-(x) = -\partial_j u(x)$ for $u(x) < 0$, $\partial_j u^-(x) = 0$ elsewhere (see Theorem 1.21.5). So, $u^- \in X$ by Proposition 1.25.2, Lemma 3.38.11, and Proposition 3.38.5. Since $F \geq 0$, $u^- \geq 0$, we have

$$0 \leq \langle F, u^- \rangle = -\int_\Omega [|\nabla u^-|^2 + c|u^-|^2]\, \mathrm{d}\mathcal{H}_m - \int_{\partial\Omega} h|u^-|^2\, \mathrm{d}\mathcal{H}_{m-1}.$$

$c, h \geq 0$ forces $\nabla u^- = 0$ a.e. in Ω, $cu^- = 0$ a.e. in Ω, $hu^- = 0$ a.e. on $\partial\Omega$. Since $\nabla u^- = 0$ on domain Ω, we infer that $u^- = a$, where a is a constant (see Lemma 1.20.1). The relations $cu^- = 0$ a.e. in Ω, $hu^- = 0$ a.e. on $\partial\Omega$ in conjunction with (7.54) imply $a = 0$. Since $u^- \equiv 0$, we deduce $u \geq 0$.

Theorem 7.21.2 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $c \in L^\infty(R^m)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\inf_{x\in\Omega} c(x) > 0$. If $F \in [W^{1,2}(\Omega)]'$, then there exists a unique weak solution $u \in W^{1,2}(\Omega)$ of the problem $-\Delta u + cu = F|\Omega$ in Ω, $\partial u/\partial n + hu = F|\partial\Omega$ on $\partial\Omega$. Moreover,*

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|F\|_{[W^{1,2}(\Omega)]'},$$

where a constant C depends only on Ω, c and h.

Proof Since the trace is a bounded linear operator form $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ by Theorem 1.24.1, the inner product

$$a(u, v) = \int_\Omega [\nabla u \cdot \nabla v + cuv] + \int_{\partial\Omega} huv\, \mathrm{d}\mathcal{H}_m$$

gives an equivalent norm on $W^{1,2}(\Omega)$. According to Riesz representation theorem (Theorem 1.9.10), there exists a unique $u \in W^{1,2}(\Omega)$ such that $a(u, v) = \langle F, v \rangle$ for all $v \in W^{1,2}(\Omega)$. Moreover, $\sqrt{a(u, u)} = \|F\|$. The uniqueness follows from Proposition 7.21.1.

Theorem 7.21.3 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $c \in L^\infty(R^m)$ have compact support, $c \geq 0$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that (7.54)* *holds. If* $F \in [\tilde{W}^{1,2}(\Omega)]'$*, then there exists a unique weak solution* $u \in \tilde{W}^{1,2}(\Omega)$ *of the problem* $-\Delta u + cu = F|\Omega$ *in* Ω*,* $\partial u/\partial n + hu = F|\partial\Omega$ *on* $\partial\Omega$*. Moreover,*

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\|F\|_{[\tilde{W}^{1,2}(\Omega)]'},$$

where a constant C *depends only on* Ω*,* c*, and* h*. If there exists* $f \in L^2_{loc}(\Omega)$ *such that* $F = f$ *in* Ω*, then* $u \in W^{2,2}_{loc}(\Omega)$*.*

Proof Define the operators L_1, L_2 from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$

$$\langle L_1 u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m, \qquad \langle L_2 u, v\rangle = \int_\Omega cuv \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} huv \, \mathrm{d}\mathcal{H}_{m-1}.$$

The operator L_1 is a Fredholm operator with index 0 by Theorem 7.20.2, and the operator L_2 is compact by Theorem 1.24.1, Lemma 1.21.8, and Theorems 1.14.3 and 1.7.4. So, $L_1 + L_2$ is a Fredholm operator with index 0 by Theorem 1.8.3. If $(L_1 + L_2)u = 0$, then

$$0 = \int_\Omega (|\nabla u|^2 + cu^2) \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} hu^2 \, \mathrm{d}\mathcal{H}_{m-1}. \tag{7.55}$$

Since $\nabla u = 0$ on Ω, there is a constant a such that $u \equiv a$ (Lemma 1.20.1). The relations (7.55) and (7.54) give that $a = 0$. Since $L_1 + L_2$ is a Fredholm operator with index 0, it is an isomorphism by Theorem 1.6.3, and $u = (L_1 + L_2)^{-1}F$ is a unique weak solution of the problem.

Suppose now that there exists $f \in L^2_{loc}(\Omega)$ such that $F = f$ in Ω. Then $-\Delta u = f - cu$ in the sense of distributions in Ω (see Theorem 3.19.3). If ω and $\tilde{\omega}$ are bounded open sets such that $\overline{\omega} \subset \tilde{\omega} \subset \overline{\tilde{\omega}} \subset \Omega$, then $f - cu \in L^2(\tilde{\omega})$ and $u \in W^{2,2}(\omega)$ by Proposition 3.18.2.

Corollary 7.21.4 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary,* $h \in L^\infty(\partial\Omega)$*,* $h \geq 0$*,* $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$*. Define*

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} huv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then, $(\ ,\ )$ *is an inner product on* $\tilde{W}^{1,2}(\Omega)$*, and the corresponding norm* $\|\ \|$ *is equivalent on* $\tilde{W}^{1,2}(\partial\Omega)$*.*

Proof If $(u, u) = 0$, then $\nabla u = 0$ in Ω, $hu = 0$ on $\partial\Omega$. Since Ω is connected, there exists a constant c such that $u \equiv c$ (see Lemma 1.20.1). Thus,

$$0 = \int_{\partial\Omega} hu \, \mathrm{d}\mathcal{H}_{m-1} = c \int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1}.$$

Since $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, we infer that $c = 0$. Therefore, $(\ ,\ )$ is an inner product.

According to Theorem 1.24.1, there exists a constant C_1 such that

$$\|u\| \le C_1 \|u\|_{\tilde{W}^{1,2}(\Omega)} \qquad \forall u \in \tilde{W}^{1,2}(\Omega).$$

Now, we prove the opposite inequality. According to Theorem 7.21.3, there exists a constant C_2 such that for all $u \in \tilde{W}^{1,2}(\Omega)$

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \le C_2 \sup\{(u,v); v \in \tilde{W}^{1,2}(\Omega), \|v\|_{\tilde{W}^{1,2}(\Omega)} \le 1\}.$$

If $v \in \tilde{W}^{1,2}(\Omega)$ and $\|v\|_{\tilde{W}^{1,2}(\Omega)} \le 1$, then Schwarz's inequality (Proposition 1.9.3) gives

$$(u,v) \le \|u\|\|v\| \le \|u\| C_1 \|v\|_{\tilde{W}^{1,2}(\Omega)} \le C_1 \|u\|.$$

Thus,

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \le C_2 C_1 \|u\|.$$

7.22 Boundary Integral Equation Method for the Neumann Problem

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary. We would like to calculate a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = f \quad \text{on } \partial\Omega. \tag{7.56}$$

Since $f \in H^{-1/2}(\partial\Omega) \subset [\tilde{W}^{1,2}(\Omega)]'$, we know whether such a solution exists (Theorem 7.20.2). There are many numerical methods that enable us to calculate a solution. One of them is the boundary element method. The core of this method is to express a solution by a potential and to solve the corresponding integral equation on the boundary. In principle, there are direct and indirect integral equation methods.

In the direct method, we utilize the representation

$$u = \mathcal{S}^{\Omega} f + \mathcal{D}^{\Omega} u \quad \text{in } \Omega.$$

So, it is enough to calculate the trace of u. By virtue of boundary behavior of potentials, we obtain the integral equation

$$u = \mathcal{S}^{\Omega} f + \frac{1}{2} u + K_{\Omega} u.$$

In the indirect integral equation method, we look for a solution in the form $u = \mathcal{S}^\Omega \varphi$ with an unknown density $\varphi \in H^{-1/2}(\partial\Omega)$. If $m = 2$, we suppose moreover that $\langle \varphi, 1 \rangle = 0$. So, we are to solve the equation

$$\frac{1}{2}\varphi - K'_\Omega \varphi = f.$$

In this section, we show that we can solve these equations by the successive approximation.

Lemma 7.22.1 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary,* $m \geq 2$. *Let* $u \in \tilde{W}^{1,2}(\Omega)$, $\Delta u = 0$. *If* Ω *is unbounded, suppose moreover* $u(x) \to 0$ *as* $|x| \to \infty$. *If we denote*

$$\langle F, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m, \qquad \varphi \in \mathcal{C}_c^\infty(R^m),$$

then $F \in H^{-1/2}(\partial\Omega) \subset [\tilde{W}^{1,2}(\Omega)]'$; *u is a weak solution of the Neumann problem* $\Delta u = 0$ *in* Ω, $\partial u/\partial n = F$ *on* $\partial\Omega$ *and*

$$u = \mathcal{D}^\Omega u + \mathcal{S}^\Omega F = \mathcal{D}^\Omega u + \mathcal{S}^\Omega(\partial u/\partial n). \tag{7.57}$$

Proof Clearly, $F \in [\tilde{W}^{1,2}(\Omega)]'$. Since $\Delta u = 0$, Theorem 2.18.2 gives that F is supported on $\partial\Omega$. So, $F \in [H^{1/2}(\partial\Omega)]' = H^{-1/2}(\partial\Omega)$ (see Theorems 1.24.2 and 1.23.3, Lemma 1.23.6, and Theorem 1.23.5). The function u is a weak solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n = F$ on $\partial\Omega$ by the definition.

Suppose first that Ω is bounded. Then, $u \in \tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega) = F_1^{2,2}(\Omega)$ is a very weak solution of the problem $\Delta u = 0$ in Ω, $\partial u/\partial n = F$ on $\partial\Omega$ (see Propositions 7.19.2, 1.25.2, and 1.23.13). Lemma 7.9.1 gives (7.57).

Let now Ω be unbounded. We have proved

$$u = \mathcal{D}^{\Omega \cap B(0;r)} u + \mathcal{S}^{\Omega \cap B(0;r)}(\partial u/\partial n) \qquad \text{in } \Omega \cap B(0;r).$$

Since $u(x) \to 0$ as $|x| \to \infty$, we have $\partial^\alpha u(x) = O(|x|^{2-m-|\alpha|})$ for $m > 2$ and $\partial^\alpha u(x) = O(|x|^{-1-|\alpha|})$ for $m = 2$ (see Proposition 2.17.3). Letting $r \to \infty$, we obtain (7.57).

Lemma 7.22.2 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary,* $m \geq 2$. *Let* $u \in \tilde{W}^{1,2}(\Omega)$, $\Delta u = 0$.

- *If* $m > 2$, *then there exists* $\psi \in H^{-1/2}(\partial\Omega)$ *such that* $u = \mathcal{S}^\Omega \psi (= V\psi)$.
- *If* $m = 2$, *then there exist a constant c and* $\psi \in H^{-1/2}(\partial\Omega)$ *with* $\langle \psi, 1 \rangle = 0$ *such that* $u = \mathcal{S}^\Omega \psi + c$.

Proof Choose $\tilde{u} \in \tilde{W}^{1,2}(R^m)$ such that $\tilde{u} = u$ on Ω. According to Theorem 7.1.2, there exists $v \in \tilde{W}^{1,2}(R^m \setminus \overline{\Omega})$ such that $\Delta v = 0$ in $R^m \setminus \overline{\Omega}$ and $v = \tilde{u}$ on $\partial\Omega$. If we define $u = v$ on $R^m \setminus \Omega$, then $u \in \tilde{W}^{1,2}(R^m)$. (Compare Proposition 1.24.3.) Denote $\psi = -\Delta u$. Then, $\psi \in \mathcal{E}_2(R^m) \subset D^{-1,2}(R^m) = [\overset{\circ}{D}{}^{1,2}(R^m)]'$ by Proposition 3.38.3. If $m = 2$, then $\langle \psi, 1 \rangle = 0$ by Corollary 2.18.6. Since $\Delta u = 0$ in $R^m \setminus \partial\Omega$, the distribution ψ is supported on $\partial\Omega$. So, $\psi \in [H^{1/2}(\partial\Omega)]' = H^{-1/2}(\partial\Omega)$ (see Theorems 1.24.2 and 1.23.3, Lemma 1.23.6, and Theorem 1.23.5).

Suppose first that $m > 2$. Then $u \in L^{2m/(m-2)}(R^m)$ by Proposition 3.38.5. Proposition 2.17.4 gives that $u(x) \to 0$ as $|x| \to \infty$. According to Proposition 3.1.8, we have $u = V\psi = \mathcal{S}^{\Omega}\psi$.

Let now $m = 2$. We have $\Delta(u - V\psi) = 0$ in the sense of distributions by Theorem 3.1.5. So, $u - V\psi \in H(R^m) \subset \mathcal{C}^{\infty}(R^m)$ (see Theorem 2.18.2 and Corollary 2.2.8). Moreover, $\partial_j(u - V\psi) \in H(R^m)$ by Corollary 2.2.9. Proposition 3.1.6 gives that $\partial_j V\psi(x) = O(|x|^{-2})$ as $|x| \to \infty$. This forces $\partial_j(u - V\psi) \in L^2(R^m)$. Theorem 2.16.2 gives $\partial_j(u - V\psi) \equiv 0$. So, the function $u - V\psi$ is constant (see Lemma 1.20.1).

Proposition 7.22.3 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $0 < s < 1$, $1 < p, q < \infty$, $h \in L^{\infty}(\partial\Omega)$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Then, $f \mapsto f/2 - K_{\Omega}f + \mathcal{S}^{\Omega}(hf)$, $f \mapsto f/2 + K_{\Omega}f + \mathcal{S}^{\Omega}(hf)$ are bounded linear Fredholm operators with index 0 on $B_s^{p,q}(\partial\Omega)$; $(1/2)I - K'_{\Omega} + h\mathcal{S}^{\Omega}$, $(1/2)I + K'_{\Omega} + h\mathcal{S}^{\Omega}$ are bounded linear Fredholm operators with index 0 on $B_{-s}^{p,q}(\partial\Omega)$. Denote $X = \{f \in B_{-s}^{p,q}(\partial\Omega); \langle f, 1\rangle = 0\}$. Then $K'_{\Omega}(X) \subset X$. If Ω is a bounded domain, then $[(1/2)I - K'_{\Omega}](B_{-s}^{p,q}(\partial\Omega)) = X$, and $(1/2)I - K'_{\Omega}$ is an isomorphism on X. If Ω is an unbounded domain, then $(1/2)I - K'_{\Omega}$ is an isomorphism on $B_{-s}^{p,q}(\partial\Omega)$ and on X.*

Proof Denote $\omega = R^m \setminus \overline{\Omega}$. For $m > 2$, denote $T_1 = (1/2)I + K_{\Omega} + \mathcal{S}^{\Omega}$ and $T_2 = (1/2)I + K_{\omega} + \mathcal{S}^{\omega}$. For $m = 2$, define $Pf = \fint_{\partial\Omega} f \, d\mathcal{H}_1$, $T_1 = (\frac{1}{2}I + K_{\Omega})(I - P) + \mathcal{S}^{\Omega}(I - P) + P$, $T_2 = (\frac{1}{2}I + K_{\omega})(I - P) + \mathcal{S}^{\omega}(I - P) + P$. Propositions 5.14.7 and 5.14.8 give that T_1 and T_2 are isomorphisms on $L^p(\partial\Omega)$ and on $W^{1,p}(\partial\Omega)$. Proposition 1.23.10 gives

$$(L^p(\partial\Omega), W^{1,p}(\partial\Omega))_{s,q} = B_s^{p,q}(\partial\Omega).$$

Using Proposition 1.10.3 on T_1, T_1^{-1}, T_2 and T_2^{-1}, we obtain that T_1 and T_2 are isomorphisms on $B_s^{p,q}(\partial\Omega)$. Since $\mathcal{S}^{\Omega} = \mathcal{S}^{\omega}$ and $f \mapsto \mathcal{S}^{\Omega}(hf)$ are compact operators on $B_s^{p,q}(\partial\Omega)$ by Corollary 5.4.10, the operators $f \mapsto f/2 + K_{\Omega}f + \mathcal{S}^{\Omega}(hf)$, $f \mapsto f/2 + K_{\omega}f + \mathcal{S}^{\Omega}(hf) = f/2 - K_{\Omega}f + \mathcal{S}^{\Omega}(hf)$ are Fredholm operators with index 0 (see Theorem 1.8.3). Since these operators are Fredholm operators with index 0 in $B_s^{p',q'}(\partial\Omega)$ with $p' = p/(p-1)$, $q' = q/(q-1)$, the adjoint operators $(1/2)I + K'_{\Omega} + h\mathcal{S}^{\Omega}$, $(1/2)I - K'_{\Omega} + h\mathcal{S}^{\Omega}$ are Fredholm operators with index 0 in $B_{-s}^{p,q}(\partial\Omega)$ (see Theorems 1.8.2 and 1.23.1).

If $f \in X \cap L^p(\partial\Omega)$, then $K'_{\Omega}f \in X$ by Lemma 5.9.6. Since $X \cap L^p(\partial\Omega)$ is a dense subset of X by Theorem 1.23.1, we infer that $K'_{\Omega}(X) \subset X$. So $(1/2)I - K'_{\Omega}$ is a Fredholm operator with index 0 on X by Lemma 1.8.8.

Suppose first that Ω is a bounded domain. The operator $(1/2)I - K'_\Omega$ is an isomorphism on the space $\{f \in W^{-1,p}(\partial\Omega); \langle f, 1\rangle = 0\}$ by Theorem 5.20.6. Lemma 1.8.4 forces that the operator $(1/2)I - K'_\Omega$ is an isomorphism on X. If $f \equiv 1$, then $[(1/2)I - K'_\Omega]f \in X$ by Lemma 5.9.6. Thus, $[(1/2)I - K'_\Omega](B^{p,q}_{-s}(\partial\Omega)) = X$.

Let now Ω be an unbounded domain. Then $(1/2)I - K'_\Omega$ is an isomorphism on $W^{-1,p}(\partial\Omega)$ by Theorem 5.20.7. Lemma 1.8.4 gives that the operator $(1/2)I - K'_\Omega$ is an isomorphism on $B^{p,q}_{-s}(\partial\Omega)$. Since $[(1/2)I - K'_\Omega](X) \subset X$, we infer that $[(1/2)I - K'_\Omega](X) = X$. Therefore, $(1/2)I - K'_\Omega$ is an isomorphism on X by Theorem 1.6.3.

Proposition 7.22.4 *Let $\Omega \subset R^2$ be an open set with compact Lipschitz boundary. Denote*

$$\mathcal{E}_2(\partial\Omega) = \{\mu \in \mathcal{E}_2(R^2); \operatorname{spt}\mu \subset \partial\Omega\}.$$

Then $\mathcal{E}_2(\partial\Omega) = \{\mu \in H^{-1/2}(\partial\Omega); \langle \mu, 1\rangle = 0\}$. If $\mu \in \mathcal{E}_2(\partial\Omega)$, then $V\mu \in D^{1,2}(R^2) \cap H^{1/2}(\partial\Omega)$. The space $\mathcal{E}_2(\partial\Omega)$ is a Hilbert space equipped with the inner product

$$(\mu, \nu)_{\mathcal{E}} = \langle \mu, V\nu\rangle = \int_{R^2} \nabla V\mu \cdot \nabla V\nu \, d\mathcal{H}_2.$$

The corresponding norm is equivalent to the norm of $H^{-1/2}(\partial\Omega)$. The space of real measures μ with finite energy in R^2 supported on $\partial\Omega$ and $\mu(R^2) = 0$ is a dense subset of $\mathcal{E}_2(\partial\Omega)$.

Proof $\mathcal{E}_2(R^2) = \{\mu \in D^{-1,2}(R^2); \langle \mu, 1\rangle = 0\}$ by Corollary 2.18.6. We have $\{\mu \in D^{-1,2}(R^2); \operatorname{spt}\mu \subset \partial\Omega\} = H^{-1/2}(\partial\Omega)$ by Theorems 1.24.2 and 1.23.5. Thus, $\mathcal{E}_2(\partial\Omega) = \{\mu \in H^{-1/2}(\partial\Omega); \langle \mu, 1\rangle = 0\}$.

Let $\mu \in \mathcal{C}'(\partial\Omega)$ with finite energy and $\mu(R^2) = 0$. Then, $V\mu \in D^{1,2}(R^2)$ by Proposition 3.41.2. According to Proposition 3.38.3, we have $-\Delta V\mu \in \mathcal{E}_2(R^2)$. Since $-\Delta V\mu = \mu$ by Theorem 3.1.5, we deduce that $\mu \in \mathcal{E}_2(R^2)$. Thus, $\mu \in \mathcal{E}_2(\partial\Omega)$.

Let $\mu \in \mathcal{E}_2(\partial\Omega)$. Then there are $f_k \in \mathcal{C}^\infty(\partial\Omega)$ such that $f_k\mathcal{H}_1|\partial\Omega \to \mu$ in $H^{-1/2}(\partial\Omega)$ by Theorems 1.23.1 and 1.23.5. Clearly

$$\int_{\partial\Omega} f_k \, d\mathcal{H}_1 \to \langle \mu, 1\rangle = 0.$$

Putting

$$\mu_k = \left(f_k - \oint_{\partial\Omega} f_k \, d\mathcal{H}_1\right) \mathcal{H}_1|\partial\Omega,$$

we have $\mu_k \to \mu$ in $H^{-1/2}(\partial\Omega)$, $\operatorname{spt}\mu_k \subset \partial\Omega$, and $\mu_k(R^2) = 0$. Moreover, μ_k have finite energy by Propositions 5.2.2 and 1.14.1.

V is a bounded linear operator from $\mathcal{E}_2(\partial\Omega)$ equipped with the norm from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ by Theorem 7.6.3. Thus, $(\mu,\nu)_{\mathcal{E}} := \langle \mu, V\nu\rangle$ is well defined. Let $\mu \in \mathcal{E}_2(\partial\Omega)$. We have proved that there exists a sequence of real measures $\mu_k \in \mathcal{C}'(\partial\Omega)$ with finite energy and $\mu_k(R^2) = 0$ such that $\mu_k \to \mu$ in $H^{-1/2}(\partial\Omega)$. Then

$$(\mu_k, \mu_n)_{\mathcal{E}} = \int\limits_{R^2} \nabla V\mu_k \cdot V\mu_n \, d\mathcal{H}_2$$

by Theorem 3.41.3. Since $\mu_k \to \mu$ in $H^{-1/2}(\partial\Omega)$ and $V\mu_k \to V\mu$ in $H^{1/2}(\partial\Omega)$, we have

$$\int\limits_{R^2} |\nabla V\mu_k - \nabla V\mu_n|^2 \, d\mathcal{H}_2 = (\mu_k - \mu_n, \mu_k - \mu_n)_{\mathcal{E}} \to 0, \quad \text{as } k, n \to \infty.$$

Thus, $\{\nabla V\mu_k\}$ is a Cauchy sequence in $L^2(R^2)$. Fix $r > 0$. Proposition 7.6.1 yields that V is a bounded linear operator from $\mathcal{E}_2(\partial\Omega)$ equipped with the norm from $H^{-1/2}(\partial\Omega)$ to $W^{1,2}(B(0;r))$. So, $V\mu_k$ is a Cauchy sequence in $D^{1,2}(R^2)$. Therefore, there exists $u \in D^{1,2}(R^2)$ such that $V\mu_k \to u$ in $D^{1,2}(R^2)$. Since $V\mu_k \to V\mu$ in $W^{1,2}(B(0;r))$ for arbitrary $r > 0$, we infer that $V\mu = u$. So, $V\mu \in D^{1,2}(R^2)$ and $V\mu_k \to V\mu$ in $D^{1,2}(R^2)$. If $\nu \in \mathcal{E}_2(\partial\Omega)$, choose a sequence of real measures $\nu_k \in \mathcal{C}'(\partial\Omega)$ with finite energy and $\nu_k(R^2) = 0$ such that $\nu_k \to \nu$ in $H^{-1/2}(\partial\Omega)$. Then

$$(\mu,\nu)_{\mathcal{E}} = \lim_{k\to\infty} (\mu_k, \nu_k)_{\mathcal{E}} = \lim_{k\to\infty} \int\limits_{R^2} \nabla V\mu_k \cdot \nabla V\nu_k \, d\mathcal{H}_2 = \int\limits_{R^2} \nabla V\mu \cdot \nabla V\nu \, d\mathcal{H}_2.$$

If

$$0 = (\mu,\mu)_{\mathcal{E}} = \int\limits_{R^2} |\nabla V\mu|^2 \, d\mathcal{H}_2,$$

then $\nabla V\mu = 0$ in R^2 and $V\mu$ is constant (see Lemma 1.20.1). According to Theorem 3.1.5, we have $\mu = -\Delta V\mu = 0$. Hence, $(\ ,\ )_{\mathcal{E}}$ is an inner product on $\mathcal{E}_2(\partial\Omega)$.

$\mathcal{E}_2(\partial\Omega)$ is a closed subspace of $\mathcal{E}_2(R^2)$. If $\mu \in \mathcal{E}_2(\partial\Omega)$ and $V\mu$ is constant, then $\mu = 0$. Proposition 3.38.3, Theorem 3.1.5, and Proposition 1.25.6 give that $\sqrt{(\mu,\mu)_{\mathcal{E}}}$ is an equivalent norm on $\mathcal{E}_2(\partial\Omega)$.

Proposition 7.22.5 *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. The space $\mathcal{E}_2(\partial\Omega)$ is a Hilbert space equipped with the inner product $(f,g)_{\mathcal{E}} = \langle f, \mathcal{S}^{\Omega} g\rangle$. If $m > 2$, then $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$ and both norms are equivalent. If $m = 2$, then $\mathcal{E}_2(\partial\Omega) = \{f \in H^{-1/2}(\partial\Omega); \langle f, 1\rangle = 1\}$ and both norms are* *equivalent. If* $f \in \mathcal{E}_2(\partial\Omega)$*, then* $u = \mathcal{S}^\Omega f \in \tilde{W}^{1,2}(\Omega)$ *is a weak solution of the Neumann problem* $\Delta u = 0$ *in* Ω*,* $\partial u/\partial n = g$ *on* $\partial\Omega$*, where* $g = f/2 - K'_\Omega f$*. The operator* $\frac{1}{2}I - K'_\Omega$ *is a bounded linear Fredholm operator with index* 0 *in* $\mathcal{E}_2(\partial\Omega)$*. The spaces* $\mathrm{Ker}((1/2)I - K'_\Omega)$ *and* $((1/2)I - K'_\Omega)(\mathcal{E}_2(\partial\Omega))$ *are orthogonal complements in* $\mathcal{E}_2(\partial\Omega)$*. Denote by* T *the restriction of* $(1/2)I + K'_\Omega$ *onto* $((1/2)I - K'_\Omega)(\mathcal{E}_2(\partial\Omega))$*. Then* $\|T\|_{\mathcal{E}} < 1$*.*

Proof The space $\mathcal{E}_2(\partial\Omega)$ is a Hilbert space equipped with the inner product $(f, g)_{\mathcal{E}} = \langle f, \mathcal{S}^\Omega g\rangle$ by Propositions 3.40.2 and 7.22.4. If $m > 2$, then $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$ and both norms are equivalent (see Proposition 3.40.3). If $m = 2$, then $\mathcal{E}_2(\partial\Omega) = \{f \in H^{-1/2}(\partial\Omega); \langle f, 1\rangle = 1\}$ and both norms are equivalent (see Proposition 7.22.4). If $f = f_1 + if_2 \in \mathcal{E}_2(\partial\Omega; C)$, we denote $\overline{f} = f_1 - if_2$. The space $\mathcal{E}_2(\partial\Omega; C)$ equipped with the inner product $(f, g)_{\mathcal{E}} = \langle f, \overline{g}\rangle$ is a Hilbert space, too. If $f = f_1 + if_2 \in \mathcal{E}_2(\partial\Omega; C)$, then

$$(f,f)_{\mathcal{E}} = \int\limits_{R^m} \nabla\mathcal{S}^\Omega f \cdot \overline{\nabla\mathcal{S}^\Omega f}\, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m} |\nabla\mathcal{S}^\Omega f|^2\, \mathrm{d}\mathcal{H}_m.$$

by Lemma 3.40.1 and Proposition 7.22.4.

Let $f \in \mathcal{E}_2(\partial\Omega)$. Then $\mathcal{S}^\Omega f \in \mathring{D}^{1,2}(R^m) \cap H(R^m \setminus \partial\Omega)$ by Propositions 3.40.2 and 7.22.4. So, $\mathcal{S}^\Omega f \in \tilde{W}^{1,2}(\Omega)$. Put $g = f/2 - K'_\Omega f$. Then $u = \mathcal{S}^\Omega f$ is a very weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ by Lemma 5.20.4. Define

$$\langle F, \varphi\rangle = \int\limits_{\Omega} \nabla u \cdot \nabla\varphi\, \mathrm{d}\mathcal{H}_m.$$

Clearly, $F \in [\tilde{W}^{1,2}(\Omega)]'$ and u is a weak solution of the Neumann problem $-\Delta u = F$ in Ω, $\partial u/\partial n = F$ on $\partial\Omega$. Then $F = g$ by Proposition 7.19.2. So, $u = \mathcal{S}^\Omega f$ is a weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$.

$\frac{1}{2}I - K'_\Omega$ is a bounded linear Fredholm operator with index 0 in the space $B^{2,2}_{-1/2}(\partial\Omega) = F^{2,2}_{-1/2}(\partial\Omega) = H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3 and Theorems 1.23.6 and 1.23.5. Since $K'_\Omega(\mathcal{E}_2(\partial\Omega)) \subset \mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.3, $\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 in $\mathcal{E}_2(\partial\Omega)$ by Lemma 1.8.8.

If $f = f_1 + if_2 \in \mathcal{E}_2(\partial\Omega, C)$, then $\mathcal{S}^\Omega f_j$ is a weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = [(1/2)I - K'_\Omega]f_j$ on $\partial\Omega$. Since $\mathcal{S}^\Omega f_j \in \tilde{W}^{1,2}(\Omega)$, we have

$$([(1/2)I - K'_\Omega]f,f)_{\mathcal{E}} = \langle [(1/2)I - K'_\Omega]f, \overline{\mathcal{S}^\Omega f_j}\rangle$$

$$= \int\limits_{\Omega} \nabla\mathcal{S}^\Omega f \cdot \overline{\nabla\mathcal{S}^\Omega f}\, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} |\nabla\mathcal{S}^\Omega f|^2\, \mathrm{d}\mathcal{H}_m \geq 0.$$

Since the operator $\frac{1}{2}I - K'_\Omega$ is positive, it is self-adjoint by Proposition 1.9.20. If $(f,f)_{\mathcal{E}} = 1$, then

$$0 \le \left(\left[\frac{1}{2}I - K'_\Omega\right]f, f\right)_{\mathcal{E}} = \int_\Omega |\nabla \mathcal{S}^\Omega f|^2 \le \int_{R^m} |\nabla \mathcal{S}^\Omega f|^2 = (f,f)_{\mathcal{E}} = 1.$$

According to Proposition 1.9.16, we have $\sigma((1/2)I - K'_\Omega) \subset \langle 0, 1\rangle$. Since the operator $(1/2)I - K'_\Omega$ is self-adjoint, it is normal. According to Proposition 1.9.21, the spaces $\operatorname{Ker}(\frac{1}{2}I - K'_\Omega)$ and $(\frac{1}{2}I - K'_\Omega)(\mathcal{E}_2(\partial\Omega))$ are orthogonal complements in $\mathcal{E}_2(\partial\Omega)$. Since $\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 and $I - T = \frac{1}{2}I - K'_\Omega$ on $\mathcal{E}_2(\partial\Omega)$, $I - T$ is a Fredholm operator with index 0 by Lemma 1.8.8. According to Proposition 1.9.16, we have $\sigma(T) \subset \langle 0, 1\rangle$. Since $\operatorname{Ker}((1/2)I - K'_\Omega) \cap [(1/2)I - K'_\Omega](\mathcal{E}_2) = \{0\}$, the operator $I - T$ is one-to-one. Since it is a Fredholm operator with index 0, it is onto. Theorem 1.6.3 forces that $I - T$ is an isomorphism. Thus, $\sigma(I - T) \subset (0, 1\rangle$. Since $\sigma(I - T)$ is a closed set by Lemma 1.6.5, there exists $\epsilon \in (0, 1)$ such that $\sigma(I - T) \subset \langle \epsilon, 1\rangle$. Hence, $\sigma(T) \subset \langle 0, 1 - \epsilon\rangle$. Since T is self-adjoint, Proposition 1.9.16 gives that $\|T\|_{\mathcal{E}} < 1$.

Theorem 7.22.6 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m > 2$, $f \in H^{-1/2}(\partial\Omega)$. Then there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56). Moreover, there exists a unique $\psi \in H^{-1/2}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega \psi$. Fix $\psi_0 \in H^{-1/2}(\partial\Omega)$. Define ψ_k by the induction by*

$$\psi_k = \frac{1}{2}\psi_{k-1} + K'_\Omega \psi_{k-1} + f. \tag{7.58}$$

Then $\psi_k \to \psi$ in $H^{-1/2}(\partial\Omega)$. Moreover,

$$\|\psi_k - \psi\|_{H^{-1/2}(\partial\Omega)} \le Cq^k\left[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{-1/2}(\partial\Omega)}\right], \tag{7.59}$$

where constants $C \in (0, \infty)$, $q \in (0, 1)$ depend only on Ω.

Proof According to Theorem 7.20.2, there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56). According to Lemma 7.22.2, there exists $\psi \in H^{-1/2}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega \psi$. Proposition 7.22.5 gives that

$$\frac{1}{2}\psi - K'_\Omega \psi = f. \tag{7.60}$$

The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega) = B^{2,2}_{-1/2}(\partial\Omega)$ by Proposition 7.22.3. This gives that ψ is a unique solution of the Eq. (7.60). Thus, there exists a unique $\psi \in H^{-1/2}(\partial\Omega)$ such that $u = \mathcal{S}^\Omega \psi$.

We rewrite the Eq. (7.60) to

$$\psi = \frac{1}{2}\psi + K'_\Omega \psi + f. \tag{7.61}$$

According to Proposition 7.22.5, there exists a constant $q \in (0, 1)$ such that

$$\left\| \frac{1}{2}\varphi + K'_\Omega \varphi \right\|_{\mathcal{E}} \leq q \|\varphi\|_{\mathcal{E}}$$

for all $\varphi \in \mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$. Define

$$T\varphi = \frac{1}{2}\varphi + K'_\Omega \varphi + f.$$

If $\varphi, \phi \in \mathcal{E}_2(\partial\Omega)$, then

$$\|T\varphi - T\phi\|_{\mathcal{E}} = \left\| \frac{1}{2}(\varphi - \phi) + K'_\Omega(\varphi - \phi) \right\|_{\mathcal{E}} \leq q\|(\varphi - \phi)\|_{\mathcal{E}}.$$

Fix $\psi_0 \in H^{-1/2}(\partial\Omega) = \mathcal{E}_2(\partial\Omega)$. Theorem 1.4.18 gives that $\psi_k \to \psi$ in $\mathcal{E}_2(\partial\Omega)$ and

$$\|\psi - \psi_k\|_{\mathcal{E}} \leq \frac{q^k}{1-q}\|\psi_1 - \psi_0\|_{\mathcal{E}} \leq \frac{q^k}{1-q}[(q+1)\|\psi_0\|_{\mathcal{E}} + \|f\|_{\mathcal{E}}].$$

According to Proposition 7.22.5, there exists positive constant C_1 such that

$$C_1^{-1}\|\varphi\|_{\mathcal{E}} \leq \|\varphi\|_{H^{-1/2}(\partial\Omega)} \leq C_1\|\varphi\|_{\mathcal{E}}$$

for all $\varphi \in H^{-1/2}(\partial\Omega)$. Therefore, $\psi_k \to \psi$ in $H^{-1/2}(\partial\Omega)$ and

$$\|\psi - \psi_k\|_{H^{-1/2}(\partial\Omega)} \leq \frac{2C_1^2}{1-q} q^k \Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{-1/2}(\partial\Omega)} \Big].$$

Theorem 7.22.7 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $f \in H^{-1/2}(\partial\Omega)$ be such that there exists a solution $u \in \tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega)$ of the Neumann problem (7.56), i.e., $\langle f, 1\rangle = 0$. Fix $\psi_0 \in H^{-1/2}(\partial\Omega)$. Define ψ_k by the induction by (7.58). Then, there exists ψ in $H^{-1/2}(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{-1/2}(\partial\Omega)$. Moreover, the estimate (7.59) holds with constants $C \in (0, \infty)$, $q \in (0, 1)$ that depend only on Ω. The function $u = \mathcal{S}^\Omega\psi$ is a solution of the Neumann problem (7.56) in $\tilde{W}^{1,2}(\Omega)$.*

Proof Denote $X = \{g \in H^{-1/2}(\partial\Omega); \langle g, 1\rangle = 0\}$. Theorem 7.20.2 forces that $f \in X$.

The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on X by Proposition 7.22.3. According to Proposition 7.22.5, there exists a constant $q \in (0, 1)$ such that

$$\left\| \frac{1}{2}\varphi + K'_\Omega \varphi \right\|_{\mathcal{E}} \leq q\|\varphi\|_{\mathcal{E}} \qquad \forall \varphi \in X.$$

Since $\frac{1}{2}I - K'_\Omega$ is a Fredholm operator with index 0 on $H^{-1/2}(\partial\Omega)$ and $[(1/2)I - K'_\Omega](H^{-1/2}(\partial\Omega)) = X$ (see Proposition 7.22.3), we deduce $\dim \mathrm{Ker}[(1/2)I - K'_\Omega] = 1$. Since $\frac{1}{2}I - K'_\Omega$ is an isomorphism on X, there exists $h \in H^{-1/2}(\partial\Omega) \setminus X$ such that $\mathrm{Ker}[(1/2)I - K'_\Omega] = \{th; t \in R^1\}$. Clearly,

$$H^{-1/2}(\partial\Omega) = X \oplus \{th; t \in R^1\}.$$

Lemma 5.20.4 gives that $\mathcal{S}^\Omega h$ is a very weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$. According to Theorem 5.20.6, there exists a constant c such that $\mathcal{S}^\Omega h = c$ in Ω.

Let now $\psi_0 \in H^{-1/2}(\partial\Omega)$ be fixed. Since $H^{-1/2}(\partial\Omega) = X \oplus \{th; t \in R^1\}$, there exist $\tilde{\psi}_0 \in X$ and $t \in R^1$ such that $\psi_0 = \tilde{\psi}_0 + \hat{\psi}_0$ with $\hat{\psi}_0 = th$. If we define

$$\tilde{\psi}_k = \frac{1}{2}\tilde{\psi}_{k-1} + K'_\Omega \tilde{\psi}_{k-1} + f, \quad \hat{\psi}_k = \frac{1}{2}\hat{\psi}_{k-1} + K'_\Omega \hat{\psi}_{k-1}, \tag{7.62}$$

then $\psi_k = \tilde{\psi}_k + \hat{\psi}_k$. Since $[(1/2)I - K'_\Omega]\hat{\psi}_0 = 0$, we have

$$\hat{\psi}_1 = \hat{\psi}_0 - [(1/2)I - K'_\Omega]\hat{\psi}_0 = \hat{\psi}_0.$$

By the induction step, we obtain $\hat{\psi}_k = \hat{\psi}_0$.

Define on X the operator T by

$$T\varphi = \frac{1}{2}\varphi + K'_\Omega \varphi + f.$$

If $\varphi, \phi \in X$, then

$$\|T\varphi - T\phi\|_{\mathcal{E}} = \left\| \frac{1}{2}(\varphi - \phi) + K'_\Omega(\varphi - \phi) \right\|_{\mathcal{E}} \le q\|(\varphi - \phi)\|_{\mathcal{E}}.$$

Remember that a norm $\| \ \|_{\mathcal{E}}$ is equivalent on X with the original norm (see Proposition 7.22.5). Theorem 1.4.18 gives that there exists $\tilde{\psi} \in X$ such that $\tilde{\psi}_k \to \tilde{\psi}$ in $H^{-1/2}(\partial\Omega)$ and

$$\|\tilde{\psi} - \tilde{\psi}_k\|_{\mathcal{E}} \le \frac{q^k}{1-q}\|\tilde{\psi}_1 - \tilde{\psi}_0\|_{\mathcal{E}} \le \frac{q^k}{1-q}[(q+1)\|\tilde{\psi}_0\|_{\mathcal{E}} + \|f\|_{\mathcal{E}}].$$

According to Proposition 7.22.5, there exists positive constant C_1 such that

$$C_1^{-1}\|\varphi\|_{\mathcal{E}} \le \|\varphi\|_{H^{-1/2}(\partial\Omega)} \le C_1\|\varphi\|_{\mathcal{E}}$$

for all $\varphi \in X$. Therefore,

$$\|\tilde{\psi} - \tilde{\psi}_k\|_{H^{-1/2}(\partial\Omega)} \le \frac{2C_1^2}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\tilde{\psi}_0\|_{H^{-1/2}(\partial\Omega)}\Big].$$

Define $\psi = \tilde{\psi} + \hat{\psi}_0$. Then $\psi_k \to \psi$ in $H^{-1/2}(\partial\Omega)$. According to Corollary 1.4.9, there exists a constant $C_2 \geq 1$ dependent only on Ω such that

$$\|\tilde{\psi}_0\|_{H^{-1/2}(\partial\Omega)} \leq C_2 \|\psi_0\|_{H^{-1/2}(\partial\Omega)}.$$

Since $\hat{\psi} = \hat{\psi}_k$, we obtain

$$\|\psi - \psi_k\|_{H^{-1/2}(\partial\Omega)} \leq \frac{2C_1^2 C_2}{1-q} q^k \Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{-1/2}(\partial\Omega)} \Big].$$

Let $k \to \infty$ in (7.62). Then, we obtain $\tilde{\psi} = \frac{1}{2}\tilde{\psi} + K'_\Omega \tilde{\psi} + f$ and therefore $\frac{1}{2}\tilde{\psi} - K'_\Omega \tilde{\psi} = f$. So, $\mathcal{S}^\Omega \tilde{\psi}$ is a solution of the Neumann problem (7.56) in $W^{1,2}(\Omega)$ (see Proposition 7.22.5). Since $\mathcal{S}^\Omega \hat{\psi} = tc$ in Ω, the function $\mathcal{S}^\Omega \psi$ is a solution of the Neumann problem (7.56) in $W^{1,2}(\Omega)$.

Theorem 7.22.8 *Let $\Omega \subset R^2$ be an unbounded domain with compact Lipschitz boundary. Let $f \in H^{-1/2}(\partial\Omega)$ be such that there exists a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56), i.e., $f \in \mathcal{E}_2(\partial\Omega) = \{g \in H^{-1/2}(\partial\Omega); \langle g, 1\rangle = 0\}$. Fix $\psi_0 \in H^{-1/2}(\partial\Omega)$. Define ψ_k by the induction by (7.58). Then there exists ψ in $\mathcal{E}_2(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{-1/2}(\partial\Omega)$. Moreover, the estimate (7.59) holds for $k \geq 1$ with constants $C \in (0, \infty)$ and $q \in (0, 1)$ that depend only on Ω. The function $u = \mathcal{S}^\Omega \psi$ is a solution of the Neumann problem (7.56) in $\tilde{W}^{1,2}(\Omega)$.*

Proof $f \in \mathcal{E}_2(\partial\Omega)$ by Theorem 7.20.2.

The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3. According to Proposition 7.22.5, there exists a constant $q \in (0, 1)$ such that

$$\left\| \frac{1}{2}\varphi + K'_\Omega \varphi \right\|_{\mathcal{E}} \leq q\|\varphi\|_{\mathcal{E}} \qquad \forall \varphi \in \mathcal{E}_2(\partial\Omega).$$

We now show that there exists $h \in L^2(\partial\Omega) \setminus \mathcal{E}_2(\partial\Omega)$ such that $\frac{1}{2}h + K'_\Omega h = 0$. Denote $\omega = R^2 \setminus \overline{\Omega}$. Let $\omega_1, \ldots, \omega_k$ be components of ω. Denote

$$Y := \{\varphi \in L^2(\partial\Omega); \int\limits_{\partial\omega_j} \varphi \, \mathrm{d}\mathcal{H}_1 = 0 \; \forall j = 1, \ldots, k\}.$$

If $\varphi \in L^2(\partial\omega)$, then $\mathcal{S}^\omega \varphi$ is an L^2-solution of the Neumann problem $\Delta u = 0$ in ω, $\partial u/\partial n = \frac{1}{2}\varphi - K'_\omega \varphi$ on $\partial\omega$. (See Lemma 5.2.1, Propositions 5.2.2 and 5.2.8, and Theorems 5.4.8 and 5.4.7.) Thus, $[(1/2)I - K'_\omega](L^2(\partial\Omega)) \subset Y$ by Lemma 5.9.1. The operator $\frac{1}{2}I - K'_\omega$ is a Fredholm operator with index 0 on $L^2(\partial\omega)$ by Proposition 5.9.9. Since the codimension of $[(1/2)I - K'_\omega](L^2(\partial\Omega))$ in $L^2(\partial\Omega)$ is at least k, there exist linearly independent $\varphi_1, \ldots, \varphi_k \in L^2(\partial\Omega)$ such that

$[(1/2)I - K'_\omega]\varphi_j = 0$. Denote

$$c_{ij} = \int\limits_{\partial\omega_j} \varphi_i \,\mathrm{d}\mathcal{H}_1.$$

Suppose first that the matrix $C = (c_{ij})$ is not regular. Then there exist constants $\alpha_1, \dots, \alpha_k$ with $|\alpha_1| + \dots + |\alpha_k| > 0$ such that $\varphi = \alpha_1\varphi_1 + \dots + \alpha_k\varphi_k \in Y$. Since $\mathcal{S}^\omega\varphi$ is an L^2-solution of the Neumann problem $\Delta u = 0$ in ω, $\partial u/\partial n = 0$ on $\partial\Omega$, there exist constants $d_1, \dots, d_k$ such that $\mathcal{S}^\omega\varphi = d_j$ on ω_j. (See Theorem 5.9.10.) Thus, $\mathcal{S}^\omega\varphi = d_j$ on $\partial\omega_j$ (see Proposition 5.2.2). Since $\varphi \in Y$, we obtain

$$\int\limits_{\partial\omega} \varphi\mathcal{S}^\omega\varphi \,\mathrm{d}\mathcal{H}_1 = 0.$$

Lemma 5.9.3 forces that $\varphi \equiv 0$. But it is a contradiction with the fact that $\varphi_1, \dots, \varphi_k$ are linearly independent elements of $L^2(\partial\omega)$. Thus, the matrix $C = (c_{ij})$ is regular. Therefore, there exist constants $\alpha_1, \dots, \alpha_k$ such that $h = \alpha_1\varphi_1 + \dots + \alpha_k\varphi_k \notin \mathcal{E}_2(\partial\Omega)$. Moreover, $0 = \frac{1}{2}h - K'_\omega h = \frac{1}{2}h + K'_\Omega h$.

Denote $Z = \{th; t \in R^1\}$. Then $Z \subset \operatorname{Ker}(\frac{1}{2}I + K'_\Omega)$ and $H^{-1/2}(\partial\Omega) = \mathcal{E}_2(\partial\Omega) \oplus Z$. According to Corollary 1.4.9, there exists a constant $C_1 \in \langle 1, \infty)$ such that

$$\|\varphi\|_{H^{-1}(\partial\Omega)} + \|\phi\|_{H^{-1}(\partial\Omega)} \le C_1\|\varphi + \phi\|_{H^{-1}(\partial\Omega)} \quad \forall\varphi \in \mathcal{E}_2(\partial\Omega),\ \forall\phi \in Z.$$

Let now $\psi_0 \in H^{-1/2}(\partial\Omega)$ be fixed. Then, there exist $\tilde{\psi}_0 \in \mathcal{E}_2(\partial\Omega)$ and $\hat{\psi}_0 \in Z$ such that $\psi_0 = \tilde{\psi}_0 + \hat{\psi}_0$. If we define

$$\tilde{\psi}_k = \frac{1}{2}\tilde{\psi}_{k-1} + K'_\Omega\tilde{\psi}_{k-1} + f, \quad \hat{\psi}_k = \frac{1}{2}\hat{\psi}_{k-1} + K'_\Omega\hat{\psi}_{k-1},$$

then $\psi_k = \tilde{\psi}_k + \hat{\psi}_k$. Since $\hat{\psi}_0 \in \operatorname{Ker}[(1/2)I + K'_\Omega]$, we have $\hat{\psi}_k = 0$ for $k \ge 1$. Thus, $\psi_k = \tilde{\psi}_k$ for $k \ge 1$.

Define on $\mathcal{E}_2(\partial\Omega)$ the operator T by

$$T\varphi = \frac{1}{2}\varphi + K'_\Omega\varphi + f.$$

If $\varphi, \phi \in \mathcal{E}_2(\partial\Omega)$, then

$$\|T\varphi - T\phi\|_{\mathcal{E}} = \left\|\frac{1}{2}(\varphi - \phi) + K'_\Omega(\varphi - \phi)\right\|_{\mathcal{E}} \le q\|(\varphi - \phi)\|_{\mathcal{E}}.$$

Remember that a norm $\| \ \|_{\mathcal{E}}$ is equivalent on $\mathcal{E}_2(\partial\Omega)$ with the original norm (see Proposition 7.22.5). Theorem 1.4.18 gives that there exists $\tilde{\psi} \in \mathcal{E}_2(\partial\Omega)$ such that

$\tilde{\psi}_k \to \tilde{\psi}$ in $H^{-1/2}(\partial\Omega)$ and

$$\|\tilde{\psi} - \tilde{\psi}_k\|_{\mathcal{E}} \le \frac{q^k}{1-q}\|\tilde{\psi}_1 - \tilde{\psi}_0\|_{\mathcal{E}} \le \frac{q^k}{1-q}[(q+1)\|\tilde{\psi}_0\|_{\mathcal{E}} + \|f\|_{\mathcal{E}}].$$

According to Proposition 7.22.5, there exists positive constant C_2 such that

$$C_2^{-1}\|\varphi\|_{\mathcal{E}} \le \|\varphi\|_{H^{-1/2}(\partial\Omega)} \le C_2\|\varphi\|_{\mathcal{E}}$$

for all $\varphi \in \mathcal{E}_2(\partial\Omega)$. Therefore, for $k \ge 1$

$$\|\psi - \psi_k\|_{H^{-1/2}(\partial\Omega)} = \|\tilde{\psi} - \tilde{\psi}_k\|_{H^{-1/2}(\partial\Omega)} \le \frac{2C_2^2}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)}$$
$$+\|\tilde{\psi}_0\|_{H^{-1/2}(\partial\Omega)}\Big] \le \frac{2C_1C_2^2}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{-1/2}(\partial\Omega)}\Big].$$

We have

$$\psi = \lim_{k\to\infty}\psi_k = \lim_{k\to\infty}\left[\frac{1}{2}\psi_k + K'_\Omega\psi_k + f\right] = \frac{1}{2}\psi + K'_\Omega\psi + f$$

and therefore $\frac{1}{2}\psi - K'_\Omega\psi = f$. Since $\psi \in \mathcal{E}_2(\partial\Omega)$, Proposition 7.22.5 gives that $\mathcal{S}^\Omega\psi$ is a solution of the Neumann problem (7.56) in $\tilde{W}^{1,2}(\Omega)$.

Theorem 7.22.9 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m > 2$, $f \in H^{-1/2}(\partial\Omega)$. Fix $\psi_0 \in H^{1/2}(\partial\Omega)$. Define ψ_k by the induction by*

$$\psi_k = \frac{1}{2}\psi_{k-1} + K_\Omega\psi_{k-1} + \mathcal{S}^\Omega f. \tag{7.63}$$

Then there exists $\psi \in H^{1/2}(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{1/2}(\partial\Omega)$. Moreover,

$$\|\psi_k - \psi\|_{H^{1/2}(\partial\Omega)} \le Cq^k\big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{1/2}(\partial\Omega)}\big], \tag{7.64}$$

where constants $C \in (0,\infty)$, $q \in (0,1)$ depend only on Ω. Define

$$u = \mathcal{S}^\Omega f + \mathcal{D}^\Omega\psi.$$

Then, $u \in \tilde{W}^{1,2}(\Omega)$ is a unique solution of the Neumann problem (7.56), and ψ is the trace of u.

Proof According to Proposition 3.40.3, we have $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$ and both norms are equivalent. Since $H^{1/2}(\partial\Omega)$ is the dual space of the space $H^{-1/2}(\partial\Omega)$, we can define on $H^{1/2}(\partial\Omega)$ the equivalent norm

$$\|f\|_E = \sup\{|\langle g,f\rangle|; g \in H^{-1/2}(\partial\Omega), \|g\|_{\mathcal{E}} \leq 1\}.$$

So, there exists a constant $C_1 \geq 1$ such that

$$C_1^{-1}\|f\|_E \leq \|f\|_{H^{1/2}(\partial\Omega)} \leq C_1\|f\|_E.$$

The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega) = B^{2,2}_{-1/2}(\partial\Omega)$ by Proposition 7.22.3. According to Proposition 7.22.5, there exists a constant $q \in (0,1)$ such that $\|(1/2)I + K'_\Omega\|_{\mathcal{E}} \leq q$. Since $(1/2)I + K_\Omega$ is the adjoint operator of $(1/2)I + K'_\Omega$, Proposition 1.5.11 gives that $\|(1/2)I + K_\Omega\|_E \leq q$.

Since $f \in H^{-1/2}(\partial\Omega) = B^{2,2}_{-1/2}(\partial\Omega)$, Theorem 7.6.3 gives that $\mathcal{S}^\Omega f \in B^{2,2}_{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)$ and

$$\|\mathcal{S}^\Omega f\|_{H^{1/2}(\partial\Omega)} \leq C_2\|f\|_{H^{-1/2}(\partial\Omega)}$$

with a constant $C_2 \geq 1$ depending only on Ω. For $\varphi \in H^{1/2}(\partial\Omega)$, define

$$T\varphi = \frac{1}{2}\varphi + K_\Omega\varphi + \mathcal{S}^\Omega f.$$

We have

$$\|T\varphi - T\phi\|_E = \left\|\frac{1}{2}(\varphi - \phi) + K_\Omega(\varphi - \phi)\right\|_E \leq q\|(\varphi - \phi)\|_E.$$

Fix $\psi_0 \in H^{1/2}(\partial\Omega)$. Theorem 1.4.18 gives that there exists $\psi \in H^{1/2}(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{1/2}(\partial\Omega)$ and

$$\|\psi - \psi_k\|_E \leq \frac{q^k}{1-q}\|\psi_1 - \psi_0\|_E \leq \frac{q^k}{1-q}[(q+1)\|\psi_0\|_E + \|\mathcal{S}^\Omega f\|_E.$$

Thus,

$$\|\psi - \psi_k\|_{H^{1/2}(\partial\Omega)} \leq \frac{2C_1^2C_2}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{1/2}(\partial\Omega)}\Big].$$

According to Theorem 7.20.2, there exists a unique solution $v \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56). According to Lemma 7.22.2, there exist $\varphi \in H^{-1/2}(\partial\Omega)$ such that $v = \mathcal{S}^\Omega\varphi$. Proposition 3.1.6 forces that $v(x) \to 0$ as $|x| \to \infty$. Lemma 7.22.1 gives

$$v = \mathcal{S}^\Omega f + \mathcal{D}^\Omega v \qquad \text{in } \Omega.$$

Using boundary behavior of boundary layer potentials (Proposition 7.6.2 and Theorem 5.3.5), we obtain

$$v = \mathcal{S}^{\Omega} f + \left[\frac{1}{2} I + K_{\Omega}\right] v.$$

Thus, $[(1/2)I - K_{\Omega}]v = \mathcal{S}^{\Omega} f$. Since

$$\psi = \lim_{k\to\infty} \psi_k = \mathcal{S}^{\Omega} f + \lim_{k\to\infty} \left[\frac{1}{2} I + K_{\Omega}\right] \psi_{k-1} = \mathcal{S}^{\Omega} f + \left[\frac{1}{2} I + K_{\Omega}\right] \psi,$$

we have $[(1/2)I - K_{\Omega}]\psi = \mathcal{S}^{\Omega} f$. Therefore,

$$[(1/2)I - K_{\Omega}](\psi - v) = 0.$$

Since $[(1/2)I - K'_{\Omega}]$ is an isomorphism on $H^{-1/2}(\partial\Omega)$, the operator $[(1/2)I - K_{\Omega}]$ is an isomorphism on $H^{1/2}(\partial\Omega)$ (see Proposition 1.6.4). Since $[(1/2)I - K_{\Omega}](\psi - v) = 0$, we infer that $\psi - v = 0$. Hence, $\psi = v$ is the trace of v and $u = v$.

Theorem 7.22.10 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $f \in H^{-1/2}(\partial\Omega)$ be such that there exists a solution $u \in \tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega)$ of the Neumann problem (7.56), i.e., $\langle f, 1\rangle = 0$. Fix $\psi_0 \in H^{1/2}(\partial\Omega)$. Define ψ_k by the induction by (7.63). Then, there exists ψ in $H^{1/2}(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{1/2}(\partial\Omega)$. Moreover, the estimate (7.64) holds with constants $C \in (0,\infty)$, $q \in (0,1)$ dependent only on Ω. Define*

$$u = \mathcal{S}^{\Omega} f + \mathcal{D}^{\Omega} \psi.$$

Then, $u \in \tilde{W}^{1,2}(\Omega)$ is a solution of the Neumann problem (7.56) and ψ is the trace of u.

Proof Denote $X = \{g \in H^{-1/2}(\partial\Omega); \langle g, 1\rangle = 0\}$. Theorem 7.20.2 forces that $f \in X$.

The operator $\frac{1}{2}I - K'_{\Omega}$ is an isomorphism on X by Proposition 7.22.3. According to Proposition 7.22.5, there exists a constant $q \in (0,1)$ such that

$$\|\frac{1}{2}\varphi + K'_{\Omega}\varphi\|_{\mathcal{E}} \le q\|\varphi\|_{\mathcal{E}} \qquad \forall \varphi \in X.$$

We have $[(1/2)I - K'_{\Omega}](H^{-1/2}(\partial\Omega)) = X$ by Proposition 7.22.3. Therefore, the codimension of the range of $[(1/2)I - K'_{\Omega}]$ is equal to 1. Since $\frac{1}{2}I - K'_{\Omega}$ is a Fredholm operator with index 0 on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3, we deduce $\dim \operatorname{Ker}[(1/2)I - K'_{\Omega}] = 1$. Since $\frac{1}{2}I - K'_{\Omega}$ is an isomorphism on X, we infer

$$H^{-1/2}(\partial\Omega) = X \oplus \operatorname{Ker}\left(\frac{1}{2}I - K'_{\Omega}\right).$$

The space of constant functions $P_0(R^m)$ is a subset of $\mathrm{Ker}[(1/2)I - K_\Omega]$ by Proposition 5.3.2. Since $(1/2)I - K_\Omega$ is a Fredholm operator with index 0 on $H^{1/2}(\partial\Omega)$ (see Proposition 7.22.3), we have $\dim \mathrm{Ker}[(1/2)I - K_\Omega] = \mathrm{codim}[(1/2)I - K'_\Omega](H^{-1/2}(\partial\Omega)) = 1$ by Theorem 1.8.2. Therefore, $\mathrm{Ker}[(1/2)I - K_\Omega] = P_0(R^m)$. Clearly, $P_0(R^m) = \{g \in H^{1/2}(\partial\Omega); \langle f, g\rangle = 0\ \forall f \in X\}$. Denote $\tilde{X} = [(1/2)I - K_\Omega](H^{1/2}(\partial\Omega))$. According to Proposition 1.5.13, we have $\tilde{X} = \{g \in H^{1/2}(\partial\Omega); \langle f, g\rangle = 0\ \forall f \in \mathrm{Ker}[(1/2)I - K'_\Omega]\}$. Since $H^{-1/2}(\partial\Omega) = X \oplus \mathrm{Ker}[(1/2)I - K'_\Omega]$, we deduce

$$H^{1/2}(\partial\Omega) = \tilde{X} \oplus P_0(R^m).$$

Moreover, $\tilde{X}$ is isomorphic with the dual space of X. We can endow $\tilde{X}$ by the norm

$$\|g\|_E = \sup\{|\langle f, g\rangle|; f \in X, \|f\|_{\mathcal{E}} \le 1\}.$$

Since $\|\ \|_{\mathcal{E}}$ is an equivalent norm on X, we deduce that $\|\ \|_E$ is an equivalent norm on $\tilde{X}$. Therefore, there exists $C_1 \ge 1$ such that

$$C_1^{-1}\|g\|_{H^{1/2}(\partial\Omega)} \le \|g\|_E \le C_1\|g\|_{H^{1/2}(\partial\Omega)}.$$

According to Proposition 7.22.5, there exists $q \in (0, 1)$ such that $\|\frac{1}{2}\varphi + K'_\Omega\varphi\|_{\mathcal{E}} \le q\|\varphi\|_{\mathcal{E}}$ for all $\varphi \in X$. Proposition 1.5.11 gives

$$\|\frac{1}{2}\varphi + K_\Omega\varphi\|_E \le q\|\varphi\|_E \qquad \forall \varphi \in \tilde{X}.$$

Fix a solution $v \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56). Then

$$v = \mathcal{S}^\Omega f + \mathcal{D}^\Omega v$$

by Lemma 7.22.1. The boundary behavior of potentials (Proposition 7.6.2 and Theorem 5.3.5) gives

$$v = \frac{1}{2}v + K_\Omega v + \mathcal{S}^\Omega f.$$

Thus, $\mathcal{S}^\Omega f = [(1/2)I - K_\Omega]v \in \tilde{X}$.

Let now $\psi_0 \in H^{1/2}(\partial\Omega)$ be fixed. Since $H^{1/2}(\partial\Omega) = \tilde{X} \oplus P_0(R^m)$, there exist $\tilde{\psi}_0 \in \tilde{X}$ and $\hat{\psi}_0 \in P_0(R^m)$ such that $\psi_0 = \tilde{\psi}_0 + \hat{\psi}_0$. If we define

$$\tilde{\psi}_k = \frac{1}{2}\tilde{\psi}_{k-1} + K_\Omega\tilde{\psi}_{k-1} + \mathcal{S}^\Omega f, \quad \hat{\psi}_k = \frac{1}{2}\hat{\psi}_{k-1} + K_\Omega\hat{\psi}_{k-1}, \tag{7.65}$$

then $\psi_k = \tilde{\psi}_k + \hat{\psi}_k$. Since $K_\Omega\hat{\psi}_0 = (1/2)\hat{\psi}_0$ by Proposition 5.3.2, we have $\hat{\psi}_1 = \hat{\psi}_0$. By the induction step, we obtain $\hat{\psi}_k = \hat{\psi}_0$.

Define on $\tilde{X}$ the operator T by

$$T\varphi = \frac{1}{2}\varphi + K_\Omega\varphi + \mathcal{S}^\Omega f.$$

If $\varphi, \phi \in \tilde{X}$, then

$$\|T\varphi - T\phi\|_E = \left\|\frac{1}{2}(\varphi - \phi) + K_\Omega(\varphi - \phi)\right\|_E \leq q\|(\varphi - \phi)\|_E.$$

Theorem 1.4.18 gives that there exists $\tilde{\psi} \in \tilde{X}$ such that $\tilde{\psi}_k \to \tilde{\psi}$ in $H^{1/2}(\partial\Omega)$ and

$$\|\tilde{\psi} - \tilde{\psi}_k\|_E \leq \frac{q^k}{1-q}\|\tilde{\psi}_1 - \tilde{\psi}_0\|_E \leq \frac{q^k}{1-q}[(q+1)\|\tilde{\psi}_0\|_E + \|\mathcal{S}^\Omega f\|_E].$$

According to Corollary 5.4.11, there exists $C_2 \geq 1$ independent of f such that

$$\|\mathcal{S}^\Omega f\|_{H^{1/2}(\partial\Omega)} \leq C_2\|f\|_{H^{-1/2}(\partial\Omega)}.$$

Therefore,

$$\|\tilde{\psi} - \tilde{\psi}_k\|_{H^{1/2}(\partial\Omega)} \leq \frac{2C_1^2C_2}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\tilde{\psi}_0\|_{H^{1/2}(\partial\Omega)}\Big].$$

Define $\psi = \tilde{\psi} + \hat{\psi}_0$. Then $\psi_k \to \psi$ in $H^{1/2}(\partial\Omega)$. According to Corollary 1.4.9, there exists a constant $C_3 \geq 1$ dependent only on Ω such that

$$\|\tilde{\psi}_0\|_{H^{1/2}(\partial\Omega)} \leq C_3\|\psi_0\|_{H^{1/2}(\partial\Omega)}.$$

Since $\hat{\psi} = \hat{\psi}_k$, we obtain

$$\|\psi - \psi_k\|_{H^{1/2}(\partial\Omega)} \leq \frac{2C_1^2C_2C_3}{1-q}q^k\Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{1/2}(\partial\Omega)}\Big].$$

Let $k \to \infty$ in (7.63). Then we obtain $\psi = \frac{1}{2}\psi + K_\Omega\psi + \mathcal{S}^\Omega f$ and therefore $\frac{1}{2}\psi - K_\Omega\psi = \mathcal{S}^\Omega f$. Since $\frac{1}{2}v - K_\Omega v = \mathcal{S}^\Omega f$, we deduce that $\psi - v \in \mathrm{Ker}[(1/2)I - K_\Omega] = P_0(R^m)$. Thus, there exists a constant c such that $\psi = v + c$ on $\partial\Omega$. By virtue of Proposition 5.3.1,

$$u = \mathcal{S}^\Omega f + \mathcal{D}^\Omega\psi = \mathcal{S}^\Omega f + \mathcal{D}^\Omega(v + c) = v + c$$

in Ω. Thus, $\psi = v + c$ is the trace of $v + c = u$. Since $v \in W^{1,2}(\Omega)$ is a solution of the Neumann problem (7.56), we infer that $u = v + c \in W^{1,2}(\Omega)$ is a solution of the Neumann problem (7.56), too.

Lemma 7.22.11 *Let $\Omega \subset R^m$ be a bounded open set with compact Lipschitz boundary. Let $1 < p, q < \infty$, $-1 < s < 0$. If $f \in B_s^{p,q}(\partial\Omega)$, then $K_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega K'_\Omega f$.*

Proof Since $\mathcal{S}^\Omega = \mathcal{S}^\omega$, $K_\Omega = -K_\omega$, $K'_\Omega = -K'_\omega$ for $\omega = R^m \setminus \overline{\Omega}$, we can suppose that Ω is bounded.

Let $f \in L^p(\partial\Omega)$. Then $K_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega K'_\Omega f$ by Proposition 5.7.1. Since $L^p(\partial\Omega)$ is a dense subset of $B_s^{p,q}(\partial\Omega)$ (compare Theorem 1.23.1), the operators $\mathcal{S}^\Omega : B_s^{p,q}(\partial\Omega) \to B_{s+1}^{p,q}(\partial\Omega)$, $K_\Omega : B_{s+1}^{p,q}(\partial\Omega) \to B_{s+1}^{p,q}(\partial\Omega)$, $K'_\Omega : B_s^{p,q}(\partial\Omega) \to B_s^{p,q}(\partial\Omega)$ are continuous (see Corollary 5.4.11 and Proposition 7.7.1); we deduce that $K_\Omega \mathcal{S}^\Omega f = \mathcal{S}^\Omega K'_\Omega f$ for all $f \in B_s^{p,q}(\partial\Omega)$.

Theorem 7.22.12 *Let $\Omega \subset R^2$ be an unbounded domain with compact Lipschitz boundary. Let $f \in H^{-1/2}(\partial\Omega)$ be such that there exists a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56), i.e., $f \in \mathcal{E}_2(\partial\Omega) = \{g \in H^{-1/2}(\partial\Omega); \langle g, 1\rangle = 0\}$. Fix $\psi_0 \in H^{1/2}(\partial\Omega)$. Define ψ_k by the induction by (7.63). Then there exists $\psi \in H^{1/2}(\partial\Omega)$ such that $\psi_k \to \psi$ in $H^{1/2}(\partial\Omega)$. Moreover, the estimate (7.64) holds for $k \geq 1$ with constants $C \in (0, \infty)$, $q \in (0, 1)$ dependent only on Ω. Define*

$$u = \mathcal{S}^\Omega f + \mathcal{D}^\Omega \psi.$$

Then, $u \in \tilde{W}^{1,2}(\Omega)$ is a solution of the Neumann problem (7.56) and ψ is the trace of u.

Proof $f \in \mathcal{E}_2(\partial\Omega)$ by Theorem 7.20.2.

The operator $\frac{1}{2}I - K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3. According to Proposition 7.22.5, there exists a constant $q \in (0, 1)$ such that

$$\left\| \frac{1}{2}\varphi + K'_\Omega \varphi \right\|_{\mathcal{E}} \leq q\|\varphi\|_{\mathcal{E}} \qquad \forall \varphi \in \mathcal{E}_2(\partial\Omega).$$

Denote $X = \{\mathcal{S}^\Omega g; g \in \mathcal{E}_2(\partial\Omega)\}$. According to Theorem 7.6.3,

$$H^{1/2}(\partial\Omega) = X \oplus P_0(R^2),$$

where $P_0(R^2)$ is the space of constant functions. Clearly, X is isomorphic to the dual of $\mathcal{E}_2(\partial\Omega)$. We define the norm $\| \; \|_E$ on X by

$$\|\varphi\|_E = \sup\{|\langle g, \varphi\rangle|; g \in \mathcal{E}_2(\partial\Omega), \|g\|_{\mathcal{E}} \leq 1\}.$$

Since $\| \; \|_{\mathcal{E}}$ and $\| \; \|_{H^{-1/2}(\partial\Omega)}$ are equivalent norms on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.4, there exists a constant $C_1 \geq 1$ such that

$$C_1^{-1}\|\varphi\|_{H^{1/2}(\partial\Omega)} \leq \|\varphi\|_E \leq C_1\|\varphi\|_{H^{1/2}(\partial\Omega)} \qquad \forall \varphi \in X.$$

K'_Ω is a bounded linear operator on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.5. If $\varphi \in X$, then there exists $g \in \mathcal{E}_2(\partial\Omega)$ such that $\varphi = \mathcal{S}^\Omega g$. Then, $K_\Omega \varphi = K_\Omega \mathcal{S}^\Omega g = \mathcal{S}^\Omega K'_\Omega g \in X$

by Lemma 7.22.11. Thus, $K_\Omega(X) \subset X$. Since K_Ω is the adjoint operator of K'_Ω, Proposition 1.5.11 gives

$$\left\| \frac{1}{2}\varphi + K_\Omega \varphi \right\|_E \le q \|\varphi\|_E \qquad \forall \varphi \in X.$$

Let now $\psi_0 \in H^{1/2}(\partial\Omega)$ be fixed. Then, there exist $\tilde{\psi}_0 \in X$ and $\hat{\psi}_0 \in P_0(R^2)$ such that $\psi_0 = \tilde{\psi}_0 + \hat{\psi}_0$. If we define

$$\tilde{\psi}_k = \frac{1}{2}\tilde{\psi}_{k-1} + K_\Omega \tilde{\psi}_{k-1} + f, \quad \hat{\psi}_k = \frac{1}{2}\hat{\psi}_{k-1} + K_\Omega \hat{\psi}_{k-1},$$

then $\psi_k = \tilde{\psi}_k + \hat{\psi}_k$.

Denote $\omega = R^2 \setminus \overline{\Omega}$. Since $\hat{\psi}_0$ is constant, we have $K_\Omega \hat{\psi}_0 = -K_\omega \hat{\psi}_0 = -\hat{\psi}_0/2$ by Proposition 5.3.2. Therefore, $\hat{\psi}_0 \in \operatorname{Ker}[(1/2)I + K'_\Omega]$. Thus, $\hat{\psi}_k = 0$ for $k \ge 1$. Hence, $\psi_k = \tilde{\psi}_k$ for $k \ge 1$.

Define the operator T on X by

$$T\varphi = \frac{1}{2}\varphi + K_\Omega \varphi + \mathcal{S}^\Omega f.$$

If $\varphi, \phi \in X$, then

$$\|T\varphi - T\phi\|_E = \left\| \frac{1}{2}(\varphi - \phi) + K_\Omega(\varphi - \phi) \right\|_E \le q\|(\varphi - \phi)\|_E.$$

Theorem 1.4.18 gives that there exists $\tilde{\psi} \in X$ such that $\tilde{\psi}_k \to \tilde{\psi}$ in $H^{1/2}(\partial\Omega)$ and

$$\|\tilde{\psi} - \tilde{\psi}_k\|_E \le \frac{q^k}{1-q}\|\tilde{\psi}_1 - \tilde{\psi}_0\|_E \le \frac{q^k}{1-q}[(q+1)\|\tilde{\psi}_0\|_E + \|\mathcal{S}^\Omega f\|_E].$$

According to Corollary 5.4.11, there exists a constant $C_2 \ge 1$ dependent only on Ω such that

$$\|\mathcal{S}^\Omega f\|_{H^{1/2}(\partial\Omega)} \le C_2 \|f\|_{H^{-1/2}(\partial\Omega)}.$$

According to Corollary 1.4.9, there exists a constant C_3 dependent only on Ω such that

$$\|\tilde{\psi}_0\|_{H^{1/2}(\partial\Omega)} \le C_3 \|\psi_0\|_{H^{1/2}(\partial\Omega)}.$$

If $k \ge 1$, then $\psi = \tilde{\psi}$, $\psi_k = \tilde{\psi}_k$, and thus

$$\|\psi - \psi_k\|_{H^{1/2}(\partial\Omega)} \le \frac{2C_2C_3C_1^2}{1-q} q^k \Big[\|f\|_{H^{-1/2}(\partial\Omega)} + \|\psi_0\|_{H^{1/2}(\partial\Omega)} \Big].$$

We have supposed that there exists a solution $v \in \tilde{W}^{1,2}(\Omega)$ of the Neumann problem (7.56). Since $|\nabla v| \in L^2(\Omega)$, there exist constants c_1 and c_2 such that for $w(x) = v(x) - c_1 - c_2 \ln|x|$, we have $w(x) \to 0$ as $|x| \to \infty$ (see Proposition 2.17.6). Since w is harmonic in $\Omega \setminus \{0\}$, Proposition 2.17.3 gives $|\nabla w(x)| = O(|x|^{-2})$ as $|x| \to \infty$. Thus, $|\nabla c_2 \ln|x|| = |\nabla(v - w)| \in L^2(\Omega \setminus \{0\})$. Since $|\nabla c_2 \ln|x|| = c_2|x|^{-1}$, we deduce that $c_2 = 0$. Thus, $w = v - c_1 \in D^{1,2}(\Omega) = \tilde{W}^{1,2}(\Omega)$ (Proposition 3.38.4) is a solution of the Neumann problem (7.56). Lemma 7.22.1 gives

$$w = \mathcal{D}^{\Omega} w + \mathcal{S}^{\Omega} f.$$

According to boundary behavior of potentials (Proposition 7.6.2 and Theorem 5.3.5), we obtain

$$w = \frac{1}{2} w + K_{\Omega} w + \mathcal{S}^{\Omega} f.$$

We have

$$\psi = \lim_{k\to\infty} \psi_k = \lim_{k\to\infty} \left[\frac{1}{2}\psi_k + K_{\Omega}\psi_k + \mathcal{S}^{\Omega} f \right] = \frac{1}{2}\psi + K_{\Omega}\psi + \mathcal{S}^{\Omega} f.$$

Thus, $(\psi - w) = \frac{1}{2}(\psi - w) + K_{\Omega}(\psi - w)$. The operator $(1/2)I - K'_{\Omega}$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3. Since $(1/2)I - K_{\Omega}$ is an isomorphism on $H^{1/2}(\partial\Omega)$ by Proposition 1.6.4 and $[(1/2)I - K_{\Omega}](\psi - w) = 0$, we infer that $\psi - w = 0$. Thus, $\psi = w$ is the trace of w, and $u = \mathcal{D}^{\Omega}\psi + \mathcal{S}^{\Omega} f = \mathcal{D}^{\Omega} w + \mathcal{S}^{\Omega} f = w$ is a solution of the Neumann problem (7.56).

7.23 Successive Approximation for Dirichlet Problem

Let $\Omega \subset R^m$ be a bounded domain with connected Lipschitz boundary. Let $g \in H^{1/2}(\partial\Omega)$. According to Theorem 7.1.2, there exists a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{7.66}$$

We would like to calculate it.

In the indirect integral equation method, we look for a solution in the form of a double layer potential with an unknown density $\psi \in H^{1/2}(\partial\Omega)$. Then, $u = \mathcal{D}^{\Omega}\psi$ is a solution of the Dirichlet problem (7.66) in $H^1(\Omega)$ if and only if

$$[(1/2)I + K_{\Omega}]\psi = g.$$

(See Proposition 7.7.2 and Theorems 7.11.2 and 5.3.5.) So, we are to solve this equation.

In the indirect integral equation method, we use the integral representation of a solution of the Dirichlet problem. Let $u \in H^1(\Omega)$ be a solution of the Dirichlet problem (7.66). Then, the normal derivative $\partial u/\partial n \in [H^1(\Omega)]'$ is given by

$$\langle \partial u/\partial n, \varphi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m, \quad \varphi \in H^1(\Omega).$$

Lemma 7.22.1 gives that $\partial u/\partial n \in H^{-1/2}(\partial\Omega)$ and

$$u = \mathcal{S}^{\Omega} \frac{\partial u}{\partial n} + \mathcal{D}^{\Omega} g \quad \text{in } \Omega. \tag{7.67}$$

So, we need to calculate $\partial u/\partial n$. (Remark that $\langle \partial u/\partial n, 1 \rangle = 0$ by Theorem 7.20.2.) Calculating the normal derivative in (7.67), we obtain

$$\frac{\partial u}{\partial n} = \left[\frac{1}{2} I - K'_{\Omega} \right] \frac{\partial u}{\partial n} + \frac{\partial \mathcal{D}^{\Omega} g}{\partial n} \tag{7.68}$$

by Proposition 7.22.5. So, we are to solve this equation.

Theorem 7.23.1 *Let $\Omega \subset R^m$ be a bounded domain with connected Lipschitz boundary. Let $g \in H^{1/2}(\partial\Omega)$ and $u \in H^1(\Omega)$ be a solution of the Dirichlet problem (7.66). Then $\partial \mathcal{D}^{\Omega} g/\partial n = H^{\Omega} g \in H^{-1/2}(\partial\Omega)$. The normal derivative $\partial u/\partial n$ is a unique solution of the equation*

$$f = [(1/2)I - K'_{\Omega}] f + H^{\Omega} g \tag{7.69}$$

in $H^{-1/2}(\Omega)$. Fix $f_0 \in H^{-1/2}(\Omega)$ and put

$$f_k = [(1/2)I - K'_{\Omega}] f_{k-1} + H^{\Omega} g, \qquad k \in N.$$

Then $f_k \to \partial u/\partial n$ in $H^{-1/2}(\partial\Omega)$. Moreover, there exist constants $q \in (0,1)$ and $C \in (0,\infty)$ dependent only on Ω such that

$$\|\partial u/\partial n - f_k\|_{H^{-1/2}(\partial\Omega)} \le C q^k \left[\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{-1/2}(\partial\Omega)} \right]. \tag{7.70}$$

The function u is given by (7.67).

Proof $\partial \mathcal{D}^{\Omega} g/\partial n = H^{\Omega} g \in H^{-1/2}(\partial\Omega)$ by Proposition 7.7.3. Moreover there exists a constant C_1 dependent only on Ω such that

$$\|H^{\Omega} g\|_{H^{-1/2}(\partial\Omega)} \le C_1 \|g\|_{H^{1/2}(\partial\Omega)}.$$

We have $\partial u/\partial n \in H^{-1/2}(\partial\Omega)$ by Lemma 7.22.1, and (7.68) forces that $\partial u/\partial n$ is a solution of (7.69). Denote $\omega = R^m \setminus \overline{\Omega}$. Since $(1/2)I - K'_\omega = (1/2)I + K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3, $\partial u/\partial n$ is a unique solution of the Eq. (7.69) in $H^{-1/2}(\Omega)$. Since $\langle \partial u/\partial n, 1\rangle = 0$ by Theorem 7.20.2, the single layer potential $\mathcal{S}^\Omega(\partial u/\partial n) \in \tilde{W}^{1,2}(\omega)$ is a weak solution of the Neumann problem

$$\Delta v = 0 \text{ in } \omega, \quad \frac{\partial v}{\partial n} = \left[\frac{1}{2}I - K'_\omega\right]\frac{\partial u}{\partial n} = H^\Omega g \text{ on } \partial\omega$$

by Proposition 7.22.5. According to Theorems 7.22.6 and 7.22.8 and Proposition 7.22.5, there exists $f \in H^{-1/2}(\partial\Omega)$ such that $f_k \to f$ in the space $H^{-1/2}(\partial\Omega)$, $[(1/2)I - K'_\omega]f = H^\Omega g$ and

$$\|f - f_k\|_{H^{-1/2}(\partial\Omega)} \le C_2 q^k \left[\|H^\Omega g\|_{H^{-1/2}(\partial\Omega)} + \|f_0\|_{H^{-1/2}(\partial\Omega)}\right]$$

where constants $q \in (0,1)$, $C_2 \in (0,\infty)$ depend only on Ω. Since $[(1/2)I - K'_\omega](\partial u/\partial n) = H^\Omega g = [(1/2)I - K'_\omega]f$ and the operator $[(1/2)I - K'_\omega]$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3, we infer that $\psi = \partial u/\partial n$. Clearly, the estimate (7.70) holds with $C = C_1 C_2$.

Theorem 7.23.2 *Let $\Omega \subset R^m$ be a bounded domain with connected Lipschitz boundary. Let $g \in H^{1/2}(\partial\Omega)$ and $u \in H^1(\Omega)$ be a solution of the Dirichlet problem (7.66). Then there exists a unique solution $f \in H^{1/2}(\partial\Omega)$ of the equation*

$$[(1/2)I + K_\Omega]f = g.$$

Fix $f_0 \in H^{1/2}(\Omega)$ and put

$$f_k = [(1/2)I - K_\Omega]f_{k-1} + g, \qquad k \in N.$$

Then $f_k \to f$ in $H^{1/2}(\partial\Omega)$. Moreover, there exist constants $q \in (0,1)$ and $C \in (0,\infty)$ dependent only on Ω such that

$$\|f - f_k\|_{H^{1/2}(\partial\Omega)} \le C q^k \left[\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{1/2}(\partial\Omega)}\right], \quad k \ge 1. \tag{7.71}$$

We have $u = \mathcal{D}^\Omega f$.

Proof Denote $\omega = R^m \setminus \overline{\Omega}$. The operator $\frac{1}{2}I - K'_\omega = \frac{1}{2}I + K'_\Omega$ is an isomorphism on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.3. So, $\frac{1}{2}I + K_\Omega$ is an isomorphism on $H^{1/2}(\partial\Omega)$ by Proposition 1.6.4. Hence, there exists a unique solution $f \in H^{1/2}(\partial\Omega)$ of the equation $[(1/2)I + K_\Omega]f = g$. The function $\mathcal{D}^\Omega f$ is a solution of the Dirichlet problem (7.66) in $H^1(\Omega)$ by Proposition 7.7.2 and Theorems 7.11.2 and 5.3.5. The uniqueness of the Dirichlet problem gives that $u = \mathcal{D}^\Omega f$ (see Theorem 7.1.2).

According to Proposition 7.22.5, there exists a constant $q \in (0,1)$ such that

$$\|(1/2)\psi - K'_\Omega\psi\|_{\mathcal{E}} = \|(1/2)\psi + K'_\omega\psi\|_{\mathcal{E}} \le q\|\psi\|_{\mathcal{E}} \quad \forall\psi \in \mathcal{E}_2(\partial\Omega). \tag{7.72}$$

Suppose first that $m > 2$. Then $\mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$. We define the norm $\| \ \|_E$ on $H^{1/2}(\partial\Omega)$ by

$$\|\varphi\|_E = \sup\{|\langle\psi,\varphi\rangle|; \psi \in \mathcal{E}_2(\partial\Omega), \|\psi\|_{\mathcal{E}} \le 1\}. \tag{7.73}$$

Since $\| \ \|_{\mathcal{E}}$ and $\| \ \|_{H^{-1/2}(\partial\Omega)}$ are equivalent norms on $H^{-1/2}(\partial\Omega)$ by Proposition 7.22.5, there exists a constant $C_1 \ge 1$ such that

$$C_1^{-1}\|\varphi\|_{H^{1/2}(\partial\Omega)} \le \|\varphi\|_E \le C_1\|\varphi\|_{H^{1/2}(\partial\Omega)} \tag{7.74}$$

for all $\varphi \in H^{1/2}(\partial\Omega)$. Since K_Ω is the adjoint operator of K'_Ω, Proposition 1.5.11 gives

$$\|(1/2)\varphi - K_\Omega\varphi\|_E \le q\|\varphi\|_E \tag{7.75}$$

for all $\varphi \in H^{1/2}(\partial\Omega)$. Define

$$T\varphi = (1/2)\varphi - K_\Omega\varphi + g.$$

If $\varphi,\psi \in H^{1/2}(\partial\Omega)$, then $\|T\varphi - T\psi\|_E \le q\|\varphi - \psi\|_E$. Theorem 1.4.18 gives that $f_k \to f$ in $H^{1/2}(\partial\Omega)$ and

$$\|f - f_k\|_E \le \frac{q^k}{1-q}\|f_1 - f_0\|_E \le \frac{q^k}{1-q}\left[\|g\|_E + \|f_0\|_E\right].$$

Thus,

$$\|f - f_k\|_{H^{1/2}(\partial\Omega)} \le \frac{C_1^2}{1-q}\, q^k \left[\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{1/2}(\partial\Omega)}\right].$$

Let now $m = 2$. Then $\mathcal{E}_2(\partial\Omega) = \{\psi \in H^{-1/2}(\partial\Omega); \langle\psi,1\rangle = 0\}$. Denote $X = \{\mathcal{S}^\Omega g; g \in \mathcal{E}_2(\partial\Omega)\}$. According to Theorem 7.6.3,

$$H^{1/2}(\partial\Omega) = X \oplus P_0(R^2),$$

where $P_0(R^2)$ is the space of constant functions. Clearly, X is isomorphic to the dual of $\mathcal{E}_2(\partial\Omega)$. We define the norm $\| \ \|_E$ on X by (7.73). Since $\| \ \|_{\mathcal{E}}$ and $\| \ \|_{H^{-1/2}(\partial\Omega)}$ are equivalent norms on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.4, there exists a constant $C_1 \ge 1$ such that (7.74) holds for all $\varphi \in X$. The operator K'_Ω is an operator on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.5. If $\varphi \in X$, then there exists $\psi \in \mathcal{E}_2(\partial\Omega)$ such that $\varphi = \mathcal{S}^\Omega\psi$.

Then, $K_\Omega\varphi = K_\Omega\mathcal{S}^\Omega\psi = \mathcal{S}^\Omega K'_\Omega\psi \in X$ by Lemma 7.22.11. Thus, $K_\Omega(X) \subset X$. Since K_Ω is the adjoint operator of K'_Ω, Proposition 1.5.11 and (7.72) give (7.75) for all $\varphi \in X$. Let now $f_0 \in H^{1/2}(\partial\Omega)$ be fixed. Then there exist $\tilde{f}_0, \tilde{g} \in X$ and $\hat{f}_0, \hat{g} \in P_0(R^2)$ such that $f_0 = \tilde{f}_0 + \hat{f}_0$, $g = \tilde{g} + \hat{g}$. If we define

$$\tilde{f}_k = \frac{1}{2}\tilde{f}_{k-1} - K_\Omega\tilde{f}_{k-1} + \tilde{g}, \quad \hat{f}_k = \frac{1}{2}\hat{f}_{k-1} - K_\Omega\hat{f}_{k-1} + \hat{g},$$

then $f_k = \tilde{f}_k + \hat{f}_k$. Since $\hat{f}_0$ is constant, we have $K_\Omega\hat{f}_0 = \hat{f}_0/2$ by Proposition 5.3.2. Therefore, $\hat{f}_k = \hat{g}$ for $k \geq 1$. Define

$$\tilde{T}\varphi = (1/2)\varphi - K_\Omega\varphi + \tilde{g}, \quad \varphi \in X.$$

If $\varphi, \psi \in X$, then $\|\tilde{T}\varphi - \tilde{T}\psi\|_E \leq q\|\varphi - \psi\|_E$. Theorem 1.4.18 gives that there exists $\tilde{f} \in X$ such that $\tilde{f}_k \to \tilde{f}$ in $H^{1/2}(\partial\Omega)$ and

$$\|\tilde{f} - \tilde{f}_k\|_E \leq \frac{q^k}{1-q}\|\tilde{f}_1 - \tilde{f}_0\|_E \leq \frac{q^k}{1-q}\left[\|\tilde{g}\|_E + \|\tilde{f}_0\|_E\right].$$

By virtue of (7.74), we get

$$\|\tilde{f} - \tilde{f}_k\|_{H^{1/2}(\partial\Omega)} \leq \frac{C_1^2 q^k}{1-q}\left[\|\tilde{g}\|_{H^{1/2}(\partial\Omega)} + \|\tilde{f}_0\|_{H^{1/2}(\partial\Omega)}\right]. \tag{7.76}$$

Since $\hat{f}_k = \hat{g}$ for $k \geq 1$, we have

$$\tilde{f} + \hat{g} = \lim_{k\to\infty} f_k = \lim_{k\to\infty}\left[\frac{1}{2}f_k - K_\Omega f_k + g\right] = \left[\frac{1}{2}I - K_\Omega\right](\tilde{f} + \hat{g}) + g.$$

Since $[(1/2)I + K_\Omega](\tilde{f} + \hat{g}) = g$, we deduce that $\tilde{f} + \hat{g} = f$. Thus, $f_k \to f$ in $H^{1/2}(\partial\Omega)$. According to Corollary 1.4.9,

$$\|\tilde{g}\|_{H^{1/2}(\partial\Omega)} \leq C_2\|g\|_{H^{1/2}(\partial\Omega)}, \quad \|\tilde{f}_0\|_{H^{1/2}(\partial\Omega)} \leq C_2\|f_0\|_{H^{1/2}(\partial\Omega)},$$

where a constant C_2 depends only on Ω. By virtue of (7.76),

$$\begin{aligned}\|f - f_k\|_{H^{1/2}(\partial\Omega)} &= \|(\tilde{f} + \hat{g}) - (\tilde{f}_k + \hat{g})\|_{H^{1/2}(\partial\Omega)} \\ &\leq \frac{C_2 C_1^2 q^k}{1-q}\left[\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{1/2}(\partial\Omega)}\right].\end{aligned}$$

7.24 The Robin Problem in $B_s^{p,q}(\Omega)$

In this section, we study the Neumann problem and the Robin problem for the Poisson equation in the Sobolev space $W^{s,p}(\Omega)$ and in the Besov space $B_s^{p,q}(\Omega)$ for a bounded domain Ω and $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. If $p = 2$, we suppose that Ω has Lipschitz boundary, if $p \neq 2$ we suppose that Ω has Ljapunov boundary. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Suppose that F is a closed subset of R^m, $1 < p, q < \infty$, and $-\infty < s < 0$. Define $B_s^{p,q}(F) = \{T \in B_s^{p,q}(R^m); \operatorname{spt} T \subset F\}$, $W^{s,p}(F) = \{T \in W^{s,p}(R^m); \operatorname{spt} T \subset F\}$.

Let now $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. Denote $p' = p/(p-1)$, $q' = q/(q-1)$. If $T \in [B_{-s}^{p',q'}(\Omega)]'$ (or $T \in [W^{-s,p'}(\Omega)]'$), define $\langle T, \varphi\rangle = \langle T, \varphi|\Omega\rangle$. Then, $T \in [B_{-s}^{p',q'}(R^m)]' = B_s^{p,q}(R^m)$ (or $T \in [W^{-s,p'}(R^m)]' = W^{s,p}(R^m)$), respectively (see Theorems 1.23.1, 1.23.3, 1.23.5, and 1.22.1). Since T is supported on $\overline{\Omega}$, one has $T \in B_s^{p,q}(\overline{\Omega})$ (or $T \in W^{s,p}(\overline{\Omega})$), respectively. Thus, $[B_{-s}^{p',q'}(\Omega)]'$ is continuously imbedded into $B_s^{p,q}(\overline{\Omega})$, and $[W^{-s,p'}(\Omega)]'$ is continuously imbedded into $W^{s,p}(\overline{\Omega})$.

Suppose now that $1/p < -s$. Let $T \in B_s^{p,q}(\overline{\Omega})$ (or $T \in W^{s,p}(\overline{\Omega})$). For $\varphi \in B_{-s}^{p',q'}(\Omega)$, choose an extension $\tilde{\varphi} \in B_{-s}^{p',q'}(R^m)$ of φ (or for $\varphi \in W^{-s,p'}(\Omega)$, choose an extension $\tilde{\varphi} \in W^{-s,p'}(R^m)$). Define $\langle T, \varphi\rangle = \langle T, \tilde{\varphi}\rangle$. If $\check{\varphi}$ is another extension, then $\tilde{\varphi} = \check{\varphi}$ on $\overline{\Omega}$ by Theorem 1.24.7. Since T is supported on $\overline{\Omega}$, one has $\langle T, \tilde{\varphi}\rangle = \langle T, \check{\varphi}\rangle$, and T is well defined as a term of $[B_{-s}^{p',q'}(\Omega)]'$ (or $[W^{-s,p'}(\Omega)]'$), respectively. So, we can identify $[W^{-s,p'}(\Omega)]' = W^{s,p}(\overline{\Omega})$, $[B_{-s}^{p',q'}(\Omega)]' = B_s^{p,q}(\overline{\Omega})$.

Proposition 7.24.1 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $0 < s < 1$, $1 < p, q < \infty$. If $\psi \in B_{-s}^{p,q}(\partial\Omega)$ then $u = \mathcal{S}^\Omega\psi \in B_{1+1/p-s}^{p,q}(\Omega)$ is a very weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \psi/2 - K'_\Omega\psi$ on $\partial\Omega$. If $\psi \in W^{-1/p,p}(\partial\Omega)$, then $u = \mathcal{S}^\Omega\psi \in W^{1,p}(\Omega)$ is a weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \psi/2 - K'_\Omega\psi$ on $\partial\Omega$.*

Proof If $\psi \in B_{-s}^{p,q}(\partial\Omega)$, then $\mathcal{S}^\Omega\psi \in B_{1+1/p-s}^{p,q}(\Omega)$ by Proposition 7.6.1. If $\psi \in L^\infty(\partial\Omega)$, then $\mathcal{S}^\Omega\psi$ is a very weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \psi/2 - K'_\Omega\psi$ on $\partial\Omega$ by Lemma 5.20.4. Let now $\psi \in B_{-s}^{p,q}(\partial\Omega)$. Choose $\psi_k \in L^\infty(\partial\Omega)$ such that $\psi_k \to \psi$ in $B_{-s}^{p,q}(\partial\Omega)$. Then $\psi_k/2 - K'_\Omega\psi_k \to \psi/2 - K'_\Omega\psi$ in $B_{-s}^{p,q}(\partial\Omega)$ by Proposition 7.7.1 and $\mathcal{S}^\Omega\psi_k \to \mathcal{S}^\Omega\psi$ in $B_{1+1/p-s}(\Omega)$ by Proposition 7.6.1. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\langle \psi/2 - K'_\Omega\psi, \varphi\rangle = \lim_{k\to\infty} \langle \psi_k/2 - K'_\Omega\psi_k, \varphi\rangle = \lim_{k\to\infty} \left[\int_\Omega (\mathcal{S}^\Omega\psi_k)(-\Delta\varphi)\, d\mathcal{H}_m \right.$$

$$\left. + \int_{\partial\Omega} (\mathcal{S}^\Omega\psi_k) \frac{\partial\varphi}{\partial n}\, d\mathcal{H}_{m-1} \right] = \int_\Omega (\mathcal{S}^\Omega\psi)(-\Delta\varphi)\, d\mathcal{H}_m + \int_{\partial\Omega} (\mathcal{S}^\Omega\psi) \frac{\partial\varphi}{\partial n}\, d\mathcal{H}_{m-1} \Big].$$

Let now $\psi \in W^{-1/p,p}(\partial\Omega)$. Then $\mathcal{S}^\Omega\psi \in W^{1,p}(\Omega)$ by Proposition 7.6.1. Therefore, there exists $F \in [W^{1,p'}(\Omega)]'$ such that $\mathcal{S}^\Omega\psi$ is a weak solution of the Neumann problem $-\Delta u = F$ in Ω, $\partial u/\partial n = F$ on $\partial\Omega$. The function $\mathcal{S}^\Omega\psi$ is a very weak solution of this problem by Proposition 7.19.2. According to Theorem 1.23.3, one has $W^{-1/p,p}(\partial\Omega) = B_{-1/p}^{p,p}(\partial\Omega)$. We have proved that $\mathcal{S}^\Omega\psi$ is a very weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \psi/2 - K'_\Omega\psi$ on $\partial\Omega$. Thus, $F = \psi/2 - K'_\Omega\psi$. So, $\mathcal{S}^\Omega\psi$ is a weak solution of the Neumann problem $\Delta u = 0$ in Ω, $\partial u/\partial n = \psi/2 - K'_\Omega\psi$ on $\partial\Omega$.

Proposition 7.24.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p, q < \infty$, $1/p < s < 1 + 1/p$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $u \in B_s^{p,q}(\Omega)$ be a very weak solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. Then, u is constant. If $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then $u \equiv 0$.*

Proof Since $\partial u/\partial n = -hu$, Lemma 7.9.1 gives that $u = \mathcal{D}^\Omega u - \mathcal{S}^\Omega(hu)$ in Ω. We have $u \in B_{s-1/p}^{p,q}(\partial\Omega) \subset L^p(\partial\Omega)$ by Theorem 1.24.7, Proposition 1.23.7, and Theorem 1.23.4. Using the nontangential limit, we obtain $u = u/2 + K_\Omega u - \mathcal{S}^\Omega hu$ (see Theorem 5.3.5 and Proposition 5.2.2). The operator $f \mapsto f/2 - K_\Omega f + \mathcal{S}^\Omega(hf)$ is a Fredholm operator with index 0 in $B_{s-1/p}^{p,q}(\partial\Omega)$ (see Proposition 7.22.3) and in $W^{1,2}(\partial\Omega)$ by Proposition 5.14.4, Corollary 5.4.10, and Theorem 1.8.3. Since $u/2 - K_\Omega u + \mathcal{S}^\Omega(hu) = 0$ on $\partial\Omega$, Lemma 1.8.4 gives $u \in W^{1,2}(\partial\Omega)$. Since $u = \mathcal{D}^\Omega u - \mathcal{S}^\Omega(hu)$ in Ω, we have $M_a(u) \in L^2(\partial\Omega)$ (see Theorem 5.3.5 and 5.2.2), $M_a(\nabla u) \in L^2(\partial\Omega)$ (see Theorem 5.5.2 and Proposition 5.2.8). Lemma 1.28.1 gives that $u \in W^{1,2}(\Omega)$. Proposition 7.19.2 forces that $u \in W^{1,2}(\Omega)$ is a weak solution of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = 0$ on $\partial\Omega$. If $h \equiv 0$, then u is constant by Theorem 7.20.2. If $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then $u \equiv 0$ by Theorem 7.21.3.

Proposition 7.24.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, $1/p < s < 1 + 1/p$, $h \in L^\infty(\partial\Omega)$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. For $u \in B_s^{p,q}(\Omega)$ or $u \in W^{s,p}(\Omega)$, define the distribution $T_h u$ by*

$$\langle T_h u, \varphi \rangle = \int_\Omega u(-\Delta\varphi) \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \left(\frac{\partial\varphi}{\partial n} + h\varphi \right) u \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T_h is a bounded linear operator from $B_s^{p,q}(\Omega)$ to $[B_{2-s}^{p',q'}(\Omega)]'$ and from $W^{1,p}(\Omega)$ to $[W^{1,p'}(\Omega)]'$.

Proof Without loss of generality, we can suppose that $\overline{\Omega} \subset B(0;1)$. Theorems 1.24.7 and 1.24.2 give that there exists a constant C_1 such that

$$\|u\|_{B_{s-1/p}^{p,q}(\partial\Omega)} \leq C_1 \|u\|_{B_s^{p,q}(\Omega)}, \quad \|u\|_{W^{1-1/p,p}(\partial\Omega)} \leq C_1 \|u\|_{W^{1,p}(\Omega)}.$$

According to Theorem 1.24.1, Proposition 1.23.7, and Theorem 1.23.4, there exists a constant C_2 such that

$$\|hu\|_{L^p(\partial\Omega)} \leq C_2\|u\|_{B_s^{p,q}(\Omega)}, \quad \|hu\|_{L^p\partial\Omega)} \leq C_2\|u\|_{W^{1,p}(\Omega)}.$$

Since the trace is a bounded linear operator from $W^{1,p'}(\Omega)$ to $L^{p'}(\partial\Omega)$ and from $B_{2-s}^{p',q'}(\Omega)$ to $L^{p'}(\partial\Omega)$, Theorem 1.14.3 gives that $T_h - T_0$ is a bounded linear operator from $B_s^{p,q}(\Omega)$ to $[B_{2-s}^{p',q'}(\Omega)]'$ and from $W^{1,p}(\Omega)$ to $[W^{1,p'}(\Omega)]'$.

Theorem 7.6.3 gives that there exists f such that $\mathcal{S}^\Omega f = u$ on $\partial\Omega$,

$$\|f\|_{B_{s-1/p-1}^{p,q}(\partial\Omega)} \leq C_3\|u\|_{B_s^{p,q}(\Omega)}, \quad \|f\|_{W^{-1/p,p}(\partial\Omega)} \leq C_3\|u\|_{W^{1,p}(\Omega)}.$$

According to Proposition 7.6.1,

$$\|\mathcal{S}^\Omega f\|_{B_s^{p,q}(\Omega)} \leq C_4\|u\|_{B_s^{p,q}(\Omega)}, \quad \|\mathcal{S}^\Omega f\|_{W^{1,p}(\Omega)} \leq C_4\|u\|_{W^{1,p}(\Omega)}.$$

Proposition 7.24.1 gives that $T_0(\mathcal{S}^\Omega f) = f/2 - K'_\Omega f$. Denote $w = u - \mathcal{S}^\Omega f$. Then $w = 0$ on $\partial\Omega$. Define $w = 0$ on $R^m \setminus \overline{\Omega}$. Then $w \in \overset{\circ}{B}{}_s^{p,q}(\Omega) \subset B_s^{p,q}(R^m)$ for $u \in B_s^{p,q}(\Omega)$ and $w \in \overset{\circ}{W}{}^{1,p}(\Omega) \subset W^{1,p}(R^m)$ for $u \in W^{1,p}(\Omega)$ (see Theorem 1.24.9 and Proposition 1.24.3). If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then

$$\langle T_0 w, \varphi\rangle = \int\limits_{R^m} (-w\Delta\varphi)\, d\mathcal{H}_m.$$

Thus, $T_0 w = -\Delta w$. Choose a bounded open set G with smooth boundary such that $\overline{\Omega} \subset G$. Theorem 1.23.8 and Proposition 1.21.11 give

$$\|T_0 w\|_{B_{s-2}^{p,q}(G)} \leq C_5\|w\|_{B_s^{p,q}(R^m)} \leq C_5(1 + C_4)\|u\|_{B^s(\Omega)},$$

$$\|T_0 w\|_{W^{-1,p}(G)} \leq C_5\|w\|_{W^{1,p}(R^m)} \leq C_5(1 + C_4)\|u\|_{W^{1,p}(\Omega)}.$$

Since $T_0 w$ is supported on $\overline{\Omega}$, Theorem 1.23.1 forces

$$\|T_0 w\|_{[B_{2-s}^{p',q'}(\Omega)]'} \leq C_5(1 + C_4)\|u\|_{B^s(\Omega)},$$

$$\|T_0 w\|_{[W^{1,p'}(\Omega)]'} \leq C_5(1 + C_4)\|u\|_{W^{1,p}(\Omega)}.$$

Proposition 7.7.1 gives that

$$\|f/2 - K'_\Omega f\|_{B_{s-1/p-1}^{p,q}(\partial\Omega)} \leq C_6\|f\|_{B_{s-1/p-1}^{p,q}(\partial\Omega)} \leq C_3 C_6\|u\|_{B_s^{p,q}(\Omega)},$$

$$\|f/2 - K'_\Omega f\|_{W^{-1/p,p}(\partial\Omega)} \leq C_6\|f\|_{W^{-1/p,p}(\partial\Omega)} \leq C_3 C_6\|u\|_{W^{1,p}(\Omega)}.$$

Since the trace is a bounded operator from $B^{p',q'}_{2-s}(\Omega)$ to $B^{p',q'}_{2-s-1/p'}(\partial\Omega)$ (Theorem 1.24.7) and $B^{p,q}_{s-1/p-1}(\partial\Omega) = [B^{p',q'}_{1+1/p-s}(\partial\Omega)]' = [B^{p',q'}_{2-s-1/p'}(\partial\Omega)]'$ by Theorem 1.23.1, we infer that

$$\|f/2 - K'_\Omega f\|_{[B^{p',q'}_{2-s}(\Omega)]'} \le C_7 \|u\|_{B^{p,q}_s(\Omega)},$$

$$\|f/2 - K'_\Omega f\|_{[W^{1,p'}(\Omega)]'} \le C_7 \|u\|_{W^{1,p}(\Omega)}.$$

Since $T_0 u = T_0 w + T_0 \mathcal{S}^\Omega f = T_0 w + f/2 - K'_\Omega f$, we obtain

$$\|T_0 u\|_{[B^{p',q'}_{2-s}(\Omega)]'} \le [C_7 + C_5(1 + C_4)]\|u\|_{B^{p,q}_s(\Omega)},$$

$$\|T_0 u\|_{[W^{1,p'}(\Omega)]'} \le [C_7 + C_5(1 + C_4)]\|u\|_{W^{1,p}(\Omega)}.$$

Lemma 7.24.4 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, $-1 < s < 0$. If $p \ne 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$. Define $Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$. Denote $T = (1/2)I - K'_\Omega + h\mathcal{S}^\Omega$, $\tilde T = [(1/2)I - K'_\Omega](I - P) + h[\mathcal{S}^\Omega(I-P) + P]$.*

- *If $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$ and $m > 2$, then T is an isomorphism on $B^{p,q}_s(\partial\Omega)$.*
- *If $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$ and $m = 2$, then $\tilde T$ is an isomorphism on $B^{p,q}_s(\partial\Omega)$.*
- *If $h \equiv 0$ and Ω is unbounded, then T is an isomorphism on $B^{p,q}_s(\partial\Omega)$.*
- *If $h \equiv 0$ and Ω is bounded, then T is an isomorphism on $\{f \in B^{p,q}_s(\partial\Omega); \langle f, 1\rangle = 0\}$.*

Proof The operators T and $\tilde T$ are Fredholm operators with index 0 in $B^{p,q}_s(\partial\Omega)$ by Proposition 7.22.3 and Theorem 1.8.3. If $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$ and $m > 2$ ($m = 2$), then T ($\tilde T$) is an isomorphism on $W^{-1,p}(\partial\Omega)$ by Theorem 5.20.8. If $h \equiv 0$ and Ω is unbounded then T is an isomorphism $W^{-1,p}(\partial\Omega)$ by Theorem 5.20.7. If $h \equiv 0$ and Ω is bounded, then T is an isomorphism on $\{f \in W^{-1,p}(\partial\Omega); \langle f, 1\rangle = 0\}$ by Theorem 5.20.6. Lemma 1.8.4 gives the proposition.

Theorem 7.24.5 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, $1/p < s < 1 + 1/p$, $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H}_{m-1} > 0$. If $p \ne 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let F be a distribution supported on $\overline{\Omega}$. Then, there exists a very weak solution $u \in B^{p,q}_s(\Omega)$ of the Robin problem*

$$-\Delta u = F|\Omega \quad \text{in } \Omega, \qquad \partial u/\partial n + hu = F|\partial\Omega \quad \text{on } \partial\Omega \tag{7.77}$$

if and only if $F \in [B^{p',q'}_{2-s}(\Omega)]'$. This solution is unique and

$$\|u\|_{B^{p,q}_s(\Omega)} \le C\|F\|_{[B^{p',q'}_{2-s}(\Omega)]'}$$

where a constant C depends only on Ω, p, q, and s.

Proof Suppose first that u is a very weak solution of the Robin problem (7.77). Proposition 7.24.3 gives that $F \in [B^{p',q'}_{2-s}(\Omega)]'$. If $F = 0$, then $u \equiv 0$ by Proposition 7.24.2.

Suppose now that $F \in [B^{p',q'}_{2-s}(\Omega)]'$ is given. Define $\tilde{F}$ by

$$\langle \tilde{F}, \varphi \rangle = \langle F, \varphi|\Omega \rangle, \qquad \varphi \in B^{p',q'}_{2-s}(R^m).$$

Then, $\tilde{F} \in [B^{p',q'}_{2-s}(R^m)]' = B^{p,q}_{s-2}(R^m)$ (see Theorem 1.23.1). Since $\tilde{F}$ is supported in $\overline{\Omega}$, we can define $v = h_m * \tilde{F}$. We have $-\Delta v = \tilde{F}$ by Theorem 3.1.5. Since $\tilde{F} \in B^{p,q}_{s-2}(R^m)$, Proposition 3.18.5 forces $v \in B^{p,q}_s(\Omega)$. Define

$$\langle G, \varphi \rangle = \int\limits_{\Omega} v(-\Delta\varphi)\, d\mathcal{H}_m + \int\limits_{\partial\Omega} v \left(\frac{\partial \varphi}{\partial n} + h\varphi \right) d\mathcal{H}_{m-1}.$$

Then v is a very weak solution of the problem $-\Delta v = G|\Omega$ in Ω, $\partial v/\partial n + hv = G|\partial\Omega$ on $\partial\Omega$. So, $G \in [B^{p',q'}_{2-s}(\Omega)]'$. Since $-\Delta v = G$ in Ω and $-\Delta v = \tilde{F}$, we deduce that $F = G$ in Ω. Thus, $F - G$ is supported on $\partial\Omega$. Using properties of traces of $B^{p',q'}_{2-s}(\Omega)$ (see Theorem 1.24.7), we deduce that $F - G \in [B^{p',q'}_{2-s-1/p'}(\partial\Omega)]' = B^{p,q}_{s-1-1/p}(\partial\Omega)$.

If $m > 2$, define $Pf = 0$; for $m = 2$, define $Pf = \oint_{\partial\Omega} f\, d\mathcal{H}_1$. We shall look for a very weak solution of the Robin problem

$$-\Delta w = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} + hw = F - G \quad \text{on } \partial\Omega \tag{7.78}$$

in the form

$$\tilde{S}f = \mathcal{S}^{\Omega}(I - P)\psi + P\psi$$

with $\psi \in B^{p,q}_{s-1-1/p}(\partial\Omega)$. Proposition 7.24.1 gives that $\tilde{S}\psi$ is a very weak solution of the problem (7.78) if and only if

$$\left(\frac{1}{2} I - K'_{\Omega} \right) (I - P)\psi + h\tilde{S}\psi = F - G. \tag{7.79}$$

The operator $[(1/2)I - K'_{\Omega}](I - P) + h\tilde{S}$ is an isomorphism on $B^{p,q}_{s-1-1/p}(\partial\Omega)$ by Lemma 7.24.4. Therefore, there exists a solution $\psi \in B^{p,q}_{s-1-1/p}(\partial\Omega)$ of the Eq. (7.79). Hence, $\tilde{\mathcal{S}}\psi \in B^{p,q}_s(\Omega)$ is a very weak solution of the problem (7.78). So, $v + \tilde{\mathcal{S}}\psi \in B^{p,q}_s(\Omega)$ is a very weak solution of the Robin problem (7.77).

Define the distribution $T_h u$ by

$$\langle T_h u, \varphi \rangle = \int\limits_{\Omega} u(-\Delta\varphi)\, d\mathcal{H}_m + \int\limits_{\partial\Omega} \left(\frac{\partial \varphi}{\partial n} + hu\varphi \right) d\mathcal{H}_{m-1}.$$

Remark that u is a very weak solution of the Robin problem (7.77) if $T_h u = F$. Proposition 7.24.3 gives that T_h is a bounded linear operator from $B_s^{p,q}(\Omega)$ to $[B_{2-s}^{p',q'}(\Omega)]'$. We have proved that T_h is one-to-one and onto. Theorem 1.6.3 gives that it is an isomorphism.

Theorem 7.24.6 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p < \infty$, $p' = p/(p-1)$, $1/p < s < 1 + 1/p$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If F is a distribution supported on $\overline{\Omega}$, then there exists a very weak solution $u \in W^{s,p}(\Omega)$ of the Robin problem (7.77) if and only if $F \in [W^{2-s,p'}(\Omega)]'$. This solution u is unique and*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \| F\|_{[W^{2-s,p'}(\Omega)]'}$$

where a constant C does not depend on F. If $s \geq 1$, then u is a unique weak solution of the Robin problem (7.77).

Proof Suppose first that $s = 1$. If there exists a very weak solution $u \in W^{s,p}(\Omega)$ of the Robin problem (7.77), then $F \in [W^{1,p'}(\Omega)]'$ by Proposition 7.19.2. Moreover, $u \in W^{1,p}(\Omega)$ is a very weak solution of (7.77) if and only if u is a weak solution of this problem.

If $u \in W^{1,p}(\Omega)$ is a very weak solution of (7.77) with $F = 0$, then $u \equiv 0$ by Corollary 7.10.7 and Proposition 7.24.2.

Suppose now that $F \in [W^{1,p'}(\Omega)]'$ is given. Define $\tilde{F}$ by $\langle \tilde{F}, \varphi\rangle = \langle F, \varphi|\Omega\rangle$ for $\varphi \in \overset{\circ}{W}{}^{1,p'}(R^m)$.Then $\tilde{F} \in [\overset{\circ}{W}{}^{1,p'}(R^m)]' = W^{-1,p}(R^m)$. Since $\tilde{F}$ is supported in $\overline{\Omega}$, we can define $v = h_m * \tilde{F}$. We have $-\Delta v = \tilde{F}$ by Theorem 3.1.5. Proposition 3.18.5 gives that $v \in W^{1,p}(\Omega)$. Define $G \in [W^{1,p'}(\Omega)]'$ by

$$\langle G, \varphi\rangle = \int_\Omega \nabla v \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} hv\varphi \, \mathrm{d}\mathcal{H}_{m-1}.$$

The function v is a weak solution of the problem $-\Delta v = G|\Omega$ in Ω, $\partial v/\partial n + hv = G|\partial\Omega$ on $\partial\Omega$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then

$$\langle G, \varphi\rangle = \int_\Omega v(-\Delta\varphi) \, \mathrm{d}\mathcal{H}_m = \langle F, \varphi\rangle$$

by Theorem 3.19.3. Thus, $F - G$ is supported on $\partial\Omega$. Using properties of traces of $W^{1,p'}(\Omega)$ (Theorem 1.24.2), we deduce that $F - G \in [W^{1-1/p',p'}(\partial\Omega)]' = W^{-1/p,p}(\partial\Omega)$. One has $F - G \in B_{-1/p}^{p,p}(\partial\Omega)$ by Theorem 1.23.3. According to Theorem 7.24.5, there exists a very weak solution $w \in B_1^{p,p}(\Omega)$ of the Robin problem

$$-\Delta w = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} + hw = F - G \quad \text{on } \partial\Omega.$$

Corollary 7.10.7 gives that $w \in W^{1,p}(\Omega)$. Thus, $u = v + w$ is a weak solution of the Robin problem (7.77) in $W^{1,p}(\Omega)$.

For $v \in W^{1,p}(\Omega)$, define $Tv \in [W^{1,p'}(\Omega)]'$ by

$$\langle Tv, \varphi \rangle = \int_{\Omega} \nabla v \cdot \nabla \varphi \, d\mathcal{H}_m + \int_{\partial\Omega} hv\varphi \, d\mathcal{H}_{m-1}.$$

T is a one-to-one bounded linear operator from $W^{1,p}(\Omega)$ onto $[W^{1,p'}(\Omega)]'$. Theorem 1.6.3 gives that T is an isomorphism.

Let now $s \neq 1$. Then $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$, $[W^{2-s,p'}(\Omega)]' = [B_{2-s}^{p',p'}(\Omega)]'$ by Theorem 1.23.3. Theorem 7.24.5 gives the proposition.

Theorem 7.24.7 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let F be a distribution supported on $\overline{\Omega}$.*

- *Then there exists a very weak solution $u \in B_s^{p,q}(\Omega)$ of the Neumann problem*

  $$-\Delta u = F|\Omega \quad in\ \Omega, \qquad \partial u/\partial n = F|\partial\Omega \quad on\ \partial\Omega \tag{7.80}$$

  if and only if $F \in [B_{2-s}^{p',q'}(\Omega)]'$, $\langle F, 1\rangle = 0$. If u is a very weak solution of the problem, then a general form of a solution is $u + c$ where c is a constant. If u is a solution of the problem such that

  $$\int_{\Omega} u \, d\mathcal{H}_m = 0 \tag{7.81}$$

  then

  $$\|u\|_{B_s^{p,q}(\Omega)} \le C \|F\|_{[B_{2-s}^{p',q'}(\Omega)]'}$$

  where a constant C depends only on Ω, p, q, and s.
- *Then there exists a very weak solution $u \in W^{s,p}(\Omega)$ of the Neumann problem (7.80) if and only if $F \in [W^{2-s,p'}(\Omega)]'$, $\langle F, 1\rangle = 0$. If u is a very weak solution of the problem, then a general form of a solution is $u + c$ where c is a constant. If $s \geq 1$, then every very weak solution is a weak solution of the problem in $W^{1,p}(\Omega)$. If u is a solution of the problem such that (7.81) holds, then*

  $$\|u\|_{W^{s,p}(\Omega)} \le C \|F\|_{[W^{2-s,p'}(\Omega)]'}$$

  where a constant C does not depend on F.

Proof If $s \neq 1$, then $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$, $[W^{2-s,p'}(\Omega)]' = [B_{2-s}^{p',p'}(\Omega)]'$ by Proposition 1.23.13.

Suppose first that u is a very weak solution of (7.80). Proposition 7.24.3 gives that $F \in [B_{2-s}^{p',q'}(\Omega)]'$ for $u \in B_s^{p,q}(\Omega)$ and $F \in [W^{1,p'}(\Omega)]'$ for $u \in W^{1,p}(\Omega)$. If $\varphi \equiv 1$ then

$$\langle F, \varphi\rangle = \int\limits_{\Omega} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

Suppose now that $F = 0$. If $u \in B_s^{p,q}(\Omega)$, then u is constant by Proposition 7.24.2. If $u \in W^{1,p}(\Omega)$, then $u \in B_1^{p,p}(\Omega)$ by Corollary 7.10.7, and thus u is constant.

Let $F = 0$. If u is constant, then u is a weak solution of the Neumann problem (7.80) in $W^{1,p}(\Omega)$. According to Proposition 7.19.2, it is a very weak solution of the problem.

For $h \in L^\infty(\partial\Omega)$ and $u \in B_s^{p,q}(\Omega)$ or $u \in W^{1,p}(\Omega)$, define the distribution $T_h u$ by

$$\langle T_h u, \varphi\rangle = \int\limits_{\Omega} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} \left[u\frac{\partial\varphi}{\partial n} + hu\varphi\right]\mathrm{d}\mathcal{H}_{m-1}.$$

The operator $T_1 - T_0 = I$ is compact from $B_s^{p,q}(\Omega)$ to $L^p(\partial\Omega)$ and from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ (see Theorems 1.24.1, 1.24.7, and 1.23.14, Proposition 1.23.15, and Theorems 1.23.4 and 1.7.4). Since the trace is a bounded linear operator from $B_{2-s}^{p',q'}(\Omega)$ to $L^{p'}(\partial\Omega)$ and from $W^{1,p'}(\Omega)$ to $L^{p'}(\partial\Omega)$, the operator $T_1 - T_0$ is a compact linear operator from $B_s^{p,q}(\Omega)$ to $[B_{2-s}^{p',q'}(\Omega)]'$ and from $W^{1,p}(\Omega)$ to $[W^{1,p'}(\Omega)]'$ (see Theorem 1.7.4). Since T_1 is an isomorphism from $B_s^{p,q}(\Omega)$ to $[B_{2-s}^{p',q'}(\Omega)]'$ and from $W^{1,p}(\Omega)$ to $[W^{1,p'}(\Omega)]'$ (see Theorems 7.24.5 and 7.24.6 and Proposition 7.19.2), the operators $T_0 : B_s^{p,q}(\Omega) \to [B_{2-s}^{p',q'}(\Omega)]'$, $T_0 : W^{1,p}(\Omega) \to [W^{1,p'}(\Omega)]'$ are Fredholm with index 0 (see Theorem 1.8.3). We know that Ker T_0 is the space of constant functions. Since $T_0(B_s^{p,q}(\Omega)) \subset \{F \in [B_{2-s}^{p',q'}(\Omega)]'; \langle f, 1\rangle = 0\}$, $T_0(W^{1,p}(\Omega)) \subset \{F \in [W^{1,p'}(\Omega)]'; \langle f, 1\rangle = 0\}$, we deduce $T_0(B_s^{p,q}(\Omega)) = \{F \in [B_{2-s}^{p',q'}(\Omega)]'; \langle f, 1\rangle = 0\}$, $T_0(W^{1,p}(\Omega)) = \{F \in [W^{1,p'}(\Omega)]'; \langle f, 1\rangle = 0\}$. Theorem 1.6.3 gives that $T_0 : \{u \in B_s^{p,q}(\Omega); \int_\Omega u\,\mathrm{d}\mathcal{H}_m = 0\} \to \{F \in [B_{2-s}^{p',q'}(\Omega)]'; \langle f, 1\rangle = 0\}$, $T_0 : \{u \in W^{1,p}(\Omega); \int_\Omega u\,\mathrm{d}\mathcal{H}_m = 0\} \to \{F \in [W^{1,p'}(\Omega)]'; \langle f, 1\rangle = 0\}$ are isomorphisms.

Let now $F \in [W^{1,p'}(\Omega)]'$ and $u \in W^{1,p}(\Omega)$ be a very weak solution of the Neumann problem (7.80). Then u is a weak solution of the problem (7.80) by Proposition 7.19.2.

Corollary 7.24.8 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$. Let $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Let ν be a finite real measure supported on $\partial\Omega$. Let μ be a finite real measure, $|\mu|(R^m \setminus \Omega) = 0$. If $\int_{\partial\Omega} h\,\mathrm{d}\mathcal{H}_{m-1} = 0$, suppose* *moreover that* $\mu(R^m) + \nu(R^m) = 0$. *Then, there exists*

$$u \in \bigcap_{1<p<m/(m-1)} W^{1,p}(\Omega)$$

that is a weak solution of the problem

$$-\Delta u = \mu \ in \ \Omega, \qquad \frac{\partial u}{\partial n} + hu = \nu \ on \ \partial\Omega.$$

If $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, *then this solution is unique. If* $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} = 0$, *then this solution is unique up to an additive constant.*

Proof Denote $\rho = \mu + \nu$. Fix $p \in (1, m/(m-1))$. Denote $p' = p/(p-1)$. Since $p' > m$, $W^{1,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,0}(\overline{\Omega})$ by Theorem 1.21.13. Thus, $\rho \in [W^{1,p'}(\Omega)]'$. According to Theorems 7.24.7 and 7.24.6, there exists a weak solution $u \in W^{1,p}(\Omega)$ of the problem. If $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, then this solution is unique. If $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} = 0$, then this solution is unique up to an additive constant. So, $u \in W^{1,q}(\Omega)$ for all $q \in (1, m/(m-1))$.

7.25 The Neumann Problem in the Half-Space

In this section, we study the Neumann problem for the Poisson equation in $\tilde{W}^{1,p}(R^m_+)$, where $1 < p < \infty$. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Lemma 7.25.1 *Let* $1 < p < \infty$, $m \in N$, $m \geq 2$. *For* $u \in D^{1,p}(R^m_+)$, *define* $Pu = u$ *in* R^m_+, $Pu(x', x_m) = u(x', -x_m)$ *for* $x_m < 0$. *Then,* $Pu \in D^{1,p}(R^m)$. *If* $u \in \tilde{W}^{1,p}(R^m_+)$, *then* $Pu \in \tilde{W}^{1,p}(R^m)$. *If* $p < m$, *then* $D^{1,p}(R^m_+) = \tilde{W}^{1,p}(R^m_+) \oplus R^1$. *If* $p \geq m$, *then* $\tilde{W}^{1,p}(R^m_+) = D^{1,p}(R^m_+)$.

Proof The characterization of functions from Sobolev spaces (Theorem 1.21.1) gives that $Pu \in D^{1,p}(R^m)$. If $p \geq m$, then $Pu \in \overset{\circ}{D}{}^{1,p}(R^m)$ by Proposition 3.38.4. So, $u \in \tilde{W}^{1,p}(R^m)$.

Let now $p < m$. If $u \in \tilde{W}^{1,p}(R^m_+)$, then u can be extended as a function from $\tilde{W}^{1,p}(R^m)$. So, $u \in L^{pm/(m-p)}(R^m_+)$ by Proposition 3.38.5. Therefore, $Pu \in L^{pm/(m-p)}(R^m)$ and $Pu \in \tilde{W}^{1,p}(R^m)$ by Proposition 3.38.5. In particular, $1 \notin D^{1,p}(R^m_+) \setminus \tilde{W}^{1,p}(R^m_+)$. If $v \in D^{1,p}(R^m_+)$, then $Pu \in D^{1,p}(R^m)$. According to Proposition 3.38.4, there exists a constant c such that $Pv - c \in \overset{\circ}{D}{}^{1,p}(R^m)$. Hence, $v - c \in \tilde{W}^{1,p}(R^m_+)$.

Proposition 7.25.2 *Let* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $m \in N$, $m \geq 2$. *Then there exists a positive constant* C *such that for all* $u \in \tilde{W}^{1,p}(R^m_+)$

$$C^{-1}\|\nabla u\|_{L^p(R^m_+)} \leq \sup_{\substack{\varphi \in \tilde{W}^{1,p'}(R^m_+) \\ |\nabla\varphi| \not\equiv 0}} \int\limits_{R^m_+} \frac{\nabla u \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m_+)}} \leq C\|\nabla u\|_{L^p(R^m_+)}.$$

Proof The right inequality is a consequence of the Hölder inequality (see Proposition 1.14.1).

Let now $u \in \tilde{W}^{1,p}(R^m_+)$. Put $u(x', x_m) = u(x', -x_m)$ for $x_m < 0$. Then $u \in \tilde{W}^{1,p}(R^m)$ by Lemma 7.25.1. For φ defined in R^m, denote $\tilde{\varphi}(x', x_m) = \varphi(x', x_m) - \varphi(x', -x_m)$, $\breve{\varphi}(x', x_m) = \varphi(x', x_m) + \varphi(x', -x_m)$. Then $\varphi = \tilde{\varphi}/2 + \breve{\varphi}/2$, $\tilde{\varphi}(x', -x_m) = -\tilde{\varphi}(x', x_m)$, $\breve{\varphi}(x', -x_m) = \breve{\varphi}(x', x_m)$. According to Proposition 3.38.2, there exists a positive constant M such that

$$\|\nabla v\|_{L^p(R^m)} \leq M \sup_{\varphi \in \mathcal{C}^\infty_c(R^m)\setminus\{0\}} \int\limits_{R^m} \frac{\nabla v \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m$$

for all $v \in D^{1,p}(R^m)$. Since $\|\nabla\breve{\varphi}\|_{L^{p'}(R^m)} \leq 2\|\nabla\varphi\|_{L^p(R^m)}$, we have

$$\begin{aligned}
\|\nabla u\|_{L^p(R^m_+)} &= \frac{1}{2}\|\nabla u\|_{L^p(R^m)} \leq \frac{M}{2} \sup_{\varphi \in \mathcal{C}^\infty_c(R^m)\setminus\{0\}} \int\limits_{R^m} \frac{\nabla u \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m \\
&= \frac{M}{4} \sup_{\substack{\varphi \in \mathcal{C}^\infty_c(R^m) \\ \varphi \not\equiv 0}} \int\limits_{R^m} \frac{\nabla u \cdot \nabla\tilde{\varphi} + \nabla u \cdot \nabla\breve{\varphi}}{\|\nabla\varphi\|_{L^{p'}(R^m)}}\, \mathrm{d}\mathcal{H}_m \\
&= \frac{M}{4} \sup_{\substack{\varphi \in \mathcal{C}^\infty_c(R^m) \\ \varphi \not\equiv 0}} \int\limits_{R^m} \frac{\nabla u \cdot \nabla\breve{\varphi}}{\|\nabla\varphi\|_{L^{p'}(R^m)}} \leq \frac{M}{2} \sup_{\substack{\varphi \in \mathcal{C}^\infty_c(R^m) \\ \varphi \not\equiv 0}} \int\limits_{R^m} \frac{\nabla u \cdot \nabla\breve{\varphi}}{\|\nabla\breve{\varphi}\|_{L^{p'}(R^m)}} \\
&= \frac{M}{2} \sup_{\substack{\varphi \in \mathcal{C}^\infty_c(R^m) \\ \varphi \not\equiv 0}} \int\limits_{R^m_+} \frac{\nabla u \cdot \nabla\breve{\varphi}}{\|\nabla\breve{\varphi}\|_{L^{p'}(R^m_+)}} \leq \frac{M}{2} \sup_{\substack{\varphi \in \tilde{W}^{1,p'}(R^m_+) \\ |\nabla\varphi| \not\equiv 0}} \int\limits_{R^m_+} \frac{\nabla u \cdot \nabla\varphi}{\|\nabla\varphi\|_{L^{p'}(R^m_+)}}.
\end{aligned}$$

Theorem 7.25.3 *Let* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $m \in N$, $m \geq 2$, $F \in [\tilde{W}^{1,p'}(R^m_+)]'$. *If* $p > m/(m-1)$ *(i.e.,* $p' < m$*), then there exists a solution* $u \in \tilde{W}^{1,p}(R^m_+)$ *of the Neumann problem*

$$-\Delta u = F \quad \text{in } R^m_+, \qquad \frac{\partial u}{\partial n} = F \quad \text{on } \partial R^m_+. \tag{7.82}$$

If $p \leq m/(m-1)$ *(i.e.,* $p' \geq m$*), then there exists a solution* $u \in \tilde{W}^{1,p}(R^m_+)$ *of the Neumann problem (7.82) if and only if* $\langle F, 1\rangle = 1$*. If* $p < m$*, then this solution is unique. If* $p \geq m$*, then this solution is unique up to an additive constant.*

Proof Fix a nonempty bounded open set ω such that $\overline{\omega} \subset R^m_+$. Denote

$$f(v) = \int_{\omega} v \, \mathrm{d}\mathcal{H}_m.$$

Then $\|\nabla v\|_{L^p(R^m)} + |f(v)|$ is an equivalent norm on $\tilde{W}^{1,p}(R^m)$ by Proposition 1.21.9. So,

$$\|u\|_\omega := \inf\{\|\nabla v\|_{L^p(R^m)} + |f(v)|; v \in \tilde{W}^{1,p}(R^m), u = v \text{ on } R^m_+\}$$

is an equivalent norm on $\tilde{W}^{1,p}(R^m_+)$. Clearly,

$$\|\nabla u\|_{L^p(R^m_+)} + |f(u)| \leq \|u\|_\omega.$$

Put $v = u$ in R^m_+, $v(x', x_m) = u(x', -x_m)$ for $x_m < 0$. Then $v \in \tilde{W}^{1,p}(R^m)$ by Lemma 7.25.1. Therefore,

$$\|u\|_\omega \leq \|\nabla v\|_{L^p(R^m)} + |f(v)| = 2\|\nabla u\|_{L^p(R^m_+)} + |f(u)|.$$

Thus, $\|\nabla u\|_{L^p(R^m_+)} + |f(u)|$ is an equivalent norm on $\tilde{W}^{1,p}(R^m_+)$. If we denote $X = \{u \in \tilde{W}^{1,p}(R^m_+); f(u) = 0\}$, then $\|\nabla u\|_{L^p(R^m_+)}$ is an equivalent norm on X.

If $u \in \tilde{W}^{1,p}(R^m_+)$, define

$$\langle T_p u, \varphi\rangle = \int_{R^m_+} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m, \qquad \varphi \in \tilde{W}^{1,p'}(R^m_+).$$

Then T_p is a bounded linear operator from $\tilde{W}^{1,p}(R^m_+)$ to $[\tilde{W}^{1,p'}(R^m_+)]'$. Proposition 7.25.2 gives that $T_p : X \to [\tilde{W}^{1,p'}(R^m_+)]'$ is a bounded below. Thus, $T_p(X)$ is a closed subspace of $[\tilde{W}^{1,p'}(R^m_+)]'$ by Proposition 1.4.13. Since X is a subspace of codimension 1, Proposition 1.3.7 forces that $T_p(\tilde{W}^{1,p}(R^m_+))$ is a closed subspace of $[\tilde{W}^{1,p'}(R^m_+)]'$.

Let now $T_p u = 0$. Then, $\|\nabla u\|_{L^p(R^m_+)} = 0$ by Proposition 7.25.2, i.e., u is constant. Denote $u(x', x_m) = u(x', -x_m)$ for $x_m < 0$. Lemma 7.25.1 gives that $u \in \tilde{W}^{1,p}(R^m)$. If $p < m$, then $u \equiv 0$ by Proposition 1.25.8 and $\operatorname{Ker} T_p = \{0\}$. If $p \geq m$, then $1 \in \tilde{W}^{1,p}(R^m_+)$ by Proposition 1.25.8 and $\operatorname{Ker} T_p = P_0(R^m)$ (i.e., the space of constant functions).

Since $\langle T_p u, v\rangle = \langle T_{p'} v, u\rangle$, we have $T_p' = T_{p'}$. Proposition 1.5.13 gives that $T_p(\tilde W^{1,p}(R^m_+)) = \{f \in [\tilde W^{1,p'}(R^m_+)]'; \langle f, u\rangle = 0\ \forall u \in \operatorname{Ker} T_{p'}\}$. If $p' < m$ (i.e., $p > m/(m-1)$), then $T_p(\tilde W^{1,p}(R^m_+)) = [\tilde W^{1,p'}(R^m_+)]'$. If $p' \ge m$ (i.e., if $p \le m/(m-1)$), then $T_p(\tilde W^{1,p}(R^m_+)) = \{F \in [\tilde W^{1,p'}(R^m_+)]'; \langle F, 1\rangle = 0\}$.

Corollary 7.25.4 *Let* $1 < p, p' < \infty$, $1/p + 1/p' = 1$, $m \in N$, $m \ge 2$, $F \in [D^{1,p'}(R^m_+)]'$. *Then there exists* $u \in D^{1,p}(R^m_+)$ *such that*

$$\int_{R^m_+} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle F, \varphi\rangle \qquad \forall \varphi \in D^{1,p'}(R^m_+) \tag{7.83}$$

if and only if $\langle F, 1\rangle = 0$. *This* u *is unique up to an additive constant.*

Proof Let u be a solution of the problem (7.83). Since $\varphi \equiv 1$ is an element of $D^{1,p'}(R^m_+)$, we have

$$\langle F, 1\rangle = \int_{R^m_+} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = 0.$$

Let now $\langle F, 1\rangle = 0$. According to Theorem 7.25.3, there exists $u \in \tilde W^{1,p}(R^m_+) \subset D^{1,p}(R^m_+)$ such that (7.83) holds for every $\varphi \in \tilde W^{1,p'}(R^m_+)$. Let now $\psi \in D^{1,p'}(R^m_+)$. According to Lemma 7.25.1, there exist $\varphi \in \tilde W^{1,p'}(R^m_+)$ and a constant c such that $\psi = \varphi + c$. Since $\langle F, 1\rangle = 0$, we have

$$\int_{R^m_+} \nabla u \cdot \nabla \psi \, \mathrm{d}\mathcal{H}_m = \int_{R^m_+} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle F, \varphi\rangle = \langle F, \psi\rangle.$$

Let now u be a solution of the problem (7.83) with $F = 0$. According to Lemma 7.25.1, there exist $\tilde u \in \tilde W^{1,p}(R^m_+)$ and a constant c such that $u = \tilde u + c$. Clearly, $\tilde u$ is a solution of the problem (7.83) with $F = 0$, too. Theorem 7.25.3 gives that $\tilde u$ is constant. Thus, u is constant, too.

7.26 The Neumann Problem and the Robin Problem in $W^{2,p}(\Omega)$

In this section, we study the Neumann problem and the Robin problem for the Poisson equation in $W^{2,p}(\Omega)$ for a bounded domain with boundary of class $\mathcal{C}^2$ and $1 < p < \infty$. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Proposition 7.26.1 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^2$, $f \in L^p(\Omega)$, $1 < p < \infty$, $p' = p/(p-1)$. If $u \in W^{1,p}(\Omega)$,*

$$\int\limits_{\Omega} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} f\varphi \, \mathrm{d}\mathcal{H}_m, \quad \forall \varphi \in \tilde{W}^{1,p'}(R^m),$$

i.e., u is a solution of the problem $\Delta u = f$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$, then $u \in W^{2,p}(\Omega)$.

Proof We can suppose that $u \in W^{1,p}(R^m)$ and u has compact support (see Lemmas 1.21.12 and 1.21.17). Proposition 3.18.2 gives that $u \in W^{2,p}_{loc}(\Omega)$. So, it is enough to prove that u is of class $W^{2,p}$ on a neighborhood of each point of the boundary. Fix $z \in \partial\Omega$. Then there exist a coordinate system, $\Phi \in \mathcal{C}^{2,0}(R^{m-1})$ and $\rho_1 > 0$ such that $z = 0$ and $\Omega \cap B(0;\rho_1) = B(0;\rho_1) \cap \{[x', x_m]; x' \in R^{m-1}, x_m > \Phi(x')\}$. We can moreover suppose that $\nabla\Phi(0) = (0, \dots, 0)$. Fix a constant $C_1 \geq 1$ such that $|\nabla\Phi| + |\nabla^2\Phi| \leq C_1$. Define $P(x', x_m) = (x', x_m + \Phi(x'))$, $P^{-1}(x', x_m) = (x', x_m - \Phi(x'))$. Easy calculation yields that the Jacobi determinants corresponding to the mappings P and P^{-1} are equal to 1. If $v \in W^{1,q}(R^m)$ has compact support and $1 < q < \infty$, then $v \circ P, v \circ P^{-1} \in W^{1,q}(R^m)$ by Proposition 1.21.14. Put $f = 0$ outside Ω, $f_1 = f \circ P$. Clearly, $f_1 \in L^p(R^m)$. Define $w = u \circ P$. Then, $w \in W^{1,p}(R^m)$ has compact support. Since $u = w \circ P^{-1}$, we have $\partial_m u = \partial_m w \circ P^{-1}$, $\partial_j u = \partial_j w \circ P^{-1} - (\partial_m w \circ P^{-1})\partial_j\Phi$ for $j = 1, \dots, m-1$. If $\varphi \in W^{1,p'}(R^m)$ is supported in $B(0;\rho_1)$, then by virtue of the substitution

$$\int\limits_{R^m_+} f_1(\varphi \circ P) \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} f\varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} \nabla u \cdot \nabla\varphi = \int\limits_{R^m_+} \Bigg\{ (\partial_m w)[\partial_m(\varphi \circ P)]$$
$$+ \sum_{j=1}^{m-1} [\partial_j w - (\partial_j\Phi)(\partial_m w)][\partial_j(\varphi \circ P) - (\partial_j\Phi)\partial_m(\varphi \circ P)] \Bigg\} \, \mathrm{d}\mathcal{H}_m.$$

We can choose $\rho_2 \in (0;\rho_1)$ such that $P^{-1}(B(0;\rho_2)) \subset B(0;\rho_1)$. So, if $\varphi \in \overset{\circ}{W}{}^{1,p'}(B(0;\rho_2))$, then

$$\int\limits_{R^m_+} \nabla w \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m_+} \varphi f_1 \, \mathrm{d}\mathcal{H}_m + \sum_{j=1}^{m-1} \int\limits_{R^m_+} [(\partial_m w)(\partial_j\Phi)(\partial_j\varphi)$$
$$+(\partial_m\varphi)(\partial_j\Phi)(\partial_j w) - (\partial_j\Phi)^2(\partial_m w)(\partial_m\varphi)] \, \mathrm{d}\mathcal{H}_m. \tag{7.84}$$

Denote

$$\mathcal{A} = \left\{ \varphi \in W^{1,p'}(R^m_+); \|\nabla\varphi\|_{L^{p'}(R^m_+)} \leq 1 \right\}.$$

According to Proposition 7.25.2 and Lemma 7.25.1, there exists a constant $C_2 \geq 1$ such that

$$\|\nabla v\|_{L^p(R^m_+)} \leq C_2 \sup_{\varphi \in \mathcal{A}} \int_{R^m_+} \nabla v \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m \qquad \forall v \in D^{1,p}(R^m_+). \tag{7.85}$$

Fix $r \in (0; \rho_2/4)$ such that $|\Phi| + |\nabla \Phi| < 1/(6mC_2 + 1)$ for $|x'| < 4r$. Choose $\psi \in \mathcal{C}_c^\infty(B(0; 2r))$ such that $\psi = 1$ on $B(0; r)$, $0 \leq \psi \leq 1$. Put $\tilde{\psi}(x', x_m) = [\psi(x', x_m) + \psi(x', -x_m)]/2$. Taking $\tilde{\psi}$ instead of ψ, we can suppose moreover that $\psi(x', -x_m) = \psi(x', x_m)$. For $\varphi \in \tilde{W}^{1,p'}(R^m_+)$, define

$$\tilde{\varphi}(x', x_m) = \varphi(x', |x_m|), \quad \tilde{w}(x', x_m) = w(x', |x_m|).$$

Then $\tilde{w} \in \tilde{W}^{1,p}(R^m)$, $\tilde{\varphi} \in \tilde{W}^{1,p'}(R^m)$ by Lemma 7.25.1. Since $\tilde{w} \in W^{1,p}(B(0; 2r)$, $\tilde{\varphi} \in W^{1,p'}(B(0; 2r))$, Theorem 1.21.13 and Lemma 1.21.12 give that there exist $q(1) \in (p, \infty)$ and $q(2) \in (p', \infty)$ such that $\tilde{w} \in L^{q(1)}(B(0; 2r))$, $\tilde{\varphi} \in L^{q(2)}(B(0; 2r))$. According to Hölder's inequality, there exists a constant $q \in (1, \infty)$ such that $\tilde{w}\tilde{\varphi}, \tilde{w}\partial_j\tilde{\varphi}, \tilde{\varphi}\partial_j\tilde{w} \in L^q(B(0; 2r))$. Since $\partial_j(\tilde{\varphi}\tilde{w}) = \tilde{\varphi}\partial_j\tilde{w} + \tilde{w}\partial_j\tilde{\varphi}$, we have $\tilde{w}\tilde{\varphi} \in W^{1,q}(B(0; 2r))$. Since ψ is supported in $B(0; 2r)$, Green formula for Sobolev spaces (Proposition 1.24.4) gives

$$0 = \int_{\partial B(0;2r)} \tilde{w}\tilde{\varphi} \frac{\partial \psi}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \int_{B(0;2r)} [(\nabla \psi) \cdot \nabla(\tilde{w}\tilde{\varphi}) + \tilde{w}\tilde{\varphi}\Delta\psi] \, \mathrm{d}\mathcal{H}_m$$

$$= 2 \int_{R^m_+} [(\nabla \psi) \cdot \nabla(\tilde{w}\tilde{\varphi}) + \tilde{w}\tilde{\varphi}\Delta\psi] \, \mathrm{d}\mathcal{H}_m.$$

Hence,

$$\int_{R^m_+} (\nabla \psi) \cdot \nabla(w\varphi) = \int_{R^m_+} -w\varphi\Delta\psi \, \mathrm{d}\mathcal{H}_m.$$

Using this equality, we obtain

$$\int_{R^m_+} \nabla\varphi \cdot \nabla(\psi w) \, \mathrm{d}\mathcal{H}_m = \int_{R^m_+} [\nabla(\psi\varphi) \cdot \nabla w + w\nabla\psi \cdot \nabla\varphi - \varphi\nabla\psi \cdot \nabla w] \, \mathrm{d}\mathcal{H}_m$$

$$= \int_{R^m_+} [\nabla(\psi\varphi) \cdot \nabla w + \nabla\psi \cdot \nabla(w\varphi) - 2\varphi\nabla\psi \cdot \nabla w] \, \mathrm{d}\mathcal{H}_m$$

$$= \int_{R^m_+} [\nabla(\psi\varphi) \cdot \nabla w - \varphi(w\Delta\psi + 2\nabla\psi \cdot \nabla w)] \, \mathrm{d}\mathcal{H}_m.$$

Put $f_2 = w\Delta\psi + 2\nabla\psi \cdot \nabla w$. Since $w \in W^{1,p}(R^m)$ and $\psi \in \mathcal{C}_c^\infty(R^m)$, we have $f_2 \in L^p(R^m)$. Moreover,

$$\int\limits_{R^m_+} \nabla\varphi \cdot \nabla(\psi w)\, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m_+} \nabla w \cdot \nabla(\psi\varphi)\, \mathrm{d}\mathcal{H}_m - \int\limits_{R^m_+} \varphi f_2\, \mathrm{d}\mathcal{H}_m. \tag{7.86}$$

Fix $k \in \{1, \dots, m-1\}$. For a nonzero real number δ, denote

$$P_k^\delta\varphi(x) = \varphi(x_1, x_2, \dots, x_{k-1}, x_k + \delta, x_{k+1}, \dots, x_m), \quad D_k^\delta\varphi = \frac{P_k^\delta\varphi - \varphi}{\delta}\, .$$

We estimate $\|D_k^\delta \nabla(w\psi)\|_{L^p(R^m)}$. According to Theorem 1.21.2, there exists a positive constant C_3 such that

$$\|D_k^\delta\varphi\|_{L^{p'}(R^m)} \leq C_3\|\nabla\varphi\|_{L^{p'}(R^m)}$$

for all k, δ, and $\varphi \in W^{1,p'}(R^m)$. If $\varphi \in W^{1,p'}(R^m_+)$, we define $\varphi(x', x_m) = \varphi(x', -x_m)$ for $x_m < 0$. Then, $\varphi \in W^{1,p'}(R^m)$ by Lemma 7.25.1. So,

$$\|D_k^\delta\varphi\|_{L^{p'}(R^m_+)} \leq C_3\|\nabla\varphi\|_{L^{p'}(R^m_+)} \qquad \forall\varphi \in W^{1,p'}(R^m_+). \tag{7.87}$$

If $\varphi \in \mathcal{A}$, then $D_k^{-\delta}\varphi \in W^{1,p'}(R^m_+)$. Using substitution, (7.86) and (7.84)

$$\int\limits_{R^m_+} [D_k^\delta \nabla(w\psi)] \cdot \nabla\varphi\, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m_+} [\nabla(w\psi)] \cdot \nabla(D_k^{-\delta}\varphi)\, \mathrm{d}\mathcal{H}_m$$

$$= \int\limits_{R^m_+} \nabla w \cdot \nabla(\psi D_k^{-\delta}\varphi)\, \mathrm{d}\mathcal{H}_m - \int\limits_{R^m_+} (D_k^{-\delta}\varphi) f_2\, \mathrm{d}\mathcal{H}_m = \int\limits_{R^m_+} (D_k^{-\delta}\varphi)(f_1\psi - f_2)\, \mathrm{d}\mathcal{H}_m$$

$$+ \sum_{j=1}^{m-1} \int\limits_{R^m_+} [(\partial_m w)(\partial_j\Phi)\partial_j(\psi D_k^{-\delta}\varphi) + (\partial_j\Phi)(\partial_j w)\partial_m(\psi D_k^{-\delta}\varphi)$$

$$-(\partial_j\Phi)^2(\partial_m w)\partial_m(\psi D_k^{-\delta}\varphi)]\, \mathrm{d}\mathcal{H}_m \leq \sum_{j=1}^{m-1} \int\limits_{R^m_+} \{(\partial_j\varphi) D_k^\delta[\psi(\partial_m w)(\partial_j\Phi)]$$

$$+(\partial_m\varphi) D_k^\delta[\psi(\partial_j\Phi)(\partial_j w)] - (\partial_m\varphi) D_k^\delta[\psi(\partial_j\Phi)^2(\partial_m w)]\}\, \mathrm{d}\mathcal{H}_m$$

$$+ \int\limits_{R^m_+} |D_k^{-\delta}\varphi|[|f_1| + |f_2| + 2(m-1)|\nabla\psi||\nabla w|(|\nabla\Phi| + |\nabla\Phi|^2)]\, \mathrm{d}\mathcal{H}_m.$$

Put $f_3 = |f_1| + |f_2| + 2(m-1)|\nabla\psi||\nabla w|(|\nabla\Phi| + |\nabla\Phi|^2)$. Then $f_3 \in L^p(R^m)$. By virtue of Hölder's inequality (Proposition 1.14.1),

$$\int_{R^m_+} [D^\delta_k \nabla(w\psi)] \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m \le \sum_{j=1}^{m-1} \int_{R^m_+} \{(\partial_j\varphi) D^\delta_k[(\partial_j\Phi)\partial_m(\psi w)]$$

$$+(\partial_m\varphi) D^\delta_k[(\partial_j\Phi)\partial_j(\psi w)] - (\partial_m\varphi) D^\delta_k[(\partial_j\Phi)^2\partial_m(\psi w)]\} \, \mathrm{d}\mathcal{H}_m$$

$$+\sum_{j=1}^{m-1} \|\nabla\varphi\|_{L^{p'}(R^m_+)} \Big\{ \|D^\delta_k[w(\partial_j\Phi)(\partial_m\psi)]\|_{L^p(R^m_+)} + \|D^\delta_k[w(\partial_j\Phi)(\partial_j\psi)]\|_{L^p(R^m_+)}$$

$$+\|D^\delta_k[w(\partial_j\Phi)^2(\partial_m\psi)]\|_{L^p(R^m_+)} \Big\} + \|f_3\|_{L^p(R^m_+)} \|D^{-\delta}_k\varphi\|_{L^{p'}(R^m_+)}.$$

Clearly, $w(\partial_j\Phi)(\partial_l\psi), w(\partial_j\Phi)^2(\partial_l\psi) \in W^{1,p}(R^m_+)$. We can choose a constant C_4 such that $\|w(\partial_j\Phi)(\partial_l\psi)\|_{W^{1,p}} \le C_4$, $\|w(\partial_j\Phi)^2(\partial_l\psi)\|_{W^{1,p}} \le C_4$, $\|f_3\|_{L^p} \le C_4$, $\|\nabla(\psi w)\|_{L^p} \le C_4$. If α and β are functions, then $D^\delta_k(\alpha\beta) = (D^\delta_k\alpha)P^\delta_k\beta + \alpha D^\delta_k\beta$. Using (7.87),

$$\int_{R^m_+} [D^\delta_k \nabla(w\psi)] \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m \le \int_{R^m_+} (m-1)|\nabla\varphi|[2|\nabla\Phi| + |\nabla\Phi|^2]|D^\delta_k\nabla(\psi w)| \, \mathrm{d}\mathcal{H}_m$$

$$+\int_{R^m_+} (m-1)[2|D^\delta_k(\nabla\Phi)| + |D^\delta_k(|\nabla\Phi|^2)]P^\delta_k[|\nabla(\psi w)||\nabla\varphi|] \, \mathrm{d}\mathcal{H}_m$$

$$+(m-1)C_3C_4 + C_3C_4\|\nabla\varphi\|_{L^{p'}(R^m_+)}.$$

Since for each x and j there exists z such that $D^\delta_k\partial_j\Phi(x) = \partial_k\partial_j\Phi(z)$, we have $|D^\delta_k\nabla\Phi| \le C_1$. Similarly, $D^\delta_k|\nabla\Phi(x')|^2 = \sum_j \partial_j\Phi(z')\partial_k\partial_j\Phi(z')$ and thus $|D^\delta_k|\nabla\Phi(x')|^2| \le mC_1^2$. If $|x'| < 4r$, then $|\nabla\Phi(x')| < 1/(6mC_2 + 1)$. Since $\operatorname{spt}\psi \subset B(0;2r)$, we have for $0 < |\delta| < r$

$$\int_{R^m_+} [D^\delta_k \nabla(w\psi)] \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m \le \frac{3(m-1)}{6mC_2+1} \|\nabla\varphi\|_{L^{p'}(R^m_+)} \|D^\delta_k\nabla(\psi w)\|_{L^p(R^m_+)}$$

$$+(m-1)(2C_1 + mC_1^2)\|\nabla(\psi w)\|_{L^p(R^m_+)} \|\nabla\varphi\|_{L^{p'}(R^m_+)} + mC_3C_4.$$

Since $\varphi \in \mathcal{A}$ was arbitrary, we obtain using (7.85) and $\|\nabla(\psi w)\|_{L^p(R^m_+)} \le C_4$

$$\|D^\delta_k \nabla(\psi w)\|_{L^p(R^m_+)} \le C_2 \sup_{\varphi\in\mathcal{A}} \int\limits_{R^m_+} [D^\delta_k \nabla(w\psi)] \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m$$

$$\le \frac{1}{2}\|D^\delta_k \nabla(\psi w)\|_{L^p(R^m_+)} + mC_2C_4(2C_1 + mC_1^2 + C_3).$$

Subtracting $(1/2)\|D^\delta_k \nabla(\psi w)\|_{L^p(R^m_+)}$ from both sides, we obtain

$$\|D^\delta_k \nabla(\psi w)\|_{L^p(R^m_+)} \le 2mC_2C_4(2C_1 + mC_1^2 + C_3).$$

If $j \in \{1, \dots, m\}$, then $\{D^\delta_k \partial_j(\psi w); 0 < \delta < r\}$ is a bounded subset of $L^p(R^m_+)$. Lemma 7.14.1 gives that $\partial_k\partial_j(\psi w) \in L^p(R^m_+)$. Hence, $\partial_j\partial_k w = \partial_k\partial_j w \in L^p(B(0;r))$.

Now, we prove $\partial^2 w/\partial x_m^2 \in L^p(B(0;r))$. Since $u(x', x_m) = w(x', x_m - \Phi(x'))$ and $\Delta u = f$, easy calculation yields

$$f(x) = \Delta w(\tilde{x}) + |\nabla\Phi(x')|^2 \frac{\partial^2 w(\tilde{x})}{\partial x_m^2} - \sum_{j=1}^{m-1} 2(\partial_j\Phi(x'))\frac{\partial^2 w(\tilde{x})}{\partial x_j \partial x_m} - \partial_m w(\tilde{x})\Delta\Phi(x')$$

where $\tilde{x} = [x', x_m - \Phi(x')]$. Thus,

$$\frac{\partial^2 w}{\partial x_m^2} = (1 + |\nabla\Phi|^2)^{-1}\left\{ f_1 + \sum_{j=1}^{m-1}\left[2(\partial_j\Phi)\frac{\partial^2 w}{\partial x_m \partial x_j} - \frac{\partial^2 w}{\partial x_j^2}\right] + \partial_m w \Delta\Phi \right\}.$$

This gives $\partial^2 w/\partial x_m^2 \in L^p(B(0;r))$. Since $w \in W^{2,p}(B(0;r))$, the function $u = w \circ P^{-1}$ is of class $W^{2,p}$ on a neighborhood of 0.

Theorem 7.26.2 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^2$, $1 < p < \infty$. Then there exists a strong solution $u \in W^{2,p}(\Omega)$ of the Neumann problem*

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{7.88}$$

if and only if $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$ and

$$\int\limits_\Omega f \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{7.89}$$

This solution is unique up to an additive constant. If $u \in W^{2,p}(\Omega)$ *is a strong solution of the Neumann problem such that*

$$\int\limits_{\Omega} u \,\mathrm{d}\mathcal{H}_m = 0, \tag{7.90}$$

then

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left[\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Omega)}\right] \tag{7.91}$$

where a constant C *depends only on* Ω *and* p.

Proof Let $u \in W^{2,p}(\Omega)$ be a strong solution of the Neumann problem (7.88). Clearly, $f \in L^p(\Omega)$. Theorem 1.24.6 forces that $g \in W^{1-1/p,p}(\partial\Omega)$. Propositions 7.19.1 and 7.19.2 give that u is a very weak solution of the problem (7.88). Hence, the relation (7.89) holds by Theorem 7.24.7. Moreover, this solution is unique up to an additive constant by Theorem 7.24.7.

Let now $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$ be such that (7.89) holds. According to Theorem 1.24.6, there exists $v \in W^{2,p}(\Omega)$ such that $\partial v/\partial n = g$ on $\partial\Omega$. Denote $\tilde{f} = -\Delta v$. Since v is a strong solution of the Neumann problem

$$-\Delta v = \tilde{f} \text{ in } \Omega, \qquad \frac{\partial v}{\partial n} = g \text{ on } \partial\Omega,$$

we have that

$$\int\limits_{\Omega} \tilde{f} \,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} g \,\mathrm{d}\mathcal{H}_{m-1} = 0.$$

Since

$$\int\limits_{\Omega} (f - \tilde{f}) \,\mathrm{d}\mathcal{H}_m = 0$$

there exists a weak solution $w \in W^{1,p}(\Omega)$ of the Neumann problem

$$-\Delta v = (f - \tilde{f}) \text{ in } \Omega, \qquad \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

(See Theorem 7.24.7.) We have $w \in W^{2,p}(\Omega)$ by Proposition 7.26.1. Theorem 1.24.6 gives that $\tilde{g} = \partial w/\partial n \in W^{1-1/p,p}(\partial\Omega)$. Proposition 7.19.1 forces that $\tilde{g} = 0$. Thus, $u = v + w \in W^{2,p}(\Omega)$ is a strong solution of (7.88).

Denote by X the set of all $u \in W^{2,p}(\Omega)$ satisfying (7.90) and by Y the space of all $[f, g] \in L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ satisfying (7.89). Then, $T : u \mapsto [-\Delta u, \partial u/\partial n]$ is a bounded linear operator from X to Y by Theorem 1.24.6. We have proved that $T(X) = Y$ and $\operatorname{Ker} T = \{0\}$. Hence, $T : X \to Y$ is an isomorphism by Theorem 1.6.3.

Theorem 7.26.3 *Let $U, \Omega \subset R^m$ be open sets, $m \geq 2$. Suppose that $L = U \cap \partial\Omega$ is a surface of class $\mathcal{C}^2$. Let $1 < p < \infty$, $u \in W^{1,p}_{loc}(\overline{\Omega})$, i.e., $u \in W^{1,p}(\omega)$ for each bounded open set ω with $\overline{\omega} \subset \overline{\Omega}$. Let $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(L)$, $h \in \mathcal{C}^1(L)$, $\Delta u = -f$ in Ω, $\partial u/\partial n + hu = g$ on L, i.e.,*

$$\int_\Omega \nabla u \cdot \nabla \varphi \; \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} hu\varphi \; \mathrm{d}\mathcal{H}_{m-1} = \int_\Omega f\varphi \; \mathrm{d}\mathcal{H}_m + \int_L g\varphi \; \mathrm{d}\mathcal{H}_{m-1}$$

for all $\varphi \in \mathcal{C}^\infty_c(R^m)$ such that $\varphi = 0$ on $\partial\Omega \setminus L$. Then, $u \in W^{2,p}_{loc}(\Omega \cup L)$, i.e., $u \in W^{2,p}(\omega)$ for each bounded open set ω with $\overline{\omega} \subset \Omega \cup L$.

Proof $-\Delta u = f$ in Ω in the sense of distribution by Theorem 3.19.3. So, $u \in W^{2,p}_{loc}(\Omega)$ by Proposition 3.18.2.

Let now $z \in L$. Choose $r > 0$ and a bounded open set $\omega \subset B(z; 3r) \cap \Omega$ with boundary of class $\mathcal{C}^{k+2}$ such that $B(z; 3r) \subset U$ and $B(z; 2r) \cap \Omega = B(z; 2r) \cap \omega$. Choose $\psi \in \mathcal{C}^\infty_c(B(z; 2r))$ such that $\psi = 1$ on $B(z; r)$. Then, $\psi u \in W^{1,p}(\Omega)$ by Lemma 1.21.17. Denote $\tilde{g} = g - hu$. Then, $-\Delta u = f$ in Ω, $\partial u/\partial n = \tilde{g}$ on L, and $\tilde{g} \in W^{1-1/p,p}_{loc}(L)$ by Theorem 1.24.5 and Lemmas 1.21.6 and 1.21.17. Since $u \in W^{1,p}(\omega)$, there exist $u_k \in \mathcal{C}^\infty_c(R^m)$ such that $u_k \to u$ in $W^{1,p}(\omega)$ as $k \to \infty$. (See Lemma 1.21.3.) Fix $\varphi \in \mathcal{C}_c(R^m)$. We can choose $\tilde{\varphi} \in \mathcal{C}^\infty_c(R^m)$ such that $\tilde{\varphi} = \varphi$ on $\overline{\omega}$ and $\tilde{\varphi} = 0$ on $\partial\Omega \setminus L$. By virtue of Lemma 1.21.17 and Theorem 1.19.1,

$$\int_\omega \nabla(\psi u) \cdot \nabla\varphi \; \mathrm{d}\mathcal{H}_m = \lim_{k\to\infty} \int_\omega \nabla(\psi u_k) \cdot \nabla\tilde{\varphi} \; \mathrm{d}\mathcal{H}_m = \lim_{k\to\infty} \Bigg[-\int_\omega \tilde{\varphi}\Delta(\psi u_k) \; \mathrm{d}\mathcal{H}_m$$
$$+ \int_{\partial\omega} \tilde{\varphi} \frac{\partial(\psi u_k)}{\partial n} \; \mathrm{d}\mathcal{H}_{m-1} \Bigg] = \lim_{k\to\infty} \Bigg[-\int_\omega \tilde{\varphi}(\psi\Delta u_k + 2\nabla\psi \cdot \nabla u_k + u_k\Delta\psi) \; \mathrm{d}\mathcal{H}_m$$
$$+ \int_{\partial\omega} \tilde{\varphi}\left(\psi\frac{\partial u_k}{\partial n} + u_k\frac{\partial\psi}{\partial n}\right) \mathrm{d}\mathcal{H}_{m-1} \Bigg] = \lim_{k\to\infty} \int_\omega \nabla u_k \cdot \nabla(\psi\tilde{\varphi}) \; \mathrm{d}\mathcal{H}_m$$
$$- \int_\omega \tilde{\varphi}(2\nabla\psi \cdot \nabla u + u\Delta\psi) \; \mathrm{d}\mathcal{H}_m + \int_{\partial\omega} \tilde{\varphi}u\frac{\partial\psi}{\partial n} \; \mathrm{d}\mathcal{H}_{m-1}.$$

Denote

$$F = -(2\nabla\psi \cdot \nabla u + u\Delta\psi) \quad \text{on } \omega,$$
$$G = u\frac{\partial\psi}{\partial n}.$$

Then $F \in L^p(\Omega)$, $G \in W^{1-1/p,p}(\partial\omega)$ by Theorem 1.24.2. Since $\psi = 0$ on $\Omega \setminus \omega$, we obtain

$$\int\limits_{\omega} \nabla(\psi u) \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} \nabla u \cdot \nabla(\psi\tilde{\varphi}) \, \mathrm{d}\mathcal{H}_m + \int\limits_{\omega} \varphi F \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\omega} G\varphi \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{\omega} (f\psi + F)\varphi \, \mathrm{d}\mathcal{H}_m + \int\limits_{L} \tilde{g}\psi\varphi \, \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\partial\omega} G\varphi \, \mathrm{d}\mathcal{H}_{m-1}.$$

Define $\tilde{G} = G + \tilde{g}\psi$ on $\partial\omega \cap L$, $\tilde{G} = G$ on $\partial\omega \setminus L$. Then $\tilde{G} \in W^{1-1/p,p}(\partial\omega)$ and

$$\int\limits_{\omega} \nabla(\psi u) \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m = \int\limits_{\omega} (f\psi + F)\varphi \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\omega} \tilde{G}\varphi \, \mathrm{d}\mathcal{H}_{m-1},$$

so $\psi u \in W^{1,p}(\omega)$ is a weak solution of the Neumann problem $-\Delta v = f\psi + F$ in ω, $\partial v/\partial n = \tilde{G}$ on $\partial\omega$. Theorems 7.26.2 and 7.24.7 give that $\psi u \in W^{2,p}(\omega)$.

Proposition 7.26.4 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^2$, $1 < p < \infty$. Let $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $h \not\equiv 0$, $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$ and $u \in W^{2,p}(\Omega)$ be a strong solution of the Robin problem*

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega. \tag{7.92}$$

If $f \geq 0$, $g \geq 0$, then $u \geq 0$.

Proof u is a weak solution of (7.92) by Proposition 7.19.1. Define $f_k = \min(f, k)$, $g_k = \min(g, k)$. According to Theorem 7.24.6, there exists a weak solution $u_k \in W^{1,p}(\Omega) \cap W^{1,2}(\Omega)$ of the problem

$$-\Delta u_k = f_k \text{ in } \Omega, \qquad \frac{\partial u_k}{\partial n} + hu_k = g_k \text{ on } \partial\Omega.$$

Moreover $u_k \to u$ in $W^{1,p}(\Omega)$ as $k \to \infty$. Since $f_k \geq 0$, $g_k \geq 0$, we have $u_k \geq 0$ by Proposition 7.21.1. According to Proposition 1.14.8, we can choose a subsequence $u_{k(j)}$ such that $u_{k(j)}(x) \to u(x)$ for almost all $x \in \Omega$. Therefore, $u \geq 0$.

Theorem 7.26.5 *Let $\Omega \subset R^m$ be a bounded open set with boundary of class $\mathcal{C}^2$, $1 < p < \infty$. Let $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $h \not\equiv 0$. Then there exists a strong solution $u \in W^{2,p}(\Omega)$ of the Robin problem (7.92) if and only if $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$. This solution is unique, and the estimate (7.91) holds with a constant C that depends only on Ω, h, and p.*

Proof Let $u \in W^{2,p}(\Omega)$ be a strong solution of the Robin problem (7.92). Clearly, $f \in L^p(\Omega)$. Theorem 1.24.6 and Lemmas 1.21.17 and 1.21.16 force that $g \in W^{1-1/p,p}(\partial\Omega)$. If $f \equiv 0$, $g \equiv 0$, then $u \equiv 0$ by Proposition 7.26.4.

Let now $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$. According to Theorem 7.24.6, there exists a weak solution $u \in W^{1,p}(\Omega)$ of the Robin problem (7.92). Theorem 7.26.3 gives that $u \in W^{2,p}(\Omega)$. By virtue of Proposition 7.19.1, we deduce that u is a strong solution of (7.92).

The operator $T : u \mapsto [-\Delta u, \partial u/\partial n + hu]$ is a bounded linear operator from $W^{2,p}(\Omega)$ onto $L^p(\Omega)\times W^{1-1/p,p}(\partial\Omega)$ (compare Theorem 1.24.6). Since $\operatorname{Ker} T = \{0\}$, the operator T is an isomorphism by Theorem 1.6.3.

7.27 The Neumann Problem in $D^{1,p}(\Omega)$ on Exterior Domains

In this section, we study the Neumann problem for the Poisson equation in $D^{1,p}(\Omega)$ for an unbounded domain Ω with compact Ljapunov boundary and $1 < p < \infty$. We find a necessary and sufficient condition for the existence of a solution. We prove that two solutions differ by a constant.

Lemma 7.27.1 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $1 < p < \infty$, $f \in W^{-1/p,p}(\partial\Omega)$. If $p \le m/(m-1)$, suppose moreover that $\langle f, 1\rangle = 0$. Put $g = f/2 - K'_\Omega f$. Then, $\mathcal{S}^\Omega f$ is a weak solution of the Neumann problem*

$$-\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega. \tag{7.93}$$

in $\tilde{W}^{1,p}(\Omega)$.

Proof $\mathcal{S}^\Omega f \in \tilde{W}^{1,p}(\Omega)$ by Proposition 7.6.4. Let $\varphi \in \mathcal{C}_c^\infty(R^m)$. Fix $r > 0$ such that $\partial\Omega \cup \operatorname{spt}\varphi \subset B(0;r)$. Denote $\omega = \Omega \cap B(0;r)$. Since $f = 0$ on $\partial B(0;r)$, we have $\mathcal{S}^\Omega f = \mathcal{S}^\omega f$. So, $\mathcal{S}^\Omega f$ is a weak solution of the problem

$$-\Delta v = 0 \text{ in } \omega, \qquad \frac{\partial v}{\partial n} = \frac{f}{2} - K'_\omega f \text{ on } \partial\omega.$$

(See Proposition 7.24.1.) Since $\varphi = 0$ on $\partial B(0,r)$, we have

$$\langle g, \varphi\rangle = \langle f/2 - K'_\omega f, \varphi\rangle = \int_\omega \nabla\varphi \cdot \nabla \mathcal{S}^\Omega f \, d\mathcal{H}_m = \int_\Omega \nabla\varphi \cdot \nabla \mathcal{S}^\Omega f \, d\mathcal{H}_m.$$

Lemma 7.27.2 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, $1 < p < \infty$. If $u \in D^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.93) with $g \equiv 0$, then u is constant.*

Proof Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$ and denote $\omega = \Omega \cap B(0;r)$. Define $g = \partial u/\partial n$ on $\partial B(0;r)$. By virtue of the Gauss-Green theorem (Proposition 1.24.4), we deduce that u is a weak solution of the Neumann problem

$$\Delta u = 0 \text{ in } \omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\omega.$$

Lemma 7.9.1 gives

$$u = \mathcal{S}^{\omega} g + \mathcal{D}^{\omega} u \quad \text{in } \omega. \tag{7.94}$$

Define

$$v = \begin{cases} \mathcal{S}^{B(0;r)} g + \mathcal{D}^{B(0;r)} u & \text{in } B(0;r), \\ u - \mathcal{D}^{\Omega} u & \text{in } \Omega. \end{cases}$$

Since $g = 0$ on $\partial\Omega$, the relation (7.94) gives that v is well defined on $\Omega \cap B(0;r)$. So, $v \in H(R^m)$. Since $|\nabla u| \in L^p(\Omega)$, we have that $u(x) = o(|x|)$ as $|x| \to \infty$ by Propositions 2.17.5 and 2.17.6. Since $\mathcal{D}^{\Omega} u(x) \to 0$ as $|x| \to \infty$, we deduce $v(x) = o(|x|)$ as $|x| \to \infty$. According to Corollary 2.16.4, there exists a constant c such that $v \equiv c$. So,

$$u = \mathcal{D}^{\Omega} u + c \quad \text{in } \Omega.$$

Using limit at the boundary (Theorem 5.3.5), we obtain

$$u = \frac{1}{2} u + K_{\Omega} u + c.$$

The trace $u \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2. The operator $(1/2)I - K_{\Omega}$ is an isomorphism on $W^{1-1/p,p}(\partial\Omega)$ by Proposition 7.22.3 and Theorem 1.23.3. Denote $G = R^m \setminus \overline{\Omega}$. Since $u/2 - K_{\Omega} u = c$ and $c/2 - K_{\Omega} c = c/2 + K_G c = c$ by Proposition 5.3.2, we deduce $u = c$ on $\partial\Omega$. Thus, $u = \mathcal{D}^{\Omega} u + c = -\mathcal{D}^{G} c + c = c$ by Proposition 5.3.1.

Theorem 7.27.3 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $C^{1,\alpha}$, $0 < \alpha < 1$, $m/(m-1) < p < \infty$, $p' = p/(p-1)$. If $F \in [\tilde{W}^{1,p'}(\Omega)]'$, then there exists a weak solution $v \in \tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$ of the Neumann problem*

$$-\Delta v = F|\Omega \text{ in } \Omega, \quad \frac{\partial v}{\partial n} = F|\partial\Omega \text{ on } \partial\Omega. \tag{7.95}$$

The general form of a solution of the problem in $D^{1,p}(\Omega)$ is $v + c$ where c is a constant.

Proof If we define $\langle F, \varphi\rangle = \langle F, \varphi|\Omega\rangle$, then $F \in D^{-1,p}(R^m)$. According to Propositions 3.38.3 and 3.38.4 and Corollary 2.18.6, there exists $w \in \tilde{W}^{1,p}(R^m)$ such that $-\Delta w = F$ in R^m. Define $g \in [\tilde{W}^{1,p'}(\Omega)]'$ by

$$\langle g, \varphi\rangle = \langle F, \varphi\rangle - \int_\Omega \nabla w \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m.$$

If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $\langle g, \varphi\rangle = 0$ by Theorem 3.19.3. So, g is supported on $\partial\Omega$. By virtue of Theorem 1.24.2, we deduce that $g \in [W^{1-1/p',p'}(\partial\Omega)]' = W^{-1/p,p}(\partial\Omega)$.

$T = (1/2)I - K'_\Omega$ is an isomorphism on $W^{-1/p,p}(\partial\Omega) = B^{p,p}_{-1/p}(\partial\Omega)$ by Proposition 7.22.3. Put $u = \mathcal{S}^\Omega T^{-1} g$. Then, $u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.93) by Lemma 7.27.1.

$v = w + u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.95). If c is constant, then $v + c \in D^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.95), too. Let now $\psi \in D^{1,p}(\Omega)$ be another weak solution of the Neumann problem (7.95). Then, $v - \psi$ is constant by Lemma 7.27.2.

Theorem 7.27.4 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, and $1 < p \le m/(m-1)$, $p' = p/(p-1)$. If $F \in [\tilde{W}^{1,p'}(\partial\Omega)]'$, then there exists a weak solution $v \in D^{1,p}(\Omega)$ of the Neumann problem for the Poisson equation (7.95) if and only if $\langle F, 1\rangle = 0$. The general form of a solution of the problem is $v + c$ where c is a constant.*

Proof Let $v \in D^{1,p}(\Omega)$ be a weak solution of the Neumann problem (7.95). Define $\varphi \equiv 1$. Since $p' \ge m$, we have that $\varphi \in \tilde{W}^{1,p'}(\Omega)$ by Lemma 3.38.11. Thus,

$$\langle F, 1\rangle = \int_\Omega \nabla v \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m = 0.$$

If $F \equiv 0$, then v is constant by Lemma 7.27.2.

Suppose now that $F \in [\tilde{W}^{1,p'}(\partial\Omega)]'$ be such that $\langle F, 1\rangle = 0$. If we define $\langle F, \varphi\rangle = \langle F, \varphi|\Omega\rangle$, then $F \in [\tilde{W}^{1,p'}(R^m)]' = D^{-1,p}(R^m)$. Since $\langle F, 1\rangle = 0$, we have $F \in \mathcal{E}_p(R^m)$ by Corollary 2.18.6. According to Propositions 3.38.3 and 3.38.4, there exists $w \in \tilde{W}^{1,p}(R^m)$ such that $-\Delta w = F$ in R^m. Define $\tilde{F} \in [\tilde{W}^{1,p'}(\partial\Omega)]'$ by

$$\langle \tilde{F}, \varphi\rangle = \int_\Omega \nabla w \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m.$$

Since w is a weak solution of the Neumann problem

$$-\Delta w = \tilde{F}|\Omega \text{ in } \Omega, \quad \frac{\partial w}{\partial n} = \tilde{F}|\partial\Omega \text{ on } \partial\Omega,$$

we deduce that $\langle \tilde{F}, 1\rangle = 0$. Define $g = F - \tilde{F}$. Then, $\langle g, 1\rangle = 0$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $\langle g, \varphi\rangle = 0$ by Theorem 3.19.3. So, g is supported on $\partial\Omega$. By virtue of Theorem 1.24.2, we deduce that $g \in [W^{1-1/p',p'}(\partial\Omega)]' = W^{-1/p,p}(\partial\Omega)$.

Denote $X = \{f \in W^{-1/p,p}(\partial\Omega); \langle f, 1\rangle = 0\}$. Proposition 7.22.3 says that $(1/2)I - K'_\Omega$ is an isomorphism on $W^{-1/p,p}(\partial\Omega) = B^{p,p}_{-1/p}(\partial\Omega)$ and on X. Put $u = \mathcal{S}^\Omega T^{-1} g$. Then $u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.93) by Lemma 7.27.1. Therefore, $v = w + u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.95). If c is constant, then $v + c \in D^{1,p}(\Omega)$ is a weak solution of the Neumann problem (7.95), too.

7.28 The Robin Problem in $D^{1,p}(\Omega)$ on Exterior Domains

In this section, we study the Robin problem for the Poisson equation in $D^{1,p}(\Omega)$ for an unbounded domain with compact Ljapunov boundary and $1 < p < \infty$. We prove the existence of a solution and discuss the uniqueness.

Lemma 7.28.1 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $u \in D^{1,p}(\Omega)$ *with* $1 < p \le m/(m-1)$, $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$. *If* u *is a weak solution of the homogeneous Robin problem* $\Delta u = 0$ *in* Ω, $\partial u/\partial n + hu = 0$ *on* $\partial\Omega$, *then* $u \equiv 0$.

Proof We show that $u \in D^{1,p'}(\Omega)$, where $p' = p/(p-1)$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Define $h = 0$ on $\partial B(0;r)$, $g = 0$ on $\partial\Omega$, $g = \partial u/\partial n^{B(0;r)}$ on $\partial\Omega$. Using the Gauss-Green theorem (Theorem 1.19.1), we deduce that u is a weak solution of the Robin problem

$$\Delta u = 0 \text{ in } \Omega \cap B(0;r), \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial[\Omega \cap B(0;r)].$$

Since $g \in W^{1-1/p',p'}(\partial[\Omega \cap B(0;r)]) \subset [W^{1,p}(\Omega)]'$, Theorem 7.24.6 gives that $u \in W^{1,p'}(\Omega \cap B(0;r)))$. Since $|\nabla u| \in L^p(\Omega)$, Proposition 2.17.7 forces $u \in D^{1,p'}(\Omega)$.

Since $p' \ge m$, we have $D^{1,p'}(\Omega) = \tilde{W}^{1,p'}(\Omega)$ by Lemma 3.38.11. Since $u \in \tilde{W}^{1,p'}(\Omega)$, the definition of a weak solution of the Robin problem gives

$$\int_\Omega \nabla u \cdot \nabla u \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} huu \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Thus, $\nabla u = 0$ in Ω, $hu = 0$ on $\partial\Omega$. Since $\nabla u \equiv 0$, there exists a constant c such that $u = c$ in Ω (see Lemma 1.20.1). Since $0 = \int_{\partial\Omega} hu \, \mathrm{d}\mathcal{H}_{m-1} = c \int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1}$ and $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, we infer that $c = 0$.

Proposition 7.28.2 *Let* $\Omega \subset R^m$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $\int_{\partial\Omega} h > 0$. *For*

$f \in W^{-1/p,p}(\partial\Omega)$, define

$$Pf = \frac{\langle f, 1\rangle}{\mathcal{H}_{m-1}(\partial\Omega)} \mathcal{H}_{m-1}|\partial\Omega, \tag{7.96}$$

$$Tf = \left[\frac{1}{2}I - K'_\Omega\right](I-P)f + h[\mathcal{S}^\Omega(I-P)f + Pf]. \tag{7.97}$$

Then T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$. If $g \in W^{-1/p,p}(\partial\Omega)$, then

$$u = \mathcal{S}^\Omega(I-P)T^{-1}g + PT^{-1}g \tag{7.98}$$

is a weak solution of the Robin problem for the Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega \tag{7.99}$$

in $D^{1,p}(\Omega)$.

Proof T is a Fredholm operator with index 0 on $B^{p,p}_{-1/p}(\partial\Omega)$ by Proposition 7.22.3 and Theorem 1.8.3. Let $f \in W^{-1/p,p}(\partial\Omega)(= B^{p,p}_{-1/p}(\partial\Omega)$ by Theorem 1.23.3). Then $u = \mathcal{S}^\Omega(I-P)f + Pf \in D^{1,p}(\Omega)$ is a weak solution of the Robin problem (7.99) with $g = Tf$. (See Lemma 7.27.1.) Let now $Tf = 0$. Fix $q \in (1, m/(m-1)$. Then $f \in W^{-1/q,q}(\partial\Omega)$ by Lemma 1.8.4. Since $u \in D^{1,q}(\partial\Omega)$ is a weak solution of the Robin problem (7.99) with $g = 0$, Lemma 7.28.1 forces that $u \equiv 0$. Thus, $\mathcal{S}^\Omega(I-P)f = -Pf$ is constant. So, $\mathcal{S}^\Omega(I-P)f \in D^{1,p}(\Omega)$ is a weak solution of the homogeneous Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = 0$ on $\partial\Omega$. Thus, $[(1/2)I - K'_\Omega](I-P)f = 0$ by Lemma 7.27.1. So, $(I-P)f = 0$ by Proposition 7.22.3. Hence, $-Pf = \mathcal{S}^\Omega(I-P)f = 0$. Thus, $f = (I-P)f + Pf = 0$. Since T is a Fredholm operator with index 0 on $W^{-1/p,p}(\partial\Omega)$ with trivial kernel, it is onto. Theorem 1.6.3 gives that T is an isomorphism.

Theorem 7.28.3 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $1 < p < \infty$, $p' = p/(p-1)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h > 0$, $F \in [\tilde{W}^{1,p'}(\Omega)]'$.*

- *There exists a weak solution $v \in D^{1,p}(\Omega)$ of the Robin problem for the Poisson equation*

$$-\Delta v = F|\Omega \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + hv = F|\partial\Omega \text{ on } \partial\Omega. \tag{7.100}$$

  If $p \leq m/(m-1)$, then this solution is unique.
- *If $m/(m-1) < p$, then there exists a weak solution $v \in \tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$ of the Robin problem for the Poisson equation (7.100). If, moreover, $p < m$, then this solution is unique.*

- *Let $F \equiv 0$, $m/(m-1) < p$. Then the space of weak solutions of the Robin problem (7.100) in $D^{1,p}(\Omega)$ has dimension* 1 *(and does not depend on p).*

Proof Fix $z \in R^m$ and $r > 0$ such that $\overline{B(z;r)} \cap \overline{\Omega} = \emptyset$. According to Corollary 2.18.6, there exists a constant c such that for

$$\langle \tilde{F}, \varphi\rangle := \langle F, \varphi|\Omega\rangle + c \int\limits_{B(z;r)} \varphi \, d\mathcal{H}_m$$

one has $\tilde{F} \in \mathcal{E}_p(R^m)$. Remark that $\tilde{F} = F$ on a neighborhood of $\overline{\Omega}$. According to Propositions 3.38.3 and 3.38.4, there exists $w \in \tilde{W}^{1,p}(R^m)$ such that $-\Delta w = \tilde{F}$ in R^m. Define g by

$$\langle g, \varphi\rangle = \langle F, \varphi\rangle - \int\limits_{\Omega} \nabla w \cdot \nabla\varphi \, d\mathcal{H}_m - \int\limits_{\partial\Omega} hw\varphi \, d\mathcal{H}_{m-1}.$$

Theorem 1.24.2 forces that $g \in [\tilde{W}^{1,p'}(\Omega)]'$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $\langle g, \varphi\rangle = 0$ by Theorem 3.19.3. So, g is supported on $\partial\Omega$. By virtue of Theorem 1.24.2, we deduce that $g \in [W^{1-1/p',p'}(\partial\Omega)]' = W^{-1/p,p}(\partial\Omega)$.

Let T be given by (7.97), where P is prescribed by (7.96). Then, T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$, and $u = \mathcal{S}^\Omega(I - P)T^{-1}g + PT^{-1}g$ is a weak solution of the Robin problem for the Laplace equation (7.99) in $D^{1,p}(\Omega)$. (See Proposition 7.28.2.) So, $v = w + u \in D^{1,p}(\Omega)$ is a weak solution of the original problem (7.100). If $p \le m/(m-1)$, then this solution is unique by Lemma 7.28.1. If $p \ge m$, then $u \in D^{1,p}(\Omega) = \tilde{W}^{1,p}(\Omega)$ by Lemma 3.38.11.

Let now $m/(m-1) < p < m$. Then, $m > 2$. We show that there exists a weak solution of the Robin problem in $\tilde{W}^{1,p}(\Omega)$. The operator $\tilde{T} := (1/2)I - K'_\Omega + h\mathcal{S}^\Omega$ is a Fredholm operator with index 0 on $B^{p,p}_{-1/p}(\partial\Omega) = W^{-1/p,p}(\partial\Omega)$ by Proposition 7.22.3 and Theorem 1.23.3. The operator $\tilde{T}$ is an isomorphism on $W^{-1,p}(\partial\Omega)$ by Theorem 5.20.8. Hence, $\tilde{T}$ is an isomorphism on $W^{-1/p,p}(\partial\Omega)$ by Lemma 1.8.4. The function $\tilde{u} = \mathcal{S}^\Omega\tilde{T}^{-1}g \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the Robin problem for the Laplace equation (7.99) by Lemma 7.27.1. Thus, $v = w + \tilde{u} \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the original problem (7.100).

Let now $F \equiv 0$, $p > m/(m-1)$, $m > 2$. If $u \in D^{1,p}(\Omega)$ is a weak solution of (7.100), then u is a very weak solution of the problem (7.100). (See Proposition 7.19.2). Remark that $\partial u/\partial n = -hu$. According to Proposition 7.9.2, there exists a constant c such that

$$u = \mathcal{D}^\Omega u - \mathcal{S}^\Omega hu + c.$$

Properties of single layer potentials and double layer potentials give that u is a very weak L^p-solution of the problem (7.100). The space of very weak L^p-solutions of the problem (7.100) has dimension 1 by Theorem 5.20.8. Let $c \in R^1$ be given. According to Theorem 5.11.5, there exists a unique L^p-solution u of the problem

(7.100) such that $u(x) \to c$ as $|x| \to \infty$. Remark that this solution does not depend on p. Choose $r \in (0, \infty)$ such that $\partial\Omega \subset B(0;r)$. Since $M_a(u), M_a(\nabla u) \in L^p(\partial\Omega)$, Lemma 1.28.1 gives that $u \in W^{1,p}(\Omega \cap B(0;2r))$. Since $u(x) \to c$ as $|x| \to \infty$, Proposition 2.17.3 gives that $|\nabla u(x)| = O(|x|^{1-m})$ as $|x| \to \infty$. So, $u \in D^{1,p}(\Omega)$. Lemma 5.8.1 gives that u is a weak solution of (7.100). If $p < m$, then $u \in \tilde{W}^{1,p}(\Omega)$ if and only if $c = 0$, i.e., if $u \equiv 0$. (See Proposition 3.38.5.)

Let now $F \equiv 0$, $m = 2$, $p > 2$. If $u \in D^{1,p}(\Omega)$ is a weak solution of (7.100), then u is a very weak solution of the problem (7.100). (See Proposition 7.19.2.) Fix $z \in R^2 \setminus \overline{\Omega}$. Since $|\nabla u| \in L^p(\Omega)$, there exist constants a and b such that $u(x) - a - b \ln|x-z| = O(|x|^{-1})$ as $|x| \to \infty$. (See Proposition 2.17.6.) Define $\rho(x) = \ln|x-z|$, $v_b(x) = u(x) - b\rho$, $\Phi = \partial\rho/\partial n + h\rho$ on $\partial\Omega$. Then $v_b \in D^{1,p}(\Omega)$, because $p > 2$. The function v_b is a very weak solution of the problem (7.99) with $g = -b\Phi$. Choose $r \in (0, \infty)$ such that $\partial\Omega \subset B(0;r)$. Since $v_b \in W^{1,p}(\Omega \cap B(0;2r))$ and v_b is bounded on $\Omega \setminus B(0;r)$, Theorem 7.11.2 gives that $M_a(v_b) \in L^p(\partial\Omega)$, and there exists a nontangential limit of v_b at almost all points of $\partial\Omega$. So, v_b is a very weak L^p-solution of the problem (7.99). According to Theorem 5.20.8, there exists a unique very weak L^p-solution of the problem (7.99). If v_1 is a very weak L^p-solution of the problem (7.99) with $g = -\Phi$, then $v_b = bv_1$. So, $u = b(v_1 + \rho)$. So, the dimension of the space of weak solutions of (7.100) in $D^{1,p}(\Omega)$ is at most 1. We now show that there exists a nontrivial solution of the problem (7.100) in $D^{1,p}(\Omega)$. Clearly $\rho \in D^{1,p}(\Omega)$ is a weak solution of the Robin problem

$$\Delta\rho = 0 \text{ in } \Omega, \quad \frac{\partial\rho}{\partial n} + h\rho = \Phi \text{ on } \partial\Omega.$$

But $\tilde{\rho} = \mathcal{S}^{\Omega}(I - P)T^{-1}\Phi + PT^{-1}\Phi$ is another weak solution of this problem. According to Proposition 3.1.6, we have $\tilde{\rho}(x) = O(1)$ as $|x| \to \infty$. Since $\rho(x) \to \infty$ as $|x| \to \infty$, we infer that $\psi = \rho - \tilde{\rho} \in D^{1,p}(\Omega)$ is a nontrivial weak solution of the Robin problem $\Delta\psi = 0$ in Ω, $\partial\psi/\partial n + h\psi = 0$ on $\partial\Omega$.

7.29 The Neumann and the Robin Problem in $D^{2,p}(\Omega)$ on Exterior Domains

In this section, we study the Neumann problem and the Robin problem for the Poisson equation in $D^{2,p}(\Omega)$, where $\Omega \subset R^m$ is an unbounded domain with compact boundary of class $\mathcal{C}^2$ and $1 < p < \infty$. We prove the existence of a solution and show that the space of solutions for the homogeneous boundary condition and the Laplace equation has dimension $m + 1$.

Lemma 7.29.1 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$ and $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $1 < p < \infty$, $g \in W^{1-1/p,p}(\partial\Omega)$. If u is a weak solution in $D^{1,p}(\Omega)$ of the Robin problem $\Delta u = 0$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$ or* *if u is an L^p-solution of this problem, then $u \in D^{2,p}(\Omega)$ is a strong solution of this problem.*

Proof If u is an L^p-solution, then $u \in W^{1,p}_{loc}(\overline{\Omega})$ by Lemma 1.28.1. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Then $u \in D^{2,p}(\Omega \cap B(0;r))$ by Theorem 7.26.3 and Lemma 5.8.1. If $u \in D^{1,p}(\Omega)$, then $u \in D^{2,p}(\Omega)$ by Proposition 2.17.7. If u is an L^p-solution of the Robin problem in Ω, then $u(x) = O(1)$ as $|x| \to \infty$ by Lemma 1.28.2. So, $|\nabla^2 u(x)| = O(|x|^{-m})$ as $|x| \to \infty$ by Proposition 2.17.3. Hence, $u \in D^{2,p}(\Omega)$. So, u is a strong solution of the problem by Proposition 7.19.1.

Theorem 7.29.2 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$ and $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $1 < p < \infty$. Then there exists a strong solution $u \in D^{2,p}(\Omega)$ of the Robin problem for the Poisson equation*

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega. \tag{7.101}$$

if and only if $f \in L^p(\Omega)$ and $g \in W^{1-1/p,p}(\partial\Omega)$.

Proof If $u \in D^{2,p}(\Omega)$ is a strong solution of the problem (7.101), then $f \in L^p(\Omega)$ and $g \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.2 and Proposition 1.25.2.

Let now $f \in L^p(\Omega)$ and $g \in W^{1-1/p,p}(\partial\Omega)$ be given. Define $f = 0$ on $R^m \setminus \Omega$. According to Proposition 7.4.2, there exists $v \in D^{2,p}(R^m)$ such that $-\Delta v = f$ in R^m. Define

$$\tilde{g} = g - \frac{\partial v}{\partial n} - hv \quad \text{on } \partial\Omega.$$

Then $\tilde{g} \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. We find a strong solution $w \in D^{2,p}(\Omega)$ of the Robin problem

$$-\Delta w = 0 \text{ in } \Omega, \qquad \frac{\partial w}{\partial n} + hw = \tilde{g} \text{ on } \partial\Omega. \tag{7.102}$$

Suppose first that $h \equiv 0$. The operator $(1/2) - K'_\Omega$ is an isomorphism on $L^p(\partial\Omega)$ by Theorem 5.11.4. Define $\psi = [(1/2)I - K'_\Omega]^{-1}\tilde{g}$, $w = \mathcal{S}^\Omega\psi$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Define $\omega = \Omega \cap B(0;r)$, $\psi = 0$ on $\partial\Omega$. Then $w = \mathcal{S}^\omega\psi \in W^{1,p}(\omega)$ is a weak solution of

$$\Delta w = 0 \text{ in } \omega, \quad \frac{\partial w}{\partial n} = \frac{1}{2}\psi - K'_\omega\psi \text{ on } \partial\Omega.$$

(See Proposition 7.24.1.) Since $\psi = 0$ on $\partial B(0;r)$, we see that $\frac{1}{2}\psi - K'_\omega\psi = \tilde{g}$ on $\partial\Omega$, $\frac{1}{2}\psi - K'_\omega\psi = \partial w/\partial n \in W^{1-1/p,p}(\partial\Omega)$ on $\partial B(0;r)$. Thus, $w \in D^{2,p}(\omega)$ by Theorem 7.26.3. Since $|\nabla^2 w(x)| = O(|x|^{-m})$ by Proposition 3.1.6, we infer that $w \in D^{2,p}(\Omega)$. So, w is a strong solution of (7.102) by Proposition 7.19.1. Thus, $u = v + w \in D^{2,p}(\Omega)$ is a strong solution of the original problem (7.101).

Let now $h \not\equiv 0$. According to Theorem 7.28.3, there exists a weak solution $w \in D^{1,p}(\Omega)$ of the problem (7.102). Lemma 7.29.1 gives that $w \in D^{2,p}(\Omega)$ is a strong solution of the problem (7.102). Thus, $u = v + w \in D^{2,p}(\Omega)$ is a strong solution of the original problem (7.101).

Proposition 7.29.3 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$ and $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $1 < p < \infty$. Then, the space of all solutions $u \in D^{2,p}(\Omega)$ of the Robin problem (7.101) with $f \equiv 0$, $g \equiv 0$ has dimension $m + 1$.*

Proof In the rest of the proof, suppose that $f \equiv 0$, $g \equiv 0$. If $u \in D^{2,p}(\Omega)$ is a solution of (7.101), then $\partial u/\partial n = -hu$ and Proposition 7.9.2 gives

$$u = \mathcal{D}^{\Omega} u - \mathcal{S}^{\Omega} hu + P, \tag{7.103}$$

where

$$P(x) = a_0 + a_1 x_1 + \cdots + a_m x_m. \tag{7.104}$$

Suppose first that $h = 0$. If a solution $u \in D^{2,p}(\Omega)$ of (7.101) is given by (7.103), then $v = \mathcal{D}^{\Omega} u$ is an L^p-solution of the Neumann problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial n} = -\frac{\partial P}{\partial n} \text{ on } \partial\Omega, \quad v(x) \to 0 \text{ as } |x| \to \infty. \tag{7.105}$$

(See Theorems 5.3.5, 5.5.2, and 7.11.2.) Let $a_0, \dots, a_m$ be given, and P have a form (7.104). Since P is harmonic in $R^m \setminus \overline{\Omega}$, Corollary 2.2.3 gives that

$$\int_{\partial\Omega} \frac{\partial P}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

According to Theorem 5.11.4, there exists a unique L^p-solution of the Neumann problem (7.105). Lemma 7.29.1 forces that $v \in D^{2,p}(\Omega)$ is a strong solution of (7.105). Hence, $u = v + P \in D^{2,p}(\Omega)$ is a strong solution of the original problem (7.101).

Let now $h \not\equiv 0$ and $m > 2$. If a solution $u \in D^{2,p}(\Omega)$ of (7.101) is given by (7.103), then $v = \mathcal{D}^{\Omega} u - \mathcal{S}^{\Omega} hu$ is an L^p-solution of the Robin problem

$$\Delta v = 0 \text{ in } \Omega, \; \frac{\partial v}{\partial n} + hv = -\frac{\partial P}{\partial n} - hP \text{ on } \partial\Omega, \; v(x) \to 0 \text{ as } |x| \to \infty. \tag{7.106}$$

(See Theorems 5.3.5 and 5.5.2; Propositions 3.1.6, 5.2.3, and 5.2.8; and Theorems 5.4.7 and 7.11.2.) Let $a_0, \dots, a_m$ be given, and P have a form (7.104). According to Theorem 5.11.5, there exists a unique L^p-solution of the Robin problem (7.106). Lemma 7.29.1 forces that $v \in D^{2,p}(\Omega)$ is a strong solution of (7.106). Hence, $u = v + P \in D^{2,p}(\Omega)$ is a strong solution of the original problem (7.101).

Suppose now $h \not\equiv 0$ and $m = 2$. Let a solution $u \in D^{2,p}(\Omega)$ of (7.101) be given by (7.103). Fix $z \in R^2 \setminus \overline{\Omega}$. According to Proposition 2.17.6, there exists a constant α such that $\alpha \ln|x-z| - \mathcal{S}^{\Omega} hu(x) = O(1)$ as $|x| \to \infty$. Put

$$\tilde{P}(x) = a_1 x_1 + \cdots + a_m x_m - \alpha \ln|x-z|, \tag{7.107}$$

$v = u - \tilde{P}$. Then v is an L^p-solution of the Robin problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + hv = -\frac{\partial \tilde{P}}{\partial n} - h\tilde{P} \text{ on } \partial\Omega. \tag{7.108}$$

(See Theorems 5.3.5 and 5.5.2; Propositions 3.1.6, 5.2.3, and 5.2.8; and Theorems 5.4.7 and 7.11.2.) Let $\alpha, a_1, \dots, a_m$ be given, and $\tilde{P}$ have a form (7.107). According to Theorem 5.11.5, there exists a unique L^p-solution of the Robin problem (7.108). Lemma 7.29.1 forces that $v \in D^{2,p}(\Omega)$ is a strong solution of (7.108). Hence, $u = v + \tilde{P} \in D^{2,p}(\Omega)$ is a strong solution of the original problem (7.101).

7.30 The Neumann and the Robin Problem in $D^{k,p}(\Omega)$ for a Planar Domain

In this section, we study the Neumann problem and the Robin problem for the Poisson equation for a domain $\Omega \subset R^2$ with compact boundary of class $\mathcal{C}^{k+2}$, $k \in N$. If Ω is bounded, we look for a solution in $W^{k+2,p}(\Omega)$ and in $B^{p,q}_s(\Omega)$, where $1 < p, q < \infty$ and $2 < s < k+2$. If Ω is unbounded, we look for a solution in $D^{k+2,p}(\Omega)$. We find a necessary and sufficient condition for the existence of a solution and characterize the space of solutions for the homogeneous boundary condition and the Laplace equation.

Theorem 7.30.1 *Let $\Omega \subset R^2$ be a bounded domain with boundary of class $\mathcal{C}^{k+2}$, $k \in N_0$, $1 < p < \infty$. Then there exists a solution $u \in W^{k+2,p}(\Omega)$ of the Neumann problem*

$$-\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \tag{7.109}$$

if and only if $f \in W^{k,p}(\Omega)$, $g \in W^{k+1-1/p,p}(\partial\Omega)$ and

$$\int_{\Omega} f \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{7.110}$$

This solution is unique up to an additive constant. If $u \in W^{k+2,p}(\Omega)$ *is a solution of the Neumann problem (*7.109*) such that*

$$\int_{\Omega} u \, d\mathcal{H}_m = 0, \tag{7.111}$$

then

$$\|u\|_{W^{k+2,p}(\Omega)} \leq C\left[\|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-1/p,p}(\partial\Omega)}\right] \tag{7.112}$$

where a constant C does not depend on g and f.

Proof Let $u \in W^{k+2,p}(\Omega)$ be a solution of the problem (7.109). Then $f \in W^{k,p}(\Omega)$ and $g \in W^{k+1-1/p,p}(\Omega)$ by Theorem 1.24.5. Theorem 7.26.2 gives (7.110). If $f \equiv 0$, $g \equiv 0$, then u is constant by Theorem 7.26.2.

Let now f and g satisfy (7.110). We show the existence of a solution. Suppose first that the boundary $\partial\Omega$ is connected and $f \equiv 0$. Denote by $\tau = (-n_2^{\Omega}, n_1^{\Omega})$ a unit tangential vector on $\partial\Omega$. The tangential derivative $\partial/\partial\tau$ is a mapping from $W^{k+2-1/p,p}(\partial\Omega)$ onto

$$\{\varphi \in W^{k+2-1/p,p}(\partial\Omega); \int_{\partial\Omega} \varphi \, d\mathcal{H}_{m-1} = 0\}.$$

(Compare Proposition 1.21.11.) So, there exists $h \in W^{k+2-1/p,p}(\partial\Omega)$ such that $\partial h/\partial\tau = f$. According to Theorem 7.14.3, there exists a solution $v \in W^{k+2,p}(\Omega)$ of the Dirichlet problem

$$\Delta v = 0 \text{ in } \Omega, \qquad v = h \text{ on } \partial\Omega.$$

The function v is an L^p-solution of this problem by Theorem 7.11.2. According to Proposition 5.15.5, there exists $\varphi \in W^{1,p}(\partial\Omega)$ such that $v = \mathcal{D}^{\Omega}\varphi$. Denote $\psi = -\partial\varphi/\partial\tau$, $u = \mathcal{S}^{\Omega}\psi$. Since $\psi \in L^p(\partial\Omega)$, there exist the nontangential limits of u and ∇u at almost all point of $\partial\Omega$, and $M_a(u) + M_a(\nabla u) \in L^p(\partial\Omega)$. (See Propositions 5.2.2 and 5.2.8 and Theorem 5.4.7.) Hence, $u \in W^{1,p}(\Omega)$ by Lemma 1.28.1. Since $\partial_1 u = \partial_2 v \in W^{k+1,p}(\Omega)$, $\partial_2 u = -\partial_1 v \in W^{k+1,p}(\Omega)$ by Corollary 5.5.3, we infer that $u \in W^{k+2,p}(\Omega)$. The trace of $\partial_j u$ is equal to the nontangential limit of $\partial_j u$ by Theorem 7.11.2. According to Corollary 5.5.3,

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial \tau} = \frac{\partial h}{\partial \tau} = f.$$

Hence, u is a solution of the Neumann problem (7.109).

Let now f and g satisfy (7.110) and the boundary $\partial\Omega$ be connected. According to Theorem 7.14.3, there exists $w \in W^{k+2,p}(\Omega)$ such that $-\Delta w = f$ in Ω.

Theorem 1.24.5 and Lemma 1.21.17 give that $\partial w/\partial n \in W^{k+1-1/p,p}(\partial\Omega)$. Since

$$\int_{\Omega} f \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \frac{\partial w}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = 0,$$

we have

$$\int_{\partial\Omega} \left[g - \frac{\partial w}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega} f + \int_{\partial\Omega} g - \int_{\Omega} f - \int_{\partial\Omega} \frac{\partial w}{\partial n} = 0.$$

We have proved the existence of a solution $v \in W^{k+2,p}(\Omega)$ of the problem

$$\Delta v = 0 \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial n} = g - \frac{\partial w}{\partial n} \quad \text{on } \partial\Omega.$$

Clearly, $u = w + v \in W^{k,p}(\Omega)$ is a weak solution of (7.109).

Let now f and g satisfy (7.110) and the boundary $\partial\Omega$ be disconnected. According to Theorem 7.26.2, there exists a solution $u \in W^{2,p}(\Omega)$ of the problem (7.109). We use the induction argument. Suppose that $k \in N$ and there exists a solution $u \in W^{k+1,p}(\Omega)$ of the problem (7.109). Since $\overline{\Omega}$ is compact, there exist open sets $U_1, \dots, U_L$ such that $\overline{\Omega} \subset U_1 \cup \cdots \cup U_L$ and $\Omega_j = U_j \cap \Omega$ is a domain with connected boundary of class $\mathcal{C}^{k+2}$. According to Lemma 1.18.1, there exist $\varphi_j \in \mathcal{C}_c^{\infty}(R^2)$ such that $\operatorname{spt}\varphi \subset U_j$ and $\varphi_1 + \cdots + \varphi_L = 1$ on $\overline{\Omega}$. We have $\varphi_j u \in W^{k+1,p}(\Omega_j)$, $\Delta(\varphi_j u) = \varphi_j \Delta u + 2\nabla\varphi_j \cdot \nabla u + u\Delta\varphi_j \in W^{k,p}(\Omega_j)$, $\partial(u\varphi_j)/\partial n = u(\partial\varphi/\partial n) + \varphi(\partial u/\partial n) \in W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. We have proved that $u\varphi_j \in W^{k+2,p}(\Omega_j)$. Therefore, $u\varphi_j \in W^{k+2,p}(\Omega)$. Hence, $u = u\varphi_1 + \cdots + u\varphi_L \in W^{k+2,p}(\Omega)$.

The operator

$$T : u \mapsto \left[-\Delta u, \frac{\partial u}{\partial n}, \int_{\Omega} u \, \mathrm{d}\mathcal{H}_2 \right]$$

is a bounded linear operator from $W^{k+2,p}(\Omega)$ onto $\{[f, g, c] \in W^{k,p}(\Omega) \times W^{k+1-1/p}(\partial\Omega) \times R^1; \int_{\Omega} f \, \mathrm{d}\mathcal{H}_2 + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$. Since it is one-to-one, it is an isomorphism by Theorem 1.6.3.

Theorem 7.30.2 *Let $\Omega \subset R^2$ be a bounded domain with boundary of class $\mathcal{C}^{k+2}$, $k \in N_0$, $1 < p < \infty$, $h \in \mathcal{C}^{k+1}(\partial\Omega)$, $h \geq 0$, $h \not\equiv 0$. Then there exists a solution $u \in W^{k+2,p}(\Omega)$ of the Robin problem*

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} + hu = g \text{ on } \partial\Omega \tag{7.113}$$

if and only if $f \in W^{k,p}(\Omega)$, $g \in W^{k+1-1/p,p}(\partial\Omega)$. *This solution is unique, and the estimate (7.112) holds with a constant* C *that does not depend on* g *and* f.

Proof Define $T_1 u = [-\Delta u, \partial u/\partial n]$, $T_2 u = [-\Delta u, \partial u/\partial n + hu]$. The operator T_1 is a Fredholm operator with index 0 from $W^{k+2,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 7.30.1. The operator $T_2 - T_1$ is a compact operator from $W^{k+2,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{k+1-1/p,p}(\partial\Omega)$ by Theorems 1.24.5, 1.23.3, and 1.23.14 and Lemma 1.21.17. Hence, T_2 is a Fredholm operator with index 0 from $W^{k+2,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 1.8.3. The operator T_2 is an isomorphism $W^{2,p}(\Omega)$ to $L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ by Theorem 7.26.5. Therefore, T_2 is an isomorphism from $W^{k+2,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{k+1-1/p,p}(\partial\Omega)$ by Lemma 1.8.4.

Theorem 7.30.3 *Let* $\Omega \subset R^2$ *be a bounded domain with boundary of class* $\mathcal{C}^{k+2}$, $k \in N$, $1 < p, q < \infty$, $2 < s < k+2$, $h \in \mathcal{C}^{k+1}(\partial\Omega)$, $h \geq 0$, $h \not\equiv 0$. *Then there exists a solution* $u \in B_s^{p,q}(\Omega)$ *of the Robin problem (7.113) if and only if* $f \in B_{s-2}^{p,q}(\Omega)$, $g \in B_{s-1-1/p}^{p,q}(\partial\Omega)$. *This solution is unique and*

$$\|u\|_{B_s^{p,q}(\Omega)} \leq C\left[\|f\|_{B_{s-2}^{p,q}(\Omega)} + \|g\|_{B_{s-1-1/p}^{p,q}(\partial\Omega)}\right] \tag{7.114}$$

where a constant C *does not depend on* g *and* f.

Proof The operator $Tu = [-\Delta u, \partial u/\partial n + hu]$ is an isomorphism from $W^{k+2,p}(\Omega)$ to $W^{k,p}(\Omega) \times W^{k+1-1/p}(\partial\Omega)$ and from $W^{2,p}(\Omega)$ to $L^p(\Omega) \times W^{1-1/p}(\partial\Omega)$ by Theorem 7.30.2. Put $\theta = (s-2)/k$. Then

$$[W^{1-1/p,p}(\partial\Omega), W^{k+1-1/p,p}(\partial\Omega)]_{\theta,q} = B_{s-1-1/p}^{p,q}(\Omega)$$

by Theorems 1.23.3 and 1.23.11. We have $W^{l,p}(\Omega) = F_l^{p,2}(\Omega)$ for $l \in N_0$ by Theorems 1.23.4, 1.23.5, and 1.22.1. Theorem 1.23.16 gives

$$[W^{2,p}(\Omega), W^{k+2,p}(\Omega)]_{\theta,q} = B_s^{p,q}(\Omega),$$
$$[L^p(\Omega), W^{k,p}(\Omega)]_{\theta,q} = B_{s-2}^{p,q}(\Omega).$$

The interpolation argument (Proposition 1.10.3) gives that T is an isomorphism from $B_s^{p,q}(\Omega)$ to $B_{s-2}^{p,q}(\Omega) \times B_{s-1-1/p}^{p,q}(\partial\Omega)$.

Theorem 7.30.4 *Let* $\Omega \subset R^2$ *be a bounded domain with boundary of class* $\mathcal{C}^{k+2}$, $k \in N$, $1 < p, q < \infty$, $2 < s < k+2$. *Then, there exists a solution* $u \in B_s^{p,q}(\Omega)$ *of the Neumann problem (7.109) if and only if* $f \in B_{s-2}^{p,q}(\Omega)$, $g \in B_{s-1-1/p}^{p,q}(\partial\Omega)$, *and (7.110) holds. This solution is unique up to an additive constant. If* $u \in B_s^{p,q}(\Omega)$ *is a solution of the Neumann problem (7.109) satisfying (7.111), then (7.114) holds with a constant* C *that does not depend on* g *and* f.

Proof The operator

$$T : u \mapsto \left[-\Delta u, \frac{\partial u}{\partial n}, \int\limits_{\Omega} u \, \mathrm{d}\mathcal{H}_2\right]$$

is an isomorphism from $W^{k+2,p}(\Omega)$ to

$$\{[f, g, c] \in W^{k,p}(\Omega) \times W^{k+1-1/p}(\partial\Omega) \times R^1; \int_{\Omega} f \, \mathrm{d}\mathcal{H}_2 + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$$

and from $W^{2,p}(\Omega)$ to

$$\{[f, g, c] \in L^p(\Omega) \times W^{1-1/p}(\partial\Omega) \times R^1; \int_{\Omega} f \, \mathrm{d}\mathcal{H}_2 + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$$

by Theorem 7.30.1.

Put $\theta = (s-2)/k$. Then

$$[W^{1-1/p,p}(\partial\Omega), W^{k+1-1/p,p}(\partial\Omega)]_{\theta,q} = B^{p,q}_{s-1-1/p}(\Omega)$$

by Theorems 1.23.3 and 1.23.11. We have $W^{l,p}(\Omega) = F^{p,2}_l(\Omega)$ for $l \in N_0$ by Theorems 1.23.4, 1.23.5, and 1.22.1. Theorem 1.23.16 gives

$$[W^{2,p}(\Omega), W^{k+2,p}(\Omega)]_{\theta,q} = B^{p,q}_s(\Omega),$$
$$[L^p(\Omega), W^{k,p}(\Omega)]_{\theta,q} = B^{p,q}_{s-2}(\Omega).$$

The interpolation argument (Propositions 1.10.1 and 1.10.3) gives that T is an isomorphism from $B^{p,q}_s(\Omega)$ to $\{[f, g, c] \in B^{p,q}_{s-2}(\Omega) \times B^{p,q}_{s-1-1/p}(\partial\Omega) \times R^1; \int_\Omega f \, \mathrm{d}\mathcal{H}_2 + \int_{\partial\Omega} g \, \mathrm{d}\mathcal{H}_1 = 0\}$.

Proposition 7.30.5 *Let $\Omega \subset R^2$ be an open set with compact boundary of class $\mathcal{C}^{k+2}$, $k \in N$, $1 < p < \infty$. Let $f \in D^{k,p}(\Omega)$, $g \in W^{k+1-1/p,p}(\partial\Omega)$ and $h \in \mathcal{C}^{k+1}(\partial\Omega)$. Suppose that $f = 0$ outside of a bounded set. If $u \in D^{k+1,p}(\Omega)$ is a solution of the Robin problem (7.113), then $u \in D^{k+2,p}(\Omega)$.*

Proof Suppose first that Ω is bounded. Then u is a solution of the Neumann problem $-\Delta u = f$ in Ω, $\partial u/\partial n = g - hu$ on $\partial\Omega$. Proposition 1.25.2, Theorem 1.24.5, and Lemma 1.21.17 give that $g - hu \in W^{k+1-1/p,p}(\partial\Omega)$. Thus, $u \in W^{k+2,p}(\Omega) = D^{k+2,p}(\Omega)$ by Theorem 7.30.1.

Suppose now that Ω is unbounded. Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$ and $f = 0$ on $\Omega \setminus B(0; r)$. Since $|\nabla^{k+1} u| \in L^p(\Omega \setminus B(0; 2r))$, Proposition 2.17.7 gives that $|\nabla^{k+2} u| \in L^p(\Omega \setminus B(0; 2r))$. Define $h = 0$ and $g = \partial u/\partial n$ on $\partial B(0; 2r)$. Then $-\Delta u = f$ in $\Omega \cap B(0; 2r)$, $\partial u/\partial n + hu = g$ on $\partial[\Omega \cap B(0; 2r)]$. We have proved that $u \in D^{k+2,p}(\Omega \cap B(0; 2r))$. Hence, $u \in D^{k+2,p}(\Omega)$.

Theorem 7.30.6 *Let $\Omega \subset R^2$ be an unbounded domain with compact boundary of class $\mathcal{C}^{k+2}$, $k \in N_0$, $h \in \mathcal{C}^{k+1}(\partial\Omega)$, $h \geq 0$ and $1 < p < \infty$. Then there exists a strong solution $u \in D^{k+2,p}(\Omega)$ of the Robin problem (7.113) if and only if $f \in D^{k,p}(\Omega)$ and $g \in W^{k+1-1/p,p}(\partial\Omega)$. If $f \equiv 0$, $g \equiv 0$, then the space of all strong solutions $u \in D^{k+2,p}(\Omega)$ of the Robin problem (7.113) has the dimension equal to* $\dim[H(R^2) \cap \tilde{P}_{k+1}(R^2)]$, *where $H(R^2)$ is the space of all harmonic function in R^2 and $\tilde{P}_{k+1}(R^2)$ is the space of all polynomials of degree at most $k+1$.*

Proof For $k = 0$, see Theorem 7.29.2 and Proposition 7.29.3. In the rest of the proof, we suppose that $k \in N$.

If $u \in D^{k+2,p}(\Omega)$ is a solution of the Robin problem (7.113), then $f \in D^{k,p}(\Omega)$ and $g \in W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5.

Let now $f \in D^{k,p}(\Omega)$ and $g \in W^{k+1-1/p,p}(\partial\Omega)$ be given. According to Theorem 7.15.3, there exists $v \in D^{k+2,p}(\Omega)$ such that $-\Delta v = f$ in Ω. Define $\tilde{g} = g - \partial v/\partial n - hv$ on $\partial\Omega$. Then, $\tilde{g} \in W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5 and Lemma 1.21.17. According to Theorem 7.29.2, there exists a strong solution $w \in D^{2,p}(\Omega)$ of the problem

$$-\Delta w = 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial n} + hw = \tilde{g} \text{ on } \partial\Omega.$$

By virtue of Proposition 7.30.5 and induction argument, we deduce that $w \in D^{k+2,p}(\Omega)$. Thus, $u = v + w \in D^{k+2,p}(\Omega)$ is a strong solution of the original problem (7.113).

Let now $f \equiv 0$, $g \equiv 0$. If $u \in D^{k+2,p}(\Omega)$ is a strong solution of the Robin problem (7.113), then $\partial u/\partial n = -hu$ and Proposition 7.9.2 give

$$u = \mathcal{D}^{\Omega} - \mathcal{S}^{\Omega} hu + P,$$

where $P \in \tilde{P}_{k+1}(R^2)$. Denote $X = \{Q \in \tilde{P}_{k+1}(R^2); Q(0) = 0, \nabla Q(0) = 0\}$. There exist $Q_1 \in \tilde{P}_1(R^2)$ and $Q_2 \in X$ such that $P = Q_1 + Q_2$. Put $v = \mathcal{D}^{\Omega} - \mathcal{S}^{\Omega} hu + Q_1$. Since $u, v \in H(\Omega)$, we infer that $Q_2 = u - v \in H(\Omega)$. Moreover, $v = u - Q_2 \in D^{k+2,p}(\Omega)$ is a strong solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + hv = -\frac{\partial Q_2}{\partial n} - hQ_2 \text{ on } \partial\Omega. \tag{7.115}$$

Let now $Q_2 \in X \cap H(\Omega)$ be given. According to Theorem 7.29.2, there exists a solution $v \in D^{2,p}(\Omega)$ of the problem (7.115). Proposition 7.30.5 gives that $v \in D^{k+2,p}(\Omega)$. So, $u = v + Q_2 \in D^{k+2,p}(\Omega)$ is a strong solution of the problem (7.113). By virtue of Proposition 7.29.3, we deduce that the space of strong solutions of (7.113) in $D^{k+2,p}(\Omega)$ has dimension equal to $m + 1 + \dim(X \cap H(\Omega)) = \dim(\tilde{P}_{k+1}(R^2) \cap H(\Omega))$.

7.31 Oblique Derivative Problem in $\tilde{W}^{1,2}(\Omega)$

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Let $a_{ij} \in \mathcal{C}^1(R^m)$ be functions with compact support such that $a_{ji} = -a_{ij}$. Define the tangential vector function

$$\tau = A(x)n^\Omega$$

and put

$$\Theta = n^\Omega(x) + \tau(x).$$

We shall study the following oblique derivative problem

$$-\Delta u = F|\Omega \text{ in } \Omega, \quad \Theta \cdot \nabla u + \frac{1}{2}\sum_{i,j=1}^m n_j(\partial_i a_{ij})u + hu = F|\partial\Omega \text{ on } \partial\Omega \tag{7.116}$$

in $\tilde{W}^{1,2}(\Omega)$. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

To define a weak solution of the problem, suppose first that $u, a_{ij} \in \mathcal{C}^2(R^m)$. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then we get by the Gauss-Green formula

$$\begin{aligned}\langle F, \varphi\rangle &= \int_\Omega \varphi(-\Delta u)\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \varphi \left\{\frac{\partial u}{\partial n} + \sum_{i,j=1}^m n_j[a_{ij}\partial_i u + \frac{1}{2}u\partial_i a_{ij}] + hu\right\}\\ &= \int_\Omega \{\nabla\varphi\cdot\nabla u + \sum_{i,j=1}^m [a_{ij}\partial_j\varphi\partial_i u + a_{ij}\varphi\partial_i\partial_j u + (\partial_j a_{ij})\varphi\partial_i u + \frac{1}{2}\varphi(\partial_i a_{ij})\partial_j u\\ &\quad + \frac{1}{2}u(\partial_i a_{ij})\partial_j\varphi + \frac{1}{2}\varphi u\partial_j\partial_i a_{ij}]\}\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega}\varphi hu\,\mathrm{d}\mathcal{H}_{m-1}.\end{aligned}$$

Since $a_{ji} = -a_{ij}$, we have

$$\sum_{i,j}(\partial_i\partial_j a_{ij})u\varphi = 0, \quad a_{ij}\varphi\partial_i\partial_j u = 0.$$

Thus,

$$\langle F, \varphi \rangle = \int_{\Omega} \{\nabla\varphi \cdot \nabla u + \sum_{i,j=1}^{m} [a_{ij}\partial_j\varphi\partial_i u + (\partial_j a_{ij})\varphi\partial_i u - \frac{1}{2}\varphi(\partial_j a_{ij})\partial_i u$$
$$-\frac{1}{2}u(\partial_j a_{ij})\partial_i\varphi]\} \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \varphi h u \, \mathrm{d}\mathcal{H}_{m-1}.$$

This motivates our definition of the weak solution of the problem (7.116).

Let $F \in [\tilde{W}^{1,2}(\Omega)]'$. We say that $u \in \tilde{W}^{1,2}(\Omega)$ is a weak solution of the oblique derivative problem (7.116) if

$$(u, \varphi)_{A,h} = \langle F, \varphi \rangle \qquad \forall \varphi \in \tilde{W}^{1,2}(\Omega),$$

where

$$(u, \varphi)_{A,h} := \int_{\Omega} \{\nabla\varphi \cdot \nabla u + \sum_{i,j=1}^{m} [a_{ij}\partial_j\varphi\partial_i u + \frac{1}{2}(\partial_j a_{ij})\varphi\partial_i u$$
$$-\frac{1}{2}(\partial_j a_{ij})u\partial_i\varphi]\} \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \varphi h u \, \mathrm{d}\mathcal{H}_{m-1}.$$

Lemma 7.31.1 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Suppose that $a_{ij} \in \mathcal{C}^1(R^m)$ are functions with compact support such that $a_{ji} = -a_{ij}$. Let $F \in [\tilde{W}^{1,2}(\Omega)]'$ and $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of the oblique derivative problem (7.116). If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then*

$$\langle F, \varphi \rangle = (u, \varphi)_{A,h} = \int_{\Omega} \nabla u \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m. \tag{7.117}$$

Therefore, $-\Delta u = F$ in Ω in the sense of distributions.

Proof Suppose first that $u \in \mathcal{C}_c^\infty(R^m)$, $a_{ij} \in \mathcal{C}^2(R^m)$. By the same reasoning as in the beginning of this section, we obtain

$$\langle F, \varphi \rangle = (u, \varphi)_{A,h} = -\int_{\Omega} \varphi\Delta u \, \mathrm{d}\mathcal{H}_m.$$

According to Theorem 3.19.3, we have

$$\langle F, \varphi \rangle = (u, \varphi)_{A,h} = \int_{\Omega} \nabla u \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m.$$

Let now $u \in \tilde{W}^{1,2}(\Omega)$, $a_{ij} \in \mathcal{C}^2(R^m)$. According to Proposition 1.25.2 and Lemma 1.21.3, there exists a sequence $u_k \in \mathcal{C}_c^\infty(R^m)$ such that $u_k \to u$ in $\tilde{W}^{1,2}(\Omega)$. Thus,

$$(u,\varphi)_{A,h} = \lim_{k\to\infty}(u_k,\varphi)_{A,h} = \lim_{k\to\infty}\int_\Omega \nabla u_k\cdot\nabla\varphi \,\mathrm{d}\mathcal{H}_m = \int_\Omega \nabla u\cdot\nabla\varphi\,\mathrm{d}\mathcal{H}_m.$$

Let now $u \in \tilde{W}^{1,2}(\Omega)$, $a_{ij} \in \mathcal{C}^1(R^m)$. Choose a nonnegative function $\phi \in \mathcal{C}^\infty(R^m)$ supported in $B(0;1)$ such that

$$\int_{R^m} \phi(x)\, d\mathcal{H}_m(x) = 1.$$

For $\epsilon > 0$, put $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^m$, $a_{ij}^\epsilon = a_{ij} * \phi_\epsilon$. Then $a_{ij}^\epsilon \in \mathcal{C}_c^\infty(R^m)$ and $a_{ij}^\epsilon \to a_{ij}$ in $\mathcal{C}^{1,0}(\overline{\Omega})$. (See Sect. 1.15.) Denote by A^ϵ the matrix (a_{ij}^ϵ). Then,

$$(u,\varphi)_{A,h} = \lim_{\epsilon\downarrow 0}(u,\varphi)_{A^\epsilon,h} = \int_\Omega \nabla u\cdot\nabla\varphi\,\mathrm{d}\mathcal{H}_m.$$

Since (7.117) holds for all $\varphi \in \mathcal{C}_c^\infty(R^m)$, Theorem 3.19.3 gives that $-\Delta u = F$ in Ω in the sense of distributions.

Lemma 7.31.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Suppose that $a_{ij} \in \mathcal{C}^1(R^m)$ are functions with compact support such that $a_{ji} = -a_{ij}$. If $u \in \tilde{W}^{1,2}(\Omega)$, then*

$$(u,u)_{A,h} = \int_\Omega |\nabla u|^2\,\mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} h|u|^2\,\mathrm{d}\mathcal{H}_{m-1}. \tag{7.118}$$

Proof Since $a_{ji} = -a_{ij}$, we have

$$\sum_{i,j=1}^m a_{ij}\partial_i u\partial_j u = 0.$$

This gives (7.118).

Theorem 7.31.3 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h\,\mathrm{d}\mathcal{H}_{m-1} > 0$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Suppose that $a_{ij} \in \mathcal{C}^1(R^m)$ are functions with compact support such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = A(x)n^\Omega(x)$, and put $\Theta = n^\Omega + \tau$. If $F \in [\tilde{W}^{1,2}(\Omega)]'$, then there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of* *the oblique derivative problem (7.116). Moreover,*

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\| F\|_{[\tilde{W}^{1,2}(\Omega)]'} \tag{7.119}$$

with a constant C that does not depend on F.

Proof If $u \in \tilde{W}^{1,2}(\Omega)$, then

$$(u, u)_{A,h} = \int\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} h|u|^2 \, \mathrm{d}\mathcal{H}_{m-1}$$

by Lemma 7.31.2. According to Corollary 7.21.4, we have that $(\, , \,)_{A,h}$ is an inner product on $\tilde{W}^{1,2}(\Omega)$, and the corresponding norm $\| \, \|_{A,h}$ is equivalent to the original norm. Let now $F \in [\tilde{W}^{1,2}(\Omega)]'$. According to the Riesz representation theorem (Theorem 1.9.10), there exists a unique $u \in \tilde{W}^{1,2}(\Omega)$ such that

$$(u, v)_{A,h} = \langle F, v\rangle \qquad \forall v \in \tilde{W}^{1,2}(\Omega).$$

Moreover,

$$\|u\|_{A,h} = \| F\|_{[\tilde{W}^{1,2}(\Omega)]'}.$$

Theorem 7.31.4 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $h \equiv 0$. Let $A = (a_{ij})$ be a matrix function of the type $m\times m$. Suppose that $a_{ij} \in \mathcal{C}^1(R^m)$ are functions with compact support such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = A(x)n^{\Omega}(x)$ and put $\Theta = n^{\Omega} + \tau$. Fix $F \in [\tilde{W}^{1,2}(\Omega)]'$.*

- *If Ω is unbounded and $m > 2$, then there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the oblique derivative problem (7.116). Moreover, the estimate (7.119) holds with a constant C that does not depend on F.*
- *If*

$$\int\limits_{\partial\Omega} |\sum_{i,j=1}^{m} n_j \partial_i a_{ij}|^2 \, \mathrm{d}\mathcal{H}_{m-1} > 0, \tag{7.120}$$

  then there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the oblique derivative problem (7.116). Moreover, the estimate (7.119) holds with a constant C that does not depend on F.
- *Let*

$$\int\limits_{\partial\Omega} |\sum_{i,j=1}^{m} n_j \partial_i a_{ij}|^2 \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{7.121}$$

Suppose that Ω *is bounded or* $m = 2$. *Then there exists a solution* $u \in \tilde{W}^{1,2}(\Omega)$ *of the oblique derivative problem (7.116) if and only if* $\langle F, 1\rangle = 0$. *If* $u \in \tilde{W}^{1,2}(\Omega)$ *is a solution of the oblique derivative problem (7.116), then a general form of a solution of the problem (7.116) in* $\tilde{W}^{1,2}(\Omega)$ *is* $v = u + c$, *where* c *is an arbitrary constant. If* $u \in \tilde{W}^{1,2}(\Omega)$ *is a solution of the oblique derivative problem (7.116), then*

$$\|\nabla u\|_{L^2(\partial\Omega)} \le C\| F\|_{[\tilde{W}^{1,2}(\Omega)]'};$$

if moreover

$$\int\limits_{\partial\Omega} u \,\mathrm{d}\mathcal{H}_{m-1} = 0 \tag{7.122}$$

then

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \le C\| F\|_{[\tilde{W}^{1,2}(\Omega)]'}.$$

Here a constant C *does not depend on* F.

Proof For $u \in \tilde{W}^{1,2}(\Omega)$, define the distributions $T_0 u$ and $T_1 u$ by

$$\langle T_0 u, \varphi\rangle = (u, \varphi)_{A,0}, \quad \langle T_1 u, \varphi\rangle = (u, \varphi)_{A,1}.$$

Then T_0 and T_1 are bounded linear operators from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ (compare Theorem 1.24.1). Since the trace γ_Ω is a compact linear operator from $\tilde{W}^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ (see Theorem 1.24.1 and Proposition 1.25.2), $T_1 - T_0$ is a compact linear operator from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ (see Theorem 1.7.4). The operator T_1 is an isomorphism by Theorem 7.31.3. So, T_0 is a Fredholm operator with index 0 by Theorem 1.8.3.

Let $u_0 \in \operatorname{Ker} T_0$. Lemma 7.31.2 gives that

$$0 = \langle T_0 u_0, u_0\rangle = \int\limits_{\Omega} |\nabla u_0|^2 \,\mathrm{d}\mathcal{H}_m.$$

Since $\nabla u_0 \equiv 0$ and Ω is connected, there exists a constant c such that $u_0 \equiv c$ (see Lemma 1.20.1).

Suppose first that Ω is unbounded and $m > 2$. The zero is the only constant function in $\tilde{W}^{1,2}(\Omega)$ by Lemma 3.38.11. Hence, the kernel $\operatorname{Ker} T_0$ is trivial. Since T_0 is a Fredholm operator with index 0, $T_0(\tilde{W}^{1,2}(\Omega)) = [\tilde{W}^{1,2}(\Omega)]'$. Thus, T_0 is an isomorphism by Theorem 1.6.3.

Suppose now that (7.120) holds. We can choose $\psi_k \in \mathcal{C}_c^\infty(R^m)$ such that ψ_k converge to $\sum_{i,j=1}^m n_j \partial_i a_{ij}$ in $L^2(\partial\Omega)$ (see Proposition 1.14.6 and Lemma 1.16.4).

Since

$$\int_{\partial\Omega} \psi_k \sum_{i,j=1}^{m} n_j \partial_i a_{ij}\, \mathrm{d}\mathcal{H}_{m-1} \to \int_{\partial\Omega} |\sum_{i,j=1}^{m} n_j \partial_i a_{ij}|^2\, \mathrm{d}\mathcal{H}_{m-1} > 0,$$

we can choose $\psi \in \mathcal{C}_c^\infty(R^m)$ such that

$$\int_{\partial\Omega} \psi \sum_{i,j=1}^{m} n_j \partial_i a_{ij}\, \mathrm{d}\mathcal{H}_{m-1} > 0. \tag{7.123}$$

According to Lemma 5.23.8 and the fact that $u_0 \equiv c$,

$$0 = \langle T_0 u_0, \psi\rangle = -\frac{1}{2}\int_{\Omega} \sum_{i,j=1}^{m} (\partial_j a_{ij}) c \partial_i \psi\, \mathrm{d}\mathcal{H}_m = \frac{c}{2}\int_{\partial\Omega} \psi \sum_{i,j=1}^{m} n_j \partial_i a_{ij}\, \mathrm{d}\mathcal{H}_{m-1}.$$

The relation (7.123) forces that $c = 0$. Since T_0 is a Fredholm operator with index 0 and trivial kernel, $T_0(\tilde{W}^{1,2}(\Omega)) = [\tilde{W}^{1,2}(\Omega)]'$. Thus, T_0 is an isomorphism by Theorem 1.6.3.

Let (7.121) hold. Suppose that Ω is bounded or $m = 2$. Then the constant functions are from $\tilde{W}^{1,2}(\Omega)$ by Lemma 3.38.11. If $u \equiv c$ and $\varphi \in \tilde{W}^{1,2}(\Omega)$, then

$$(u, \varphi)_{A,h} = -\frac{c}{2}\int_{\Omega} \sum_{i,j=1}^{m} (\partial_j a_{ij}) \partial_i \varphi\, \mathrm{d}\mathcal{H}_m = \frac{c}{2}\int_{\partial\Omega} \varphi \sum_{i,j=1}^{m} n_j (\partial_i a_{ij})\, \mathrm{d}\mathcal{H}_{m-1} = 0$$

by Lemma 5.23.8 and (7.121). Thus, u is a solution of the problem (7.116) with $F = 0$ if and only if u is constant.

Let now $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of the problem (7.116). By virtue of Lemma 5.23.8 and (7.121),

$$\langle F, 1\rangle = \int_{\Omega} \frac{1}{2} \sum_{ij=1}^{m} (\partial_j a_{ij}) \partial_i u\, \mathrm{d}\mathcal{H}_m = -\frac{1}{2}\int_{\partial\Omega} u \sum_{i,j=1}^{m} n_j (\partial_i a_{ij})\, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Since T_0 is a Fredholm operator with one-dimensional kernel, we deduce that $T_0(\tilde{W}^{1,2}(\Omega)) = X := \{ F \in [\tilde{W}^{1,2}(\Omega)]'; \langle F, 1\rangle = 1\}$.

Define

$$Tu = [T_0 u, \int_{\partial\Omega} u\, \mathrm{d}\mathcal{H}_{m-1}].$$

Then T is a bounded linear operator from $\tilde{W}^{1,2}(\Omega)$ to $X \times R^1$ by Theorem 1.24.1. Since T is one-to-one and onto, there exists a constant C such that

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C[\|T_0 u\|_{[\tilde{W}^{1,2}(\Omega)]'} + |\int_{\partial\Omega} u \, \mathrm{d}\mathcal{H}_{m-1}|].$$

(see Theorem 1.6.3). Let now $v \in \tilde{W}^{1,2}(\Omega)$ be a solution of the problem (7.116). Then there exists a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the problem (7.116) satisfying (7.122). Since $u - v$ is constant, we have

$$\|\nabla v\|_{L^2(\Omega)} \leq \|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\|F\|_{[\tilde{W}^{1,2}(\Omega)]'}.$$

7.32 Transmission Problem in $D^{1,p}$

In this section, we study the transmission problem for the Laplace equation in $D^{1,p}(\Omega_\pm)$. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. (We do not suppose that Ω_+ is connected.) Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Denote by n the exterior unit normal of Ω_+. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega_+)$, $1 < p < \infty$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$. *We say that* $u_\pm \in D^{1,p}(\Omega_\pm)$ *is a weak solution of the transmission problem*

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{7.124}$$

$$a_+ u_+ - a_- u_- = g, \quad b_+ \frac{\partial u_+}{\partial n} - b_- \frac{\partial u_-}{\partial n} + h u_+ = f \quad \text{on } \partial\Omega_+ \tag{7.125}$$

if $u_\pm \in H(\Omega_\pm)$, *the condition* $a_+ u_+ - a_- u_- = g$ *is fulfilled in the sense of traces and*

$$b_+ \int_{\Omega_+} \nabla u_+ \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m + b_- \int_{\Omega_-} \nabla u_- \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega_+} h u_+ \varphi \, \mathrm{d}\mathcal{H}_{m-1} = \langle f, \varphi \rangle$$

for all $\varphi \in \mathcal{C}_c^\infty(R^m)$ *(or equivalently for all* $\varphi \in \tilde{W}^{1,p'}(R^m)$ *with* $p' = p/(p-1)$*).*

If $\tilde{u}_\pm = a_\pm u$, $\tilde{b}_\pm = b_\pm / a_\pm$, $\tilde{h} = h/a_+$, then $u_\pm$ is a weak solution of the transmission problem (7.124), (7.125) if and only if $\tilde{u}_\pm$ is a weak solution of the transmission problem

$$\Delta \tilde{u}_\pm = 0 \quad \text{in } \Omega_\pm, \tag{7.126}$$

$$\tilde{u}_+ - \tilde{u}_- = g, \quad \tilde{b}_+ \frac{\partial \tilde{u}_+}{\partial n} - \tilde{b}_- \frac{\partial \tilde{u}_-}{\partial n} + \tilde{h} \tilde{u}_+ = f \quad \text{on } \partial\Omega_+. \tag{7.127}$$

Lemma 7.32.1 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary. Set* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Let* $1 < p < \infty$. *If* $p \neq 2$, *suppose that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $\tilde{b}_\pm$ *be positive constants,* $\tilde{h} \in L^\infty(\partial\Omega_+)$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$. *Then* $\mathcal{D}^\Omega g \in D^{1,p}(\Omega_\pm)$,

$$[\partial\mathcal{D}^\Omega g/\partial n^\Omega]_\pm = H^\Omega g \in W^{-1/p,p}(\partial\Omega).$$

If

$$\tilde{u}_\pm = v_\pm + \mathcal{D}^\Omega g$$

then $\tilde{u}_\pm$ *is a weak solution of the transmission problem (7.126), (7.127) in* $D^{1,p}(\Omega_\pm)$ *if and only if* $v_\pm$ *is a weak solution of the transmission problem*

$$\Delta v_\pm = 0 \quad in\ \Omega_\pm, \tag{7.128}$$

$$v_+ - v_- = 0, \quad \tilde{b}_+\frac{\partial v_+}{\partial n} - \tilde{b}_-\frac{\partial v_-}{\partial n} + \tilde{h}v_+ = F \quad on\ \partial\Omega_+. \tag{7.129}$$

in $D^{1,p}(\Omega_\pm)$, *where*

$$F = f + (\tilde{b}_- - \tilde{b}_+)H^\Omega g - \tilde{h}(g/2 + K_\Omega g) \in W^{-1/p,p}(\partial\Omega).$$

Proof $\mathcal{D}^\Omega g \in D^{1,p}(\Omega_+)$ by Proposition 7.7.2. Define $g = 0$ on $R^m \setminus \partial\Omega$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. For $\omega(r) = \Omega_- \cap B(0;r)$, we have $\mathcal{D}^\Omega g = -\mathcal{D}^{\omega(r)}g \in D^{1,p}(\omega(r))$ by Proposition 7.7.2. Since $|\nabla\mathcal{D}^\Omega g(x)| = O(|x|^{-m})$ as $|x| \to \infty$, we deduce that $\mathcal{D}^\Omega g \in D^{1,p}(\Omega_-)$.

Proposition 7.7.3 gives that $[\partial\mathcal{D}^\Omega g/\partial n]_+ = H^\Omega g \in W^{-1/p,p}(\partial\Omega)$ and H^Ω is a bounded linear operator on $W^{-1/p,p}(\partial\Omega)$. If $g \in W^{1,p}(\partial\Omega)$, then $[\partial\mathcal{D}^\Omega g/\partial n]_- = H^\Omega g$ by Theorem 5.5.2. Let now $g \in W^{1-1/p,p}(\partial\Omega)$. Choose $g_k \in W^{1,p}(\partial\Omega)$ such that $g_k \to g$ in $W^{1-1/p,p}(\partial\Omega)$. Define $g_k = 0$ in $R^m \setminus \partial\Omega$. Let $\varphi \in \mathcal{C}_c^\infty(R^m)$. Choose $r > 0$ such that $\partial\Omega \cup \operatorname{spt}\varphi \subset B(0;r)$. By virtue of Proposition 7.7.3,

$$\langle[\partial\mathcal{D}^\Omega g/\partial n]_-, \varphi\rangle = \int\limits_{\omega(r)} \nabla\varphi \cdot \nabla\mathcal{D}^{\omega(r)}g \,\mathrm{d}\mathcal{H}_m = \lim_{k\to\infty} \int\limits_{\omega(r)} \nabla\varphi \cdot \nabla\mathcal{D}^{\omega(r)}g_k \,\mathrm{d}\mathcal{H}_m$$

$$= \lim_{k\to\infty} \int\limits_{\Omega_-} \nabla\varphi \cdot \nabla\mathcal{D}^\Omega g_k \,\mathrm{d}\mathcal{H}_m = \lim_{k\to\infty} \langle H^\Omega g_k, \varphi\rangle = \langle H^\Omega g, \varphi\rangle.$$

Hence, $[\partial\mathcal{D}^\Omega g/\partial n]_- = H^\Omega g$.

If $\tilde{u}_\pm = v_\pm + \mathcal{D}^\Omega g$, then $\tilde{u}_\pm$ is a weak solution of the transmission problem (7.126), (7.127) in $D^{1,p}(\Omega_\pm)$ if and only if $v_\pm$ is a weak solution of the transmission problem (7.128), (7.129) in $D^{1,p}(\Omega_\pm)$. (See Theorems 5.3.5 and 7.11.2.)

Lemma 7.32.2 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary. Set* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Suppose that* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $a_\pm$, $b_\pm$ *be positive constants,* $h \in L^\infty(\partial\Omega_+)$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$, $u_\pm \in D^{1,p}(\Omega_\pm)$ *be a weak solution of the transmission problem (7.124), (7.125).*

- *If* $u_-(x) = O(1)$ *as* $|x| \to \infty$, *then* $u_\pm$ *is a very weak* L^p*-solution of the transmission problem (7.124), (7.125).*
- *If* $m > 2$, *then there exists* $u_\infty \in R^1$ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$.
- *If* $p \leq m/(m-1)$, *then there exists* $u_\infty \in R^1$ *such that* $u_-(x) \to u_\infty$ *and* $|\nabla u_-(x)| = O(|x|^{-m})$ *as* $|x| \to \infty$.

Proof Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$. Denote $\omega = \Omega_- \cap B(0; r)$. According to Theorem 7.11.2, we have $M_a^\Omega(u_+), M_a^\omega(u_-) \in L^p(\partial\Omega)$, and there exist the nontangential limits of $u_\pm$ at almost all points of $\partial\Omega$. Moreover, the nontangential limit of $u_\pm$ is equal to the trace of $u_\pm$. If $u_-(x) = O(1)$ as $|x| \to \infty$, then $M_a^{\Omega_-}(u_-) \in L^p(\partial\Omega)$. Proposition 7.19.2 forces that $u_\pm$ is a very weak L^p-solution of (7.124), (7.125).

Let now $m > 2$. Since $|\nabla u_-| \in L^p(\Omega_-)$, Proposition 2.17.5 gives that there exists $u_\infty \in R^1$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$.

Let now $p \leq m/(m-1)$. We can suppose that $0 \notin \overline{\Omega_-}$. Since $|\nabla u_-| \in L^p(\Omega_-)$, Propositions 2.17.5 and 2.17.6 give that there exist constants c and u_∞ such that for $v = u_- - u_\infty - ch_m$, we have $v(x) = O(|x|^{1-m})$ as $|x| \to \infty$. According to Proposition 2.12.1, we have $|\nabla v(x)| = O(|x|^{-m})$ as $|x| \to \infty$. Thus, $|\nabla v| \in L^p(\Omega_-)$. Since there exists a positive constant c_1 such that $|\nabla h_m(x)| = c_1|x|^{1-m}$, we infer $|\nabla h_m| \notin L^p(\Omega_-)$. But $|c\nabla v| = |\nabla u - \nabla v| \in L^p(\Omega_-)$. This forces $c = 0$.

Lemma 7.32.3 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary. Set* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Suppose that* $1 < p < \infty$. *Let* $\tilde{b}_\pm$ *be positive constants,* $\tilde{h} \in L^\infty(\partial\Omega_+)$. *Let* $\psi \in W^{-1/p,p}(\partial\Omega)$. *If* $p \leq m/(m-1)$, *suppose moreover* $\langle\psi, 1\rangle = 0$. *Then,* $v_\pm = \mathcal{S}^\Omega\psi$ *is a weak solution of the transmission problem (7.128), (7.129) in* $\tilde{W}^{1,p}(\Omega_\pm) \subset D^{1,p}(\Omega_\pm)$ *with* $F = (\tilde{b}_+ + \tilde{b}_-)\psi/2 - (\tilde{b}_+ - \tilde{b}_-)K'_\Omega\psi + \tilde{h}\mathcal{S}^\Omega\psi$.

Proof Lemma is a consequence of Lemma 7.27.1 and Propositions 7.24.1 and 7.6.2.

Lemma 7.32.4 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary. Set* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Suppose that* $1 < p < \infty$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $b_\pm$ *be positive constants,* $h \in L^\infty(\partial\Omega_+)$, $h \geq 0$. *For* $f \in W^{-1/p,p}(\partial\Omega)$, *define* $Pf = \langle f, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$, $T = [(b_+ + b_-)/2]I - (b_+ - b_-)K'_\Omega + h\mathcal{S}^\Omega$, $\tilde{T} = T(I - P) + hP$.

- T *and* $\tilde{T}$ *are Fredholm operators with index* 0 *on* $W^{-1/p,p}(\partial\Omega)$.
- *If* $m > 2$, *then the operator* T *is an isomorphism on* $W^{-1/p,p}(\partial\Omega)$.
- *If* $m = 2$ *and* $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$, *then the operator* $\tilde{T}$ *is an isomorphism on* $W^{-1/p,p}(\partial\Omega)$.
- *If* $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} = 0$, *then the operator* T *is an isomorphism on* $W^{-1/p,p}(\partial\Omega)$ *and on* $X_0 = \{f \in W^{-1/p,p}(\partial\Omega); \langle f, 1\rangle = 0\}$.

Proof According to Lemma 5.21.1, the operators T and $\tilde{T}$ are Fredholm with index 0 in $W^{-1,p}(\partial\Omega)$. The operators T and $\tilde{T}$ are Fredholm operators with index 0 in $L^p(\partial\Omega)$ by Propositions 5.9.9, 5.11.1, and 5.2.2 and Theorem 1.8.3. By the interpolation, we obtain that T and $\tilde{T}$ are Fredholm operators with index 0 on $W^{-1/p,p}(\partial\Omega)$. (See Propositions 1.10.4 and 1.23.12 and Theorems 1.23.3 and 1.22.1.)

If $m > 2$, then T is an isomorphism on $L^p(\partial\Omega)$ by Propositions 5.12.2 and 5.12.4. Lemma 1.8.4 gives that T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$.

Let now $m = 2$, $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$. Then $\tilde{T}$ is an isomorphism on $L^p(\partial\Omega)$ by Proposition 5.12.5. Lemma 1.8.4 forces that $\tilde{T}$ is an isomorphism on $W^{-1/p,p}(\partial\Omega)$.

Let now $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} = 0$. Proposition 7.22.3 gives that $T(X_0) \subset X_0$. If $m > 2$, then T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$, and thus it is an isomorphism on X_0 by Lemma 1.8.8. Let now $m = 2$. Lemma 1.8.8 forces that T is a Fredholm operator with index 0 on X_0. Since T is an isomorphism on $Y_0 = \{f \in L^p(\partial\Omega); \int_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0\}$ by Proposition 5.12.3, Lemma 1.8.4 gives that T is an isomorphism on X_0. If $f \equiv 1$ on $\partial\Omega$, then $Tf \notin X_0$ by Lemma 5.9.6. Hence, $\operatorname{Ker} T = \{0\}$ and $T(W^{-1/p,p}(\partial\Omega)) = W^{-1/p,p}(\partial\Omega)$. So, T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$ by Theorem 1.6.3.

Theorem 7.32.5 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $m > 2$. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Suppose that $m/(m-1) < p < \infty$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega_+)$, $h \geq 0$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$.*

- *If $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125), then there exists $u_\infty \in R^1$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$.*
- *If $u_\infty \in R^1$ is given, then there exists a unique weak solution $u_\pm \in D^{1,p}(\Omega_\pm)$ of the transmission problem (7.124), (7.125) with $u_-(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \leq C\big[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{W^{-1/p,p}(\partial\Omega)} + |u_\infty|\big],$$

  where a constant C does not depend on f, g, and u_∞.
- *If $h \equiv 0$ and $v_\pm \in D^{1,p}(\Omega)$ is a particular weak solution of the transmission problem (7.124), (7.125), then the general form of a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ is $u_\pm = v_\pm + c/a_\pm$, where c is a constant.*
- *If $p < m$, then there exists a unique weak solution $u_\pm \in \tilde{W}^{1,p}(\Omega_\pm)$ of the transmission problem (7.124), (7.125). Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \leq C\big[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{W^{-1/p,p}(\partial\Omega)}\big],$$

  where a constant C does not depend on f and g.

Proof If $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125), then $u_\pm$ is a very weak L^p-solution of the problem, and there exists $u_\infty \in R^1$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. (See Lemma 7.32.2.) If $u_\infty = 0$, $g \equiv 0, f \equiv 0$, then $u_\pm \equiv 0$ by Theorem 5.21.2.

We now show the existence of a solution. Suppose first that $a_\pm = 1$, $g \equiv 0$. The operator $T = [(b_+ + b_-)/2]I - (b_+ - b_-)K'_\Omega + h\mathcal{S}^\Omega$ is an isomorphism on $W^{-1/p,p}(\partial\Omega)$ by Lemma 7.32.4. Define $u_\pm = \mathcal{S}^\Omega T^{-1} f$. Then $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125) by Lemma 7.32.3. Moreover, $u_-(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6. Let now $u_\infty \in R^1$ be given. We have proved that there exists a weak solution $v_\pm \in D^{1,p}(\Omega_\pm)$ of the problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$v_+ - v_- = 0, \quad b_+ \frac{\partial v_+}{\partial n} - b_- \frac{\partial v_-}{\partial n} + h v_+ = f - u_\infty h \quad \text{on } \partial\Omega,$$

$$v_-(x) \to 0 \quad \text{as } |x| \to \infty.$$

Clearly, $u_\pm = v_\pm + u_\infty \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125) such that $u_-(x) \to u_\infty$ as $|x| \to \infty$.

Let now $a_\pm$ and g be general. Reasoning at the beginning of this section and Lemma 7.32.1 give that there exists a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$.

Denote by $u_\pm^{f,g,u_\infty}$ a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Then $[f, g, u_\infty] \mapsto [u_+^{f,g,u_\infty}, u_-^{f,g,u_\infty}]$ is a linear mapping from $W^{-1/p,p}(\partial\Omega) \times W^{1-1/p,p}(\partial\Omega) \times R^1$ to $D^{1,p}(\Omega_+) \times D^{1,p}(\Omega_-)$ that is closed by Theorem 5.21.2. Closed-graph theorem (Theorem 1.4.15) gives that this mapping is bounded.

Let now $p < m$. Let $u_\pm$ be a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to 0$ as $|x| \to \infty$. Then $u_\pm \in \tilde{W}^{1,p}(\Omega_\pm)$ by Propositions 3.38.5 and 1.25.2.

Let $h \equiv 0$ and $v_\pm \in D^{1,p}(\Omega)$ be a particular weak solution of the transmission problem (7.124), (7.125). If c is a constant, then $u_\pm = v_\pm + c/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$, too. Let now $w_\pm \in D^{1,p}(\Omega_\pm)$ be a weak solution of the transmission problem (7.124), (7.125). Then there exist constants w_∞, v_∞ such that $w_-(x) \to w_\infty$, $v_-(x) \to v_\infty$ as $|x| \to \infty$. Put $c = a_-(w_\infty - v_\infty)$.Then $u_\pm = v_\pm + c/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to w_\infty$ as $|x| \to \infty$. Hence, $w_\pm = u_\pm$.

Theorem 7.32.6 *Let* $\Omega = \Omega_+ \subset R^m$ *be a bounded open set with Lipschitz boundary. Set* $\Omega_- = R^m \setminus \overline{\Omega_+}$. *Let* $1 < p \le m/(m-1)$. *If* $p \neq 2$, *suppose that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $a_\pm$, $b_\pm$ *be positive constants,* $h \equiv 0$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$.

- *If* $u_\pm \in D^{1,p}(\Omega_\pm)$ *is a weak solution of the transmission problem (7.124), (7.125), then* $\langle f, 1\rangle = 0$, *and there exists* $u_\infty \in R^1$ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$.
- *If* $\langle f, 1\rangle = 0$ *and* $u_\infty \in R^1$ *is given, then there exists a unique weak solution* $u_\pm \in D^{1,p}(\Omega_\pm)$ *of the transmission problem (7.124), (7.125) with* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \le C\big[\|g\|_{W^{1-1/p,p}(\partial\Omega)} \\ + \|f\|_{W^{-1/p,p}(\partial\Omega)} + |u_\infty|\big],$$

  where a constant C *does not depend on* f, g, *and* u_∞.
- *If* $v_\pm \in D^{1,p}(\Omega)$ *is a particular weak solution of the transmission problem (7.124), (7.125), then the general form of a weak solution of the transmission problem (7.124), (7.125) in* $D^{1,p}(\Omega_\pm)$ *is* $u_\pm = v_\pm + c/a_\pm$, *where* c *is a constant.*
- *If* $\langle f, 1\rangle = 0$ *and* $p < m$ *(i.e., if* $p \neq 2$*), then there exists a unique weak solution* $u_\pm \in \tilde W^{1,p}(\Omega_\pm)$ *of the transmission problem (7.124), (7.125). Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \le C\big[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{W^{-1/p,p}(\partial\Omega)}\big],$$

  where a constant C *does not depend on* f *and* g.

Proof Let $u_\pm \in D^{1,p}(\Omega_\pm)$ be a weak solution of the transmission problem (7.124), (7.125). Since $\langle \partial u_\pm/\partial n, 1\rangle = 1$ by Theorem 7.27.4, Proposition 7.20.1, and Theorem 7.24.7, we infer that $\langle f, 1\rangle = 0$. Lemma 7.32.2 gives that $u_-(x) \to u_\infty$ as $|x| \to \infty$ for some $u_\infty \in R^1$, and $u_\pm$ is a weak L^p-solution of the transmission problem (7.124), (7.125). If $u_\infty = 0$, $f \equiv 0$, and $g \equiv 0$, then $u_\pm \equiv 0$ by Theorems 5.21.2 and 5.21.4.

Let now f, g, and u_∞ be given, $\langle f, 1\rangle = 0$. We show the existence of a solution. Suppose first that $a_\pm = 1, g \equiv 0$. The operator $T = [(b_+ + b_-)/2]I - (b_+ - b_-)K'_\Omega$ is an isomorphism on $X_0 = \{\varphi \in W^{-1/p,p}(\partial\Omega); \langle \varphi, 1\rangle = 0\}$ by Lemma 7.32.4. Define $u_\pm = \mathcal{S}^\Omega T^{-1} f + u_\infty$. Then $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125) by Lemma 7.32.3. Moreover, $u_-(x) \to u_\infty$ as $|x| \to \infty$ by Proposition 3.1.6.

Let now $a_\pm$ and g be general. Reasoning at the beginning of this section and Lemma 7.32.1 give that there exists a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$.

Denote by $u_\pm^{f,g,u_\infty}$ a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Then $[f, g, u_\infty] \mapsto [u_+^{f,g,u_\infty}, u_-^{f,g,u_\infty}]$ is a linear mapping from $X_0 \times W^{1-1/p,p}(\partial\Omega) \times R^1$ to $D^{1,p}(\Omega_+) \times D^{1,p}(\Omega_-)$ that is closed by Theorems 5.21.2 and 5.21.4. Closed-graph theorem (Theorem 1.4.15) gives that this mapping is bounded.

Let $v_\pm \in D^{1,p}(\Omega)$ be a particular weak solution of the transmission problem (7.124), (7.125). If c is a constant, then $u_\pm = v_\pm + c/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$, too. Let now $w_\pm \in D^{1,p}(\Omega_\pm)$ be a weak solution of the transmission problem (7.124), (7.125). Then there exist constants w_∞, v_∞ such that $w_-(x) \to w_\infty$, $v_-(x) \to v_\infty$ as $|x| \to \infty$. Put $c = a_-(w_\infty - v_\infty)$.Then $u_\pm = v_\pm + c/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to w_\infty$ as $|x| \to \infty$. Hence, $w_\pm = u_\pm$.

Let now $p < m$. Let $u_\pm$ be a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to 0$ as $|x| \to \infty$. Then $u_\pm \in \tilde{W}^{1,p}(\Omega_\pm)$ by Propositions 3.38.5 and 1.25.2.

Theorem 7.32.7 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary. Set $\Omega_- = R^m \setminus \overline{\Omega_+}$. Let $1 < p \le m/(m-1)$. If $p \neq 2$, suppose that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $\int_{\partial\Omega} h \, d\mathcal{H}_{m-1} > 0$, $g \in W^{1-1/p,p}(\partial\Omega_+)$, $f \in W^{-1/p,p}(\partial\Omega_+)$. Then the following assertions hold:*

- *There exists a unique weak solution $u_\pm \in D^{1,p}(\Omega_\pm)$ of the transmission problem (7.124), (7.125). Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \le C\big[\|g\|_{W^{1-1/p,p}(\partial\Omega)} + \|f\|_{W^{-1/p,p}(\partial\Omega)}\big],$$

  where a constant C does not depend on f, g, and u_∞.
- *For $\psi \in W^{-1/p,p}(\partial\Omega)$, define $P\psi = \langle\psi, 1\rangle/\mathcal{H}_{m-1}(\partial\Omega)$,*

$$\tilde{T} = [(b_+ + b_-)/2](I - P) - (b_+ - b_-)K'_\Omega(I - P) + h[\mathcal{S}^\Omega(I - P) + P].$$

  Then $\tilde{T}$ is an isomorphism on $W^{-1/p,p}(\partial\Omega)$.
- *If $a_\pm = 1$, then*

$$u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega \tilde{T}^{-1} F$$

  where

$$F = f + (b_+ - b_-)H^\Omega g - h(g/2 + K_\Omega g).$$

Proof By virtue of reasoning at the beginning of this section, we can suppose that $a_\pm = 1$.

Let $u_\pm \in D^{1,p}(\Omega_\pm)$ be a weak solution of the transmission problem (7.124), (7.125) with $g \equiv 0, f \equiv 0$. Lemma 7.32.2 gives that $u_\pm$ is a weak L^p-solution of the transmission problem (7.124), (7.125). Theorems 5.21.2, 5.21.3, and 5.12.7 force that $u_\pm$ is an L^2-solution of the transmission problem (7.124), (7.125). Define $u = u_\pm$ on $\Omega_\pm$. Since $u_+ = u_-$ on $\partial\Omega$, we can define u on $\partial\Omega$ as the nontangential limit of $u_\pm$. According to Lemma 7.32.2, we have $u_-(x) = O(1)$, $|\nabla u_-(x)| = O(|x|^{-m})$

as $|x| \to \infty$. By virtue of Lemma 5.8.1,

$$0 = \int\limits_{\partial\Omega} u \left[hu + b_+ \frac{\partial u_+}{\partial n} - b_- \frac{\partial u_-}{\partial n} \right] \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\Omega} hu^2 \, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ b_+ \int\limits_{\Omega_+} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + b_- \lim_{r \to \infty} \int\limits_{\partial[B(0;r) \cap \Omega_-]} u_-(n^{\Omega_-} \cdot \nabla u_-) \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int\limits_{\partial\Omega} hu^2 \, \mathrm{d}\mathcal{H}_{m-1} + b_+ \int\limits_{\Omega_+} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + b_- \int\limits_{\Omega_-} |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m.$$

Since $h \geq 0$, we deduce that $\nabla u = 0$ on $\Omega_\pm$ and $hu = 0$ on $\partial\Omega$. Since $\nabla u = 0$ on $R^m \setminus \partial\Omega$, the function u is constant on each component of $R^m \setminus \partial\Omega$. Since the nontangential limit of u with respect to Ω_+ is equal to the nontangential limit of u with respect to Ω_+, the function u is constant on R^m. Since $hu = 0$ on $\partial\Omega$ and $h \not\equiv 0$, we infer that $u \equiv 0$.

Now we show that $\tilde{T}$ is an isomorphism on $W^{-1/p,p}(\partial\Omega)$. Let now $g \equiv 0$ and $\psi \in W^{-1/p,p}(\partial\Omega)$. Then $u_\pm = \mathcal{S}^\Omega(I-P)\psi + P\psi$ in $\Omega_\pm$ is a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ with $a_\pm = 1$, $g \equiv 0$ if and only if $\tilde{T}\psi = f$. (See Lemma 7.32.3.) Suppose that $\tilde{T}\psi = 0$. Then $u_\pm$ is a solution of the problem (7.124), (7.125) with $f \equiv 0$. We have proved that $u_\pm \equiv 0$. According to Proposition 7.24.1 and Lemma 7.27.1,

$$0 = \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} = \left[\frac{1}{2} I - K'_\Omega \right] (I - P)\psi + \left[\frac{1}{2} I + K'_\Omega \right] (I - P)\psi = (I - P)\psi.$$

Hence, $0 = u_\pm = P\psi$, and $\psi = (I - P)\psi + P\psi = 0$. The operator $\tilde{T}$ is Fredholm with index 0 on $W^{-1/p,p}(\partial\Omega)$ by Lemma 7.32.4. Therefore, $\tilde{T}(W^{-1/p,p}(\partial\Omega)) = W^{-1/p,p}(\partial\Omega)$. So, $\tilde{T}$ is an isomorphism by Theorem 1.6.3.

Let now $a_\pm = 1$ and f and g be given. Put

$$F_{f,g} = f + (b_+ - b_-)H^\Omega g - h(g/2 + K_\Omega g)$$

Then, $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega \tilde{T}^{-1} F_{f,g}$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ by Lemma 7.32.1. The mapping

$$Q : [g,f] \mapsto \mathcal{D}^\Omega g + \mathcal{S}^\Omega \tilde{T}^{-1} F_{f,g}$$

is a linear operator from $W^{1-1/p,p}(\partial\Omega) \times W^{1-1/p,p}(\partial\Omega)$ to $D^{1,p}(\Omega_+) \times D^{1,p}(\Omega_-)$. Since $H^\Omega : W^{1-1/p,p}(\partial\Omega) \to W^{-1/p,p}(\partial\Omega)$ is a bounded linear operator by Proposition 7.7.3, the mapping Q is closed. Closed-graph theorem (Theorem 1.4.15) gives that the mapping Q is bounded.

Theorem 7.32.8 *Let* $\Omega = \Omega_+ \subset R^2$ *be a bounded open set with boundary of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $\Omega_- = R^2 \setminus \overline{\Omega}$. *Let* $a_\pm$, $b_\pm$ *be positive constants,* $h \equiv 0$, $2 < p < \infty$. *Let* $g \in W^{1-1/p,p}(\partial\Omega)$, $f \in W^{-1/p,p}(\partial\Omega)$.

- *Then there exists a weak solution* $u_\pm \in D^{1,p}(\Omega)$ *of the transmission problem (7.124), (7.125). The general form of a weak solution of the transmission problem (7.124), (7.125) in* $D^{1,p}(\Omega_\pm)$ *is* $u_\pm + c/a_\pm$, *where* c *is a constant.*
- *Let* $\langle f, 1\rangle = 0$. *If* $u_\pm$ *is a weak solution in* $D^{1,p}(\Omega)$ *of the transmission problem (7.124), (7.125), then*

$$u_\pm \in \bigcap_{1<q\le p} D^{1,q}(\Omega_\pm) \tag{7.130}$$

and there exists a constant u_∞ *such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *On the other hand, if* u_∞ *is given, then there exists a unique weak solution* $u_\pm \in D^{1,p}(\Omega_\pm)$ *of the transmission problem (7.124), (7.125) such that* $u_-(x) \to u_\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\|u_+\|_{D^{1,p}(\Omega_+)} + \|u_-\|_{D^{1,p}(\Omega_-)} \le C\Big[\|g\|_{W^{1-1/p,p}(\partial\Omega)}$$
$$+\|f\|_{W^{-1/p,p}(\partial\Omega)} + |u_\infty|\Big]$$

where a constant C *does not depend on* f, g, *and* u_∞.

Proof Define $\tilde b_\pm = b_\pm/a_\pm$, $T = [(\tilde b_+ + \tilde b_-)/2]I - (\tilde b_+ - \tilde b_-)K'_\Omega$. Then T is an isomorphism on $W^{-1/p,p}(\partial\Omega)$ by Lemma 7.32.4. Define

$$F = f + (\tilde b_- - \tilde b_+)H^\Omega g,$$
$$u_\pm = a_\pm^{-1}[\mathcal{D}^\Omega g + \mathcal{S}^\Omega T^{-1} F].$$

Then $u_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ by Lemmas 7.32.1 and 7.32.3. If c is a constant, then $u_\pm + c/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$, too.

Suppose that $\langle f, 1\rangle = 0$. Since $H^\Omega g$ is the normal derivative of $\mathcal{D}^\Omega g$ by Proposition 7.7.3, Theorem 7.24.7 forces $\langle H^\Omega g, 1\rangle = 0$. Hence, $\langle F, 1\rangle = 0$. So, $\langle T^{-1}F, 1\rangle = 0$ by Lemma 7.32.4. According to Proposition 3.1.6, one has $u_-(x) \to 0$, $|\nabla u_-(x) = O(|x|^{-2})|$ as $|x| \to \infty$. This and Hölder's inequality (Proposition 1.14.1) give that $u_\pm \in D^{1,q}(\Omega_\pm)$ for all $q \in (1, p\rangle$. If $u_\infty \in R^1$ is given, then $v_\pm = u_\pm + u_\infty a_-/a_\pm$ is a weak solution of the transmission problem (7.124), (7.125) such that $v_-(x) \to u_\infty$ as $|x| \to \infty$.

Let now $v_\pm$ be a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ with $g \equiv 0, f \equiv 0$. One has $\langle \partial v_+/\partial n, 1\rangle = 0$ by Theorem 7.24.7. So, $\langle \partial v_-/\partial n, 1\rangle = (b_+/b_-)\langle \partial v_+/\partial n, 1\rangle = 0$. Fix $z \in \Omega_+$. Since $|\nabla v_-| \in L^p(\Omega_-)$, there exist constants c_1 and c_2 such that for $w(x) = v_-(x) - c_1 \ln(x-z) - c_2$, we have

$w(x) = O(|x|^{-1})$ as $|x| \to \infty$. (See Proposition 2.17.6.) Thus, $|\nabla w(x)| = O(|x|^{-2})$ as $|x| \to \infty$ by Proposition 2.12.1. This and Hölder's inequality (Proposition 1.14.1) give that $w \in D^{1,2}(\Omega_-)$. Theorem 7.27.4 yields $\langle \partial w/\partial n, 1\rangle = 0$. So,

$$c_1 \int\limits_{\partial\Omega} \frac{\partial \ln(\cdot - z)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \langle \partial v_-/\partial n, 1\rangle - \langle \partial w/\partial n, 1\rangle = 0. \tag{7.131}$$

Fix $r \in (0, \infty)$ such that $\partial\Omega \subset B(z; r)$. Then

$$\int\limits_{\partial B(z;r)} \frac{\partial \ln(\cdot - z)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial B(z;r)} r^{-1} \, \mathrm{d}\mathcal{H}_{m-1} = 2\pi r \neq 0.$$

Since

$$\int\limits_{\partial(\Omega\cap B(z;r))} \frac{\partial \ln(\cdot - z)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} = 0$$

by Corollary 2.2.3, we infer that

$$\int\limits_{\partial\Omega} \frac{\partial \ln(\cdot - z)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1} \neq 0.$$

This and the equality (7.131) force that $c_1 = 0$. Thus, $v_\pm \in D^{1,2}(\Omega_\pm)$. According to Theorem 7.32.6, there exists a constant c such that $v_\pm = c/a_\pm$.

For $f \in X_0 = \{\psi \in W^{-1/p,p}(\partial\Omega); \langle \psi, 1\rangle = 0\}$, denote by $u_\pm^{f,g,u_\infty}$ a weak solution of the problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. Then $[f, g, u_\infty] \mapsto [u_+^{f,g,u_\infty}, u_-^{f,g,u_\infty}]$ is a linear mapping from $X_0 \times W^{1-1/p,p}(\partial\Omega) \times R^1$ to $D^{1,p}(\Omega_+) \times D^{1,p}(\Omega_-)$ that is closed by Theorem 5.21.4. Closed-graph theorem (Theorem 1.4.15) gives that this mapping is bounded.

Theorem 7.32.9 *Let $\Omega = \Omega_+ \subset R^2$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $\Omega_- = R^2 \setminus \overline{\Omega}$. Let $a_\pm$, $b_\pm$ be positive constants, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\int_{\partial\Omega} h \, \mathrm{d}\mathcal{H}_{m-1} > 0$ Let $2 < p < \infty$, $g \in W^{1-1/p,p}(\partial\Omega)$, $f \in W^{-1/p,p}(\partial\Omega)$. Fix $z \in \Omega_+$.*

- *If $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125), then there exists a constant c such that*

$$u_-(x) - c \cdot \ln|x - z| = O(1) \quad \text{as } |x| \to \infty. \tag{7.132}$$

- *If $c \in R^1$ is given, then there exists a unique weak solution $u_\pm \in D^{1,p}(\Omega)$ of the transmission problem (7.124), (7.125) with (7.132).*

Proof Let $u_\pm \in D^{1,p}(\Omega_\pm)$ be a weak solution of the transmission problem (7.124), (7.125). According to Proposition 2.17.6, there exists a constant c such that (7.132) holds. Let now $c = 0$, $g \equiv 0$, and $f \equiv 0$. Proposition 2.17.3 gives that $|\nabla u_-(x)| = O(|x|^{-2})$ as $|x| \to \infty$. This and Hölder's inequality (Proposition 1.14.1) give that $u_\pm \in D^{1,2}(\Omega_\pm)$. Theorem 7.32.7 forces that $u_\pm \equiv 0$.

We now show that there exists a weak solution $u_\pm \in D^{1,p}(\Omega)$ of the transmission problem (7.124), (7.125) such that $u_-(x) = O(1)$ as $|x| \to \infty$. According to Lemma 7.32.1 and reasoning at the beginning of this section, we can suppose that $a_\pm = 1$ and $g \equiv 0$. For $\varphi \in W^{-1/p,p}(\partial\Omega)$, define $P\varphi = \langle \varphi, 1\rangle / \mathcal{H}_{m-1}(\partial\Omega)$,

$$\tilde{T}\varphi = \frac{b_+ + b_-}{2}(I-P)\varphi - (b_+ - b_-)K'_\Omega(I-P)\varphi + h[\mathcal{S}^\Omega(I-P)\varphi + P\varphi].$$

The operator $\tilde{T}$ is an isomorphism in $W^{-1/p,p}(\partial\Omega)$ by Lemma 7.32.4. The function

$$u_\pm = \mathcal{S}^\Omega(I-P)\tilde{T}^{-1}f + P\tilde{T}^{-1}f$$

is a weak solution of the transmission problem (7.124), (7.125) in $D^{1,p}(\Omega_\pm)$ by Lemma 7.32.3. Moreover, $u_-(x) = O(1)$ as $|x| \to \infty$ by Proposition 3.1.6.

Let now $c \in R^1$ be given. Put

$$\tilde{g}(x) = g(x) + a_- \cdot \ln|x-z|,$$

$$\tilde{f}(x) = f(x) + b_- \frac{n(x)\cdot(x-z)}{|x-z|}.$$

We have shown that there exists a weak solution $v_\pm \in D^{1,p}(\Omega_\pm)$ of the transmission problem

$$\Delta v_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$a_+ v_+ - a_- v_- = \tilde{g}, \quad b_+ \frac{\partial v_+}{\partial n} - b_- \frac{\partial v_-}{\partial n} + h v_+ = \tilde{f} \quad \text{on } \partial\Omega$$

such that $v_-(x) = O(1)$ as $|x| \to \infty$. Define $u_-(x) = v_-(x) + c \cdot \ln(x-z)$, $u_+ = v_+$. Clearly, $u_\pm \in D^{1,p}(\Omega_\pm)$ is a weak solution of the transmission problem (7.124), (7.125), (7.132).

7.33 Successive Approximation for the Transmission Problem

Theorem 7.33.1 *Let $\Omega = \Omega_+ \subset R^m$ be a bounded open set with Lipschitz boundary, $\Omega_- = R^m \setminus \overline{\Omega}$. Let b_+, b_- be positive constants, $g \in H^{1/2}(\partial\Omega)$.*

- *If $m > 2$, then there exists a solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of the transmission problem*

$$\Delta u_\pm = 0 \text{ in } \Omega_\pm, \quad u_+ - u_- = g, \; b_+ \frac{\partial u_+}{\partial n^\Omega} - b_- \frac{\partial u_-}{\partial n^\Omega} = f \text{ on } \partial\Omega \tag{7.133}$$

if and only if $f \in \mathcal{E}_2(\partial\Omega) = H^{-1/2}(\partial\Omega)$. This solution is unique and

$$u_-(x) \to 0 \quad \text{as } |x| \to \infty. \tag{7.134}$$

- *If $m = 2$, then there is a solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of the transmission problem (7.133) if and only if $f \in \mathcal{E}_2(\partial\Omega) = \{\psi \in H^{-1/2}(\partial\Omega); \langle\psi, 1\rangle = 0\}$. Moreover, there exists a unique solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of (7.133), (7.134).*
- *Let $f \in \mathcal{E}_2(\partial\Omega)$ and $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ be a solution of the transmission problem (7.133), (7.134). Then*

$$u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega \Phi$$

where

$$\Phi = [\partial u_+/\partial n^\Omega]_+ - [\partial u_-/\partial n^\Omega]_-.$$

Denote

$$F = f - (b_+ - b_-) H^\Omega g$$

where $H^\Omega g = \partial \mathcal{D}^\Omega g/\partial n^\Omega$. For a fixed $\Phi_0 \in \mathcal{E}_2(\partial\Omega)$, define

$$\Phi_k = \Phi_{k-1} - \frac{2}{b_+ + b_-}\Phi_{k-1} + \frac{2}{b_+ + b_-} F, \quad k \in N.$$

Then $\Phi_k \to \Phi$ in $\mathcal{E}_2(\partial\Omega)$ (and therefore in $H^{-1/2}(\partial\Omega)$) and

$$\|\Phi - \Phi_k\|_{\mathcal{E}} \le \frac{1}{\min(b_+, b_-)} \left[\frac{|b_+ - b_-|}{b_+ + b_-}\right]^k [\|F\|_{\mathcal{E}} + \|\Phi_0\|_{\mathcal{E}}].$$

Proof If $m = 2$, then there exists a solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of the transmission problem (7.133) if and only if $f \in \mathcal{E}_2(\partial\Omega)$. Moreover, there exists a unique solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of (7.133), (7.134). (See Theorem 7.32.6.)

Let $m > 2$. Then $\tilde{W}^{1,2}(\Omega_+) = D^{1,2}(\Omega_+)$, $\tilde{W}^{1,2}(\Omega_-) \subset D^{1,2}(\Omega_+)$ by Proposition 1.25.2. Let now $u_- \in D^{1,2}(\Omega_-)$, $\Delta u_- = 0$. Since $|\nabla u_-| \in L^2(\Omega_-)$, there exists a constant u_∞ such that $u_-(x) \to u_\infty$ as $|x| \to \infty$. (See Proposition 2.17.5.) According to Proposition 3.38.5, we have $u_- \in \tilde{W}^{1,2}(\Omega_-)$ if and only if $u_\infty = 0$. According to Theorem 7.32.5, there exists a solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of the transmission problem (7.133) if and only if $f \in H^{-1/2}(\partial\Omega)$ and this solution is unique.

Let $f \in \mathcal{E}_2(\partial\Omega)$ and $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ be a solution of the transmission problem (7.133), (7.134). Then $u_\pm$ is a very weak L^2-solution of the transmission problem (7.133), (7.134) by Theorem 7.11.2 and Proposition 7.19.2. Corollary 5.6.7 gives the representation $u_\pm = \mathcal{D}^\Omega g + \mathcal{S}^\Omega \Phi$. We use this representation for the calculation of the normal derivative of $u_\pm$. According to Lemma 7.32.1 and Proposition 7.22.5,

$$f = b_+ \frac{\partial u_+}{\partial n^\Omega} - b_- \frac{\partial u_-}{\partial n^\Omega}$$
$$= (b_+ - b_-)H^\Omega g + b_+ \left(\frac{1}{2}I - K'_\Omega\right)\Phi - b_- \left(-\frac{1}{2}I - K'_\Omega\right)\Phi.$$

Denoting

$$T = b_+ \left(\frac{1}{2}I - K'_\Omega\right) - b_- \left(-\frac{1}{2}I - K'_\Omega\right).$$

we obtain

$$T\Phi = F. \tag{7.135}$$

T is a bounded linear operator on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.5. We now show

$$\|I - [2/(b_+ + b_-)]T\|_{\mathcal{E}} \le \frac{|b_+ - b_-|}{b_+ + b_1} < 1. \tag{7.136}$$

Denote

$$B = \{\varphi \in \mathcal{E}_2(\partial\Omega); \|\varphi\|_{\mathcal{E}} \le 1\}.$$

Fix $\psi \in B$. Using Theorem 1.9.10, Lemma 3.40.1, and Propositions 3.40.3, 7.22.4, 7.22.5, and 1.14.1,

$$\|[I - [2/(b_+ + b_-)]T]\psi\|_{\mathcal{E}} = \sup_{\varphi\in B}([I - [2/(b_+ + b_-)]T]\psi, \varphi)_{\mathcal{E}}$$
$$= \sup_{\varphi\in B}\left[\int_{R^m} \nabla\mathcal{S}^\Omega\psi \cdot \nabla\mathcal{S}^\Omega\varphi \,\mathrm{d}\mathcal{H}_m - \frac{2b_+}{b_+ + b_-}\int_{\Omega_+} \nabla\mathcal{S}^\Omega\psi \cdot \nabla\mathcal{S}^\Omega\varphi \,\mathrm{d}\mathcal{H}_m\right.$$
$$\left. - \frac{2b_-}{b_+ + b_-}\int_{\Omega_-} \nabla\mathcal{S}^\Omega\psi \cdot \mathcal{S}^\Omega\nabla\varphi \,\mathrm{d}\mathcal{H}_m\right] \le \sup_{\varphi\in B}\int_{R^m} \frac{|b_+ - b_-|}{b_+ + b_-}|\nabla\mathcal{S}^\Omega\psi| \cdot |\nabla\mathcal{S}^\Omega\varphi| \,\mathrm{d}\mathcal{H}_m$$
$$\le \sup_{\varphi\in B} \frac{|b_+ - b_-|}{b_+ + b_-}\|\nabla\mathcal{S}^\Omega\psi\|_{L^2(R^m)}\|\nabla\mathcal{S}^\Omega\varphi\|_{L^2(R^m)} \le \frac{|b_+ - b_-|}{b_+ + b_-}.$$

So, (7.136) holds true. According to (7.135), we have

$$\Phi = \Phi - \frac{2}{b_+ + b_-} T\Phi + \frac{2}{b_+ + b_-} F. \tag{7.137}$$

Define

$$\tilde{T}(\varphi) = \varphi - \frac{2}{b_+ + b_-} T\varphi + \frac{2}{b_+ + b_-} F.$$

Since $H^\Omega g = [\partial \mathcal{D}^\Omega g/\partial n]_+$, Proposition 7.7.3 and Theorem 7.20.2 give that $F \in \mathcal{E}_2(\partial\Omega)$. Thus, $\tilde{T}$ is an operator on $\mathcal{E}_2(\partial\Omega)$. If $\psi, \varphi \in \mathcal{E}_2(\partial\Omega)$, then (7.136) yields that

$$1 > q := \frac{|b_+ - b_-|}{b_+ + b_-} \geq \|\tilde{T}(\varphi) - \tilde{T}(\psi)\|_{\mathcal{E}}.$$

Theorem 1.4.18 and (7.137) give that Φ is a unique solution of $\varphi = \tilde{T}(\varphi)$ in $\mathcal{E}_2(\partial\Omega)$. Moreover, if $\Phi_0 \in \mathcal{E}_2(\partial\Omega)$ and $\Phi_k = \tilde{T}(\Phi_{k-1})$ for $k \in N$, then

$$\begin{aligned}\|\Phi - \Phi_k\|_{\mathcal{E}} &\leq \frac{q^k}{1-q}\|\Phi_1 - \Phi_0\|_{\mathcal{E}} \leq \frac{q^k}{1-q}\frac{2}{b_+ + b_-}\left[\|F\|_{\mathcal{E}} + \|\Phi_0\|_{\mathcal{E}}\right] \\ &= \frac{1}{\min(b_+, b_-)}\left[\frac{|b_+ - b_-|}{b_+ + b_-}\right]^k \left[\|F\|_{\mathcal{E}} + \|\Phi_0\|_{\mathcal{E}}\right].\end{aligned}$$

Remark that $\| \; \|_{\mathcal{E}}$ and $\| \; \|_{H^{-1/2}(\partial\Omega)}$ are equivalent norms on $\mathcal{E}_2(\partial\Omega)$ by Proposition 7.22.5.

7.34 Obstacle Problem

In this section, we prove that there exists a unique solution of the obstacle problem in $W^{1,2}(\Omega)$, where Ω is a bounded domain. We show that this solution is a superharmonic function. We also discuss the relation between the obstacle problem and the Dirichlet problem.

Let $\Omega \subset R^m$ be a bounded open set, $g, f \in W^{1,2}(\Omega)$, $g \geq f$. Denote

$$\mathcal{K}_{g,f} = \{v \in W^{1,2}(\Omega); v - g \in \overset{\circ}{W}{}^{1,2}(\Omega), v - f \geq 0\}.$$

We say that u is a solution of the obstacle problem if

$$u \in \mathcal{K}_{g,f}, \quad \|\nabla u\|_{L^2(\Omega)} = \inf_{v \in \mathcal{K}_{g,f}} \|\nabla v\|_{L^2(\Omega)}. \tag{7.138}$$

Theorem 7.34.1 *Let $\Omega \subset R^m$ be a bounded open set, $g,f \in W^{1,2}(\Omega)$, $g \geq f$. Then there exists a unique solution u of the obstacle problem (7.138).*

Proof Denote

$$a = \inf_{v \in \mathcal{K}_{g,f}} \|\nabla v\|_{L^2(\Omega)}.$$

Choose a sequence $u_n \in \mathcal{K}_{g,f}$ such that $\|\nabla u_n\|_{L^2(\Omega)} \to a$ as $n \to \infty$. We can suppose that $\|\nabla u_n\|_{L^2(\Omega)} \leq a + 1$. Denote $v_n = u_n - g$. Then $v_n \in \overset{\circ}{W}{}^{1,2}(\Omega)$ and

$$\begin{aligned} \|\nabla v_n\|^2_{L^2(\Omega)} &= \|\nabla u_n\|^2_{L^2(\Omega)} + \|\nabla g\|^2_{L^2(\Omega)} - 2\int_\Omega \nabla u_n \cdot \nabla g \, \mathrm{d}\mathcal{H}_m \\ &\leq (a+1)^2 + \|\nabla g\|^2_{L^2(\Omega)} + 2(a+1)\|\nabla g\|_{L^2(\Omega)} \end{aligned}$$

by Hölder's inequality (Proposition 1.14.1). Lemma 1.21.10 gives that $\|\nabla w\|_{L^2(\Omega)}$ is an equivalent norm on $\overset{\circ}{W}{}^{1,2}(\Omega)$. Since v_n is a bounded sequence in the reflexive space $W^{1,2}(\Omega)$, there exists $v \in W^{1,2}(\Omega)$ such that v_n converges weakly to v (see Theorem 1.5.9). Since $v_n \in \overset{\circ}{W}{}^{1,2}(\Omega)$, Proposition 1.5.14 gives that $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$. Put $u = g + v$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, $\varphi \geq 0$, then

$$\int_\Omega (u - f)\varphi \, \mathrm{d}\mathcal{H}_m = \lim_{n\to\infty} \int_\Omega (u_n - f)\varphi \, \mathrm{d}\mathcal{H}_m \geq 0.$$

Thus, $u \geq f$. Hence, $u \in \mathcal{K}_{g,f}$. Since v_n converges weakly to v in $\overset{\circ}{W}{}^{1,2}(\Omega)$, Lemma 1.5.8 gives

$$\|\nabla v\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\nabla v_n\|_{L^2(\Omega)}.$$

Since v_n converges weakly to v in $W^{1,2}(\Omega)$, we have

$$\begin{aligned} a^2 &\leq \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m = \int_\Omega |\nabla g|^2 \, \mathrm{d}\mathcal{H}_m + \int_\Omega |\nabla v|^2 \, \mathrm{d}\mathcal{H}_m + 2\int_\Omega \nabla v \cdot \nabla g \, \mathrm{d}\mathcal{H}_m \\ &\leq \int_\Omega |\nabla g|^2 \, \mathrm{d}\mathcal{H}_m + \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 \, \mathrm{d}\mathcal{H}_m + 2\lim_{n\to\infty} \int_\Omega \nabla v_n \cdot \nabla g \, \mathrm{d}\mathcal{H}_m \\ &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, \mathrm{d}\mathcal{H}_m = a^2. \end{aligned}$$

Thus, u is a solution of the obstacle problem (7.138).

Let now u and v be two solutions of the obstacle problem. Then there exists $t \in R^1$ and $w \in L^2(\Omega, R^m)$ such that w is orthogonal to ∇u in $L^2(\Omega, R^m)$ and $\nabla v = t\nabla u + w$. Since

$$a^2 = \|\nabla u\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} = t^2\|\nabla u\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} = t^2 a^2 + \|w\|^2_{L^2(\Omega)},$$

we have $|t| \le 1$. Clearly, $(u+v)/2 \in \mathcal{K}_{g,f}$. So,

$$\begin{aligned} a^2 \le \left\| \frac{\nabla(u+v)}{2} \right\|^2_{L^2(\Omega)} &= \frac{1}{4}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla v\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2}\int\limits_\Omega \nabla u \cdot (t\nabla u + w)\, \mathrm{d}\mathcal{H}_m = a^2\frac{1+t}{2}. \end{aligned}$$

This forces that $t = 1$. Since

$$a^2 = \|\nabla v\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} = a^2 + \|w\|^2_{L^2(\Omega)},$$

we infer that $w \equiv 0$. Thus, $\nabla u - \nabla v \equiv 0$ and $u - v$ is constant on each component of Ω. Since $u - v \in \overset{\circ}{W}{}^{1,2}(\Omega)$, we deduce that $u - v \equiv 0$.

Proposition 7.34.2 *Let $\Omega \subset R^m$ be a bounded open set, $g, f \in W^{1,2}(\Omega)$, $g \ge f$. If $u \in W^{1,2}(\Omega)$ is a solution of the obstacle problem (7.138), then u is superharmonic in Ω.*

Proof Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$, $\varphi \ge 0$. If $t > 0$, then $u + t\varphi \in \mathcal{K}_{g,f}$. Since $\|\nabla(u+t\varphi)\|_{L^2(\Omega)} \ge \|\nabla u\|_{L^2(\Omega)}$, we have

$$0 \le \lim_{t \downarrow 0} \frac{\|\nabla(u+t\varphi)\|^2_{L^2(\Omega)} - \|\nabla u\|^2_{L^2(\Omega)}}{t} = 2\int\limits_\Omega \nabla u \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m.$$

Theorem 3.19.4 gives that u is superharmonic.

Proposition 7.34.3 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $g, f \in W^{1,2}(\Omega)$, $g \ge f$, and $u \in W^{1,2}(\Omega)$ be a solution of the obstacle problem (7.138). If $v \in W^{1,2}(\Omega)$ is superharmonic in Ω, $v \ge f$ in Ω, $v \ge g$ on $\partial\Omega$, then $v \ge u$. If $g = f$, then u is the least superharmonic majorant of f.*

Proof Denote $w = \min(u, v)$. Then $w \in W^{1,2}(\Omega)$ by Theorem 1.21.5. Clearly, $w \ge f$ in Ω. Since $w = g$ on $\partial\Omega$, we have $w - g \in \overset{\circ}{W}{}^{1,2}(\Omega)$. (See Proposition 1.24.3.) Thus, $w \in \mathcal{K}_{g,f}$.

The function u is superharmonic by Proposition 7.34.2. So, w is superharmonic by Lemma 3.3.3. Theorem 3.19.4 gives that

$$\int\limits_\Omega \nabla w \cdot \nabla\varphi \, \mathrm{d}\mathcal{H}_m \ge 0$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$ with $\varphi \geq 0$. We have $u - w \in \overset{\circ}{W}{}^{1,2}(\Omega)$, $u - w \geq 0$. According to Lemma 1.25.10, there exists a sequence $\{\varphi_n\} \subset \overset{\circ}{W}{}^{1,2}(\Omega)$ such that $\varphi_n \geq 0$ and $\varphi_n \to u - w$ in $W^{1,2}(\Omega)$. Thus,

$$\int_\Omega \nabla w \cdot \nabla(u-w)\, \mathrm{d}\mathcal{H}_m = \lim_{n\to\infty} \int_\Omega \nabla w \cdot \nabla \varphi_n \, \mathrm{d}\mathcal{H}_m \geq 0.$$

This inequality and $w \in \mathcal{K}_{g,f}$ give

$$\int_\Omega |\nabla u|^2\, \mathrm{d}\mathcal{H}_m = \int_\Omega |\nabla w|^2\, \mathrm{d}\mathcal{H}_m + 2\int_\Omega \nabla w \cdot \nabla(u-w)\, \mathrm{d}\mathcal{H}_m$$

$$+ \int_\Omega |\nabla(u-w)|^2\, \mathrm{d}\mathcal{H}_m \geq \int_\Omega |\nabla u|^2\, \mathrm{d}\mathcal{H}_m + \int_\Omega |\nabla(u-w)|^2\, \mathrm{d}\mathcal{H}_m.$$

Thus, $\nabla(u - w) = 0$. So, $u - w$ is constant on each component of Ω (see Lemma 1.20.1). Since $u - w \in \overset{\circ}{W}{}^{1,2}(\Omega)$, we infer that $u - w \equiv 0$. Hence, $v \geq u$.

Proposition 7.34.4 *Let $\Omega \subset R^m$ be a bounded open set with Lipschitz boundary, $g,f \in W^{1,2}(\Omega)$, $g \geq f$, and $u \in W^{1,2}(\Omega)$ be a solution of the obstacle problem (7.138). If f is subharmonic in Ω, then u is a unique solution of the Dirichlet problem $v \in W^{1,2}(\Omega)$,*

$$\Delta v = 0 \quad in\ \Omega, \qquad v = g \quad on\ \partial\Omega.$$

Proof According to Theorem 7.1.2, there exists a unique solution $v \in W^{1,2}(\Omega)$ of the Dirichlet problem. Since f is subharmonic, Theorem 3.19.4 gives $-\Delta f \leq 0 = \Delta v$ in Ω. Since $v = g \geq f$ on $\partial\Omega$, Theorem 7.3.2 gives that $v \geq f$ in Ω. According to Proposition 7.34.3, we have $v \geq u$. The function u is superharmonic in Ω by Proposition 7.34.2. Thus, $-\Delta u \geq 0 = \Delta v$ in Ω by Theorem 3.19.4. Since $v = g \leq u$ on $\partial\Omega$, Theorem 7.3.2 gives that $u \geq v$ in Ω. Hence, $u = v$.

Proposition 7.34.5 *Let $\Omega \subset R^m$ be a bounded open set, $f \in W^{1,2}(\Omega)$, $g = f$, and $u \in W^{1,2}(\Omega)$ be a solution of the obstacle problem (7.138). If f is superharmonic in Ω, then $u = f$.*

Proof Since f is superharmonic, Theorem 3.19.4 gives that

$$\int_\Omega \nabla f \cdot \nabla\varphi\, \mathrm{d}\mathcal{H}_m \geq 0$$

for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$ with $\varphi \geq 0$. Since $u \in \mathcal{K}_{ff}$, we have $u - f \in \overset{\circ}{W}{}^{1,2}(\Omega)$, $u - f \geq 0$. According to Lemma 1.25.10 and Proposition 1.25.2, there exists a sequence $\{\varphi_n\} \subset$

$\overset{\circ}{W}{}^{1,2}(\Omega)$ such that $\varphi_n \geq 0$ and $\varphi_n \to u - f$ in $W^{1,2}(\Omega)$. Thus,

$$\int_\Omega \nabla f \cdot \nabla(u-f)\,\mathrm{d}\mathcal{H}_m = \lim_{n\to\infty} \int_\Omega \nabla f \cdot \nabla \varphi_n\,\mathrm{d}\mathcal{H}_m \geq 0.$$

Since $f \in \mathcal{K}_{f,f}$, we have

$$\int_\Omega |\nabla u|^2\,\mathrm{d}\mathcal{H}_m = \int_\Omega |\nabla f|^2\,\mathrm{d}\mathcal{H}_m + \int_\Omega |\nabla(u-f)|^2\,\mathrm{d}\mathcal{H}_m + 2\int_\Omega \nabla f \cdot \nabla(u-f)\,\mathrm{d}\mathcal{H}_m$$
$$\geq \int_\Omega |\nabla u|^2\,\mathrm{d}\mathcal{H}_m + \int_\Omega |\nabla(u-f)|^2\,\mathrm{d}\mathcal{H}_m + 0.$$

Thus, $\nabla(u-f) = 0$. So, $u-f$ is constant on each component of Ω (see Lemma 1.20.1). Since $u - f \in \overset{\circ}{W}{}^{1,2}(\Omega)$, we infer that $u - f \equiv 0$.

7.35 Dirichlet: Oblique Derivative Problem in $\tilde{W}^{1,2}(\Omega)$

In this section, we prove that there exists a unique solution of the mixed Dirichlet-oblique derivative problem for the Poisson equation in $\tilde{W}^{1,2}(\Omega)$ where Ω is a domain with compact Lipschitz boundary. We also show the monotony principle for this problem.

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary. Let Γ_R be a nonempty closed subset of $\partial\Omega$ such that $\Gamma_D = \partial\Omega \setminus \Gamma_R$ is nonempty. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Let $a_{ij} \in \mathcal{C}^1(R^m)$ be functions with compact support such that $a_{ji} = -a_{ij}$. Define the tangential vector function

$$\tau = A(x)n^\Omega$$

and put

$$\Theta = n^\Omega(x) + \tau(x).$$

Let $F \in [\tilde{W}^{1,2}(\Omega)]'$, $g \in W^{1/2,2}(\partial\Omega) = H^{1/2}(\partial\Omega)$. *We say that* $u \in \tilde{W}^{1,2}(\Omega)$ *is a weak solution of the mixed problem*

$$-\Delta u = F \quad \text{in } \Omega, \tag{7.139}$$

$$\Theta \cdot \nabla u + \frac{1}{2}\sum_{i,j=1}^m n_j(\partial_i a_{ij})u + hu = F \quad \text{on } \Gamma_R, \tag{7.140}$$

$$u = g \quad \text{on } \Gamma_D \tag{7.141}$$

if $u = g$ *on* Γ_D *in the sense of traces and*

$$(u, \varphi)_{A,h} = \langle F, \varphi \rangle$$

for all $\varphi \in \tilde{W}^{1,2}(\Omega)$ *such that* $\varphi = 0$ *on* Γ_D. *Here*

$$(u, \varphi)_{A,h} := \int_\Omega \{\nabla \varphi \cdot \nabla u + \sum_{i,j=1}^{m} [a_{ij} \partial_j \varphi \partial_i u + \frac{1}{2} (\partial_j a_{ij}) \varphi \partial_i u$$
$$- \frac{1}{2} (\partial_j a_{ij}) u \partial_i \varphi]\} \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \varphi h u \, \mathrm{d}\mathcal{H}_{m-1}.$$

If $A \equiv 0$, *we say about a mixed Dirichlet-Robin problem.*

Theorem 7.35.1 *Let* $\Omega \subset R^m$ *be a domain with compact Lipschitz boundary. Let* Γ_R *be a nonempty closed subset of* $\partial\Omega$ *such that* $\Gamma_D = \partial\Omega \setminus \Gamma_R$ *is nonempty. Let* $A = (a_{ij})$ *be a matrix function of the type* $m \times m$. *Let* $a_{ij} \in \mathcal{C}^1(R^m)$ *be functions with compact support such that* $a_{ji} = -a_{ij}$. *Define the tangential vector function* $\tau(x) = A(x) n^\Omega(x)$ *and put* $\Theta(x) = n^\Omega(x) + \tau(x)$. *Let* $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $F \in [\tilde{W}^{1,2}(\Omega)]'$, $g \in H^{1/2}(\partial\Omega)$. *Then, there exists a unique weak solution* $u \in \tilde{W}^{1,2}(\Omega)$ *of the mixed Dirichlet-derivative oblique problem (7.139)–(7.141). Moreover,*

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\{\|F\|_{[\tilde{W}^{1,2}(\Omega)]'} + \|g\|_{H^{1/2}(\partial\Omega)}\}$$

with a constant C that does not depend on F and g.

Proof Define $H = h$ on Γ_R, $H = 1$ on Γ_D. Then $(\ ,\ )_{A,H}$ is an inner product on $\tilde{W}^{1,2}(\Omega)$, and the corresponding norm is equivalent to the original norm (see Lemma 7.31.2 and Corollary 7.21.4).

Define

$$\tilde{W}^{1,2}(\Omega, \Gamma_D) = \{u \in \tilde{W}^{1,2}(\Omega); \gamma_\Omega u = 0 \text{ on } \Gamma_D\}.$$

Theorem 1.24.1 gives that $\tilde{W}^{1,2}(\Omega; \Gamma_D)$ is a closed subspace of $\tilde{W}^{1,2}(\Omega)$. If $u, v \in \tilde{W}^{1,2}(\Omega; \Gamma_D)$, then $(u, v)_{A,h} = (u, v)_{A,H}$. Thus, $(\ ,\ )_{A,h}$ is an inner product on $\tilde{W}^{1,2}(\Omega; \Gamma_D)$, and the corresponding norm is equivalent to the original norm. So, if $F \in [\tilde{W}^{1,2}(\Omega)]'$ and $g \equiv 0$, then the Riesz representation theorem (Theorem 1.9.10) forces that there exists a unique weak solution $u \in \tilde{W}^{1,2}(\Omega)$ of the problem (7.139)–(7.141), and

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C_1 \|F\|_{[\tilde{W}^{1,2}(\Omega)]'}$$

with a constant C_1 that does not depend on F.

Let now $g \in H^{1/2}(\partial\Omega)$ and $F \in [\tilde{W}^{1,2}(\Omega)]'$. According to Theorems 1.24.2 and Lemma 1.21.12, there exists $G \in W^{1,2}(R^m) \hookrightarrow \tilde{W}^{1,2}(\Omega)$, such that g is the trace of G and

$$\|G\|_{\tilde{W}^{1,2}(\Omega)} \le C_2 \|g\|_{H^{1/2}(\partial\Omega)},$$

where C_2 does not depend on g. Define $\tilde{F} \in [\tilde{W}^{1,2}(\Omega)]'$ by

$$\tilde{F}(\varphi) = (G, \varphi)_{A,h}.$$

Then

$$\|\tilde{F}\|_{[\tilde{W}^{1,2}(\Omega)]'} \le C_3 \|G\|_{\tilde{W}^{1,2}(\Omega)}$$

with C_3 independent on G (see Theorems 1.24.1 and 1.14.3). We have proved that there exists a unique solution $v \in \tilde{W}^{1,2}(\Omega)$ of the problem

$$-\Delta v = F - \tilde{F} \quad \text{in } \Omega,$$

$$\Theta \cdot \nabla v + \frac{1}{2} \sum_{i,j=1}^{m} n_j (\partial_i a_{ij}) v + hv = F - \tilde{F} \quad \text{on } \Gamma_R,$$

$$v = 0 \quad \text{on } \Gamma_D$$

and

$$\|v\|_{\tilde{W}^{1,2}(\Omega)} \le C_1 \| F - \tilde{F}\|_{[\tilde{W}^{1,2}(\Omega)]'}.$$

The function $u = G + v$ is a unique solution of the problem (7.139)–(7.141) in $\tilde{W}^{1,2}(\Omega)$ and

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} \le [C_2 + C_1(1 + C_3 C_2)]\{\| F\|_{[\tilde{W}^{1,2}(\Omega)]'} + \|g\|_{H^{1/2}(\partial\Omega)}\} .$$

Proposition 7.35.2 *Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary and Γ_R be a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \ne \emptyset$. Let $A = (a_{ij})$ be a matrix function of the type $m \times m$. Let $a_{ij} \in \mathcal{C}^1(R^m)$ be functions with compact support such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = A(x) n^\Omega(x)$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $F \in [\tilde{W}^{1,2}(\Omega)]'$, $g \in H^{1/2}(\partial\Omega)$. Let $u \in \tilde{W}^{1,2}(\Omega)$ be a weak solution of the mixed problem (7.139)–(7.141). If $F \ge 0$ and $g \ge 0$ on Γ_D, then $u \ge 0$.*

Proof Define $v = -\min(u, 0)$. Then $v \ge 0$, $v \in \tilde{W}^{1,2}(\Omega)$, $v = 0$ on Γ_D, $\nabla v(x) = -\nabla u(x)$ for $u(x) < 0$, $\nabla v(x) = 0$ for $u(x) \ge 0$. (See Theorem 1.21.5.) Since $F \ge 0$

$$0 \le \langle F, v\rangle = (u, v)_{A,h} = (-v, v)_{A,h} = -\int_\Omega |\nabla v|^2 \, d\mathcal{H}_m - \int_{\partial\Omega} hv^2 \, d\mathcal{H}_{m-1}.$$

by Lemma 7.31.2. Thus, $\nabla v = 0$ in Ω. Since Ω is connected, the function v is constant. Since $v = 0$ on Γ_D, we deduce that $v \equiv 0$. Hence, $u \ge 0$.

Proposition 7.35.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary and Γ_R be a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \ne \emptyset$. Let $A = (a_ij)$ be a matrix of the type $m \times m$ such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = An^\Omega(x)$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $g \in W^{1/2,2}(\partial\Omega)$, $F \in [W^{1,2}(\Omega)]'$, $u \in W^{1,2}(\Omega)$ be a solution of the mixed problem (7.139)–(7.141). Let $F \ge 0$, $M \le g$ on Γ_D. If $M \le 0$ or $h = 0$ on Γ_R, then $u \ge M$. If $M \le g \le N$ on Γ_D and $F \equiv 0$, $h \equiv 0$, then $M \le u \le N$.*

Proof Let $F \ge 0$, $M \le g$ on Γ_D. Define $v = u - M$, $h = 0$ on $R^m \setminus \Gamma_R$. Then

$$-\Delta v = G \text{ in } \Omega, \quad v = g - M \ge 0 \text{ on } \Gamma_D, \quad \frac{\partial v}{\partial \Theta} + hv = G \text{ on } \Gamma_R,$$

where $G = F - Mh\mathcal{H}_{m-1}|\Gamma_R$. If $M \le 0$ or $h = 0$ on Γ_R, then $G \ge 0$. Proposition 7.35.2 gives that $v \ge 0$. Hence, $u \ge M$.

Let now $g \le N$ on Γ_D and $F \equiv 0$, $h \equiv 0$. Then $w = -u$ is a solution of

$$-\Delta w = 0 \text{ in } \Omega, \quad w = -g \ge -N \text{ on } \Gamma_D, \quad \frac{\partial w}{\partial \Theta} = 0 \text{ on } \Gamma_R.$$

We have proved that $w \ge -N$. Hence, $u \le N$.

Proposition 7.35.4 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m > 2$, and Γ_R be a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \ne \emptyset$. Let $A = (a_{ij})$ be a matrix of the type $m\times m$ such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = An^\Omega(x)$, and put $\Theta(x) = n^\Omega(x) + \tau(x)$. Let $h \equiv 0$, $F \equiv 0$, $g \in W^{1/2,2}(\partial\Omega)$, $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of the mixed problem (7.139)–(7.141). Then, $u(x) \to 0$ as $|x| \to \infty$. If $|g| \le M$ on Γ_D, then $|u| \le M$.*

Proof Since $\Delta u = 0$ in Ω and $\partial_j u \in L^2(\Omega)$, there exists a constant c such that $u(x) \to c$ as $|x| \to \infty$. (See Proposition 2.17.5.) Since $u \in \tilde{W}^{1,2}(\Omega)$, Proposition 3.38.5 gives that $c = 0$.

Let now $|g| \le M$ on Γ_D. If $M = 0$, then $u \equiv 0$ by Theorem 7.35.1. So, we can suppose that $M > 0$. Fix $r > 0$ such that $|u(x)| \le M/2$ for $|x| \ge r$. Define $g = u$ on $\partial B(0; r)$. Then u is a solution of the mixed problem

$$\Delta u = 0 \text{ in } \Omega_r, \quad \frac{\partial u}{\partial \Theta} = 0 \text{ on } \Gamma_R, \quad u = g \text{ on } \partial\Omega_r \setminus \Gamma_R,$$

where $\Omega_r = B(0; r) \cap \Omega$. Proposition 7.35.3 gives that $|u| \le M$ on Ω_r. Therefore, $|u| \le M$ on Ω.

Proposition 7.35.5 *Let $\Omega \subset R^2$ be an unbounded domain with compact Lipschitz boundary and Γ_R be a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \neq \emptyset$. Let $A = (a_{ij})$ be a matrix of the type $m \times m$ such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = An^{\Omega}(x)$, and put $\Theta(x) = n^{\Omega}(x) + \tau(x)$. Let $h \equiv 0$, $F \equiv 0$, $g \in W^{1/2,2}(\partial\Omega)$, $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of the mixed problem (7.139)–(7.141). Then there exists a constant c such that $u(x) \to c$ as $|x| \to \infty$. If $g \geq M$ on Γ_D, then $u \geq M$. If $g \leq N$ on Γ_D, then $u \leq N$.*

Proof Fix $z \in R^2 \setminus \overline{\Omega}$. Since $|\nabla u| \in L^2(\Omega)$, there exist constants $\tilde{c}$ and c such that for $w(x) = u(x) - \tilde{c}h_2(x - z) - c$, we have $w(x) = O(|x|^{-1})$ as $|x| \to \infty$. (See Proposition 2.17.6.) Remark 2.1.3 gives that $w \in H(\Omega)$. Since $w(x) = O(|x|^{-1})$ as $|x| \to \infty$, Proposition 2.12.1 forces $|\nabla w(x)| = O(|x|^{-2})$ as $|x| \to \infty$. So, $|\nabla w| \in L^2(\Omega)$. Thus,

$$\frac{|\tilde{c}|}{2\pi|x - z|} = |\nabla \tilde{c}h_2(x - z)| = |\nabla u(x) - \nabla w(x)| \in L^2(\Omega).$$

This forces that $\tilde{c} = 0$. Hence, $u(x) \to c$ as $|x| \to \infty$.

Let $g \leq N$ on Γ_D. Suppose first that $c > N$. Choose $a \in (N, c)$. Define $v = \max(u - a, 0)$. Then $v \in \tilde{W}^{1,2}(\Omega)$, $\nabla v(x) = \nabla u(x)$ for $u(x) > a$, $\nabla v(x) = 0$ for $u(x) \leq a$ (see Theorem 1.21.5). Since $u < a$ on Γ_D, we have $v = 0$ on Γ_D. Since $h = 0$, $F = 0$, we have from the definition of the problem

$$0 = \langle F, v\rangle = \int_{\Omega} [\nabla u \cdot \nabla v + \sum_{i,j=1}^{m} a_{ij}\partial_i u \partial_j v] = \int_{\Omega} [|\nabla v|^2 + \sum_{i,j=1}^{m} a_{ij}\partial_i v \partial_j v] \, \mathrm{d}\mathcal{H}_m.$$

Since $a_{ji} = -a_{ij}$, we infer

$$0 = \int_{\Omega} |\nabla v|^2 \, \mathrm{d}\mathcal{H}_m.$$

Since $\nabla v \equiv 0$, the function v is constant. Since $v = 0$ on Γ_D, we infer that $v \equiv 0$. But $v(x) \to c - a > 0$ as $|x| \to \infty$, what is a contradiction. Thus, $c \leq N$. Let $\tilde{N} > N$. Since $u(x) \to c$ as $|x| \to \infty$, there exists $r > 0$ such that $u(x) < \tilde{N}$ for $|x| \geq r$. Define $g = u$ on $\partial B(0; r)$. Then u is a solution of the problem $\Delta u = 0$ in $\Omega_r = \Omega \cap B(0; r)$, $\partial u/\partial\Theta = 0$ on Γ_R and $u = g$ on $\partial\Omega_r \setminus \Gamma_R$. Since $g \leq \tilde{N}$ on $\partial\Omega_r \setminus \Gamma_R$, Proposition 7.35.3 gives that $u \leq \tilde{N}$ on Ω_r. Hence, $u \leq \tilde{N}$ on Ω. Since $\tilde{N} > N$ was arbitrary, we infer that $u \leq N$.

Let $g \geq M$ on Γ_D. Then $\Delta(-u) = 0$ in Ω, $\partial(-u)/\partial n = 0$ on Γ_R and $-u = -g \leq -M$ on Γ_D. We have proved that $-u \leq -M$. Thus, $u \geq M$.

Theorem 7.35.6 *Let $\Omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary and Γ_R be a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \neq \emptyset$, $m > 2$. Let $A = (a_{ij})$ be a matrix of the type $m \times m$ such that $a_{ji} = -a_{ij}$. Define the tangential vector function $\tau(x) = An^{\Omega}(x)$, and put $\Theta(x) = n^{\Omega}(x) + \tau(x)$. Let $h \in L^{\infty}(\partial\Omega)$, $h \geq 0$, $g \in W^{1/2,2}(\partial\Omega)$, $F \in [\tilde{W}^{1,2}(\Omega)]'$. If $u \in D^{1,2}(\Omega)$, then there exists a constant*

u_∞ such that

$$\lim_{r\to\infty} \oint\limits_{\partial B(0;r)} u \, \mathrm{d}\mathcal{H}_{m-1} = u_\infty. \tag{7.142}$$

Let u_∞ be given. Then there exists a unique solution $u \in D^{1,2}(\Omega)$ of the mixed problem (7.139)–(7.142). Moreover,

$$\|u\|_{D^{1,2}(\Omega)} \le C\left[\|g\|_{W^{1/2,2}(\partial\Omega)} + \|F\|_{[\tilde{W}^{1,2}(\Omega)]'} + |u_\infty|\right], \tag{7.143}$$

where C depends only on Ω, Γ_D, and h.

Proof Let $u \in D^{1,2}(\Omega)$. According to Corollary 3.38.4, there exists a constant u_∞ and $v \in \tilde{W}^{1,2}(\Omega)$ such that $u = v + u_\infty$. Proposition 3.38.5 gives

$$\lim_{r\to\infty} \oint\limits_{\partial B(0;r)} v \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{7.144}$$

So, (7.142) holds.

Let now u_∞ be given. Denote $\tilde{g} = g - u_\infty$, $\tilde{F} = F - u_\infty h \mathcal{H}_{m-1}|\Gamma_R$. Then $u \in D^{1,2}(\Omega)$ is a solution of the problem (7.139)–(7.142) if $u = v + u_\infty$, where $v \in D^{1,2}(\Omega)$ is a solution of the problem (7.144), $-\Delta v = \tilde{F}|\Omega$ in Ω, $v = \tilde{g}$ on Γ_D and $\partial v/\partial\Theta + hv = \tilde{F}|\Gamma_R$ on Γ_R. The function $v \in D^{1,2}(\Omega)$ satisfies the relation (7.144) if and only if $v \in \tilde{W}^{1,2}(\Omega)$ (see Propositions 3.38.4 and 3.38.5). Theorem 7.35.1 gives that there exists a unique solution $v \in \tilde{W}^{1,2}(\Omega)$ of the problem. Moreover,

$$\|v\|_{D^{1,2}(\Omega)} \le c_1\left[\|\tilde{g}\|_{W^{1/2,2}(\partial\Omega)} + \|\tilde{F}\|_{[\tilde{W}^{1,2}(\Omega)]'}\right],$$

where c_1 depends only on Ω, Γ_D, and h. Remember that

$$\|u\|_{D^{1,2}(\Omega)} = \left[\int\limits_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m + \int\limits_\omega |u|^2 \, \mathrm{d}\mathcal{H}_m\right]^{1/2}$$

where ω is a bounded open subset of Ω. Clearly, there exists a constant C such that (7.143) holds.

7.36 Robin-Dirichlet Problem in $W^{s,p}(\Omega)$ and $B^{p,q}_s(\Omega)$ for Ω Bounded

In this section, we prove the existence of a unique solution of the Robin-Dirichlet mixed problem in $W^{s,p}(\Omega)$ and $B^{p,q}_s(\Omega)$, where $\Omega \subset R^m$ is a bounded domain. If $\partial\Omega$ is Lipschitz, we suppose that $p = 2$, $1 < q < \infty$ and $1/2 < s < 3/2$. If $\partial\Omega$ is

Ljapunov, we suppose that $1 < p, q < \infty$ and $1/p < s < 1 + 1/p$. If $\partial\Omega$ is of class $\mathcal{C}^2$, we look for a solution in $W^{2,p}(\Omega)$ with $1 < p < \infty$. If Ω is a planar domain with boundary of class $\mathcal{C}^{k+2}$, $k \in N$, then we study solutions in $W^{k+2,p}(\Omega)$.

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $1 < p, q < \infty$, $1/p < s < \infty$. Let $h \in L^\infty(\partial\Omega)$ and F be a distribution supported on $\overline{\Omega}$. Let $u \in D^{s,p}(\Omega)$ or $u \in B_s^{p,q}(\Omega)$. *We say that u is a very weak solution of the mixed problem*

$$-\Delta u = F|\Omega \qquad \text{in } \Omega, \tag{7.145}$$

$$u = g \qquad \text{on } \Gamma_D, \tag{7.146}$$

$$\frac{\partial u}{\partial n} + hu = F|\Gamma_R \qquad \text{on } \Gamma_R. \tag{7.147}$$

if $u = g$ *on* Γ_D *in the sense of traces and*

$$\int\limits_{\Omega} u(-\Delta\varphi)\, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} \left[u \frac{\partial\varphi}{\partial n} + bu\varphi \right] \mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi \rangle \tag{7.148}$$

for all $\varphi \in \mathcal{C}_c^\infty(R^m)$ *with* $\varphi = 0$ *on* Γ_D.

Proposition 7.36.1 *Let* $\Omega \subset R^m$ *be an open set with compact Lipschitz boundary. Let* Γ_D *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_D \cup \Gamma_R$. *Let* $1 < p < \infty$, $p' = p/(p-1)$, $h \in L^\infty(\Gamma_R)$, $g \in W^{1-1/p}(\Gamma_D)$, $F \in [W^{1,p'}(\Omega)]'$. *If* $u \in D^{1,p}(\Omega)$ *is a weak solution of the Dirichlet-Robin mixed problem (7.145), (7.146), (7.147), then u is a very weak solution of the problem (7.145), (7.146), (7.147).*

Proof It is a consequence of the Green formula for Sobolev spaces (Proposition 1.24.4).

Lemma 7.36.2 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. Let* Γ_D *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_D \cup \Gamma_R$. *Let* $1 < p, q < \infty$, $p' = p/(p-1)$, $q' = q/(q-1)$, $1/p < s < 1 + 1/p$. *If* $p \neq 2$, *suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$. *Let* $h \in L^\infty(\Gamma_R)$, *u be a very weak solution of the problem (7.145), (7.146), (7.147). If* $u \in B_s^{p,q}(\Omega)$, *then* $g \in B_{s-1/p}^{p,q}(\Gamma_D)$ *and* $F \in [B_{2-s}^{p',q'}(\Omega)]'$. *If* $u \in W^{s,p}(\Omega)$, *then* $g \in W^{s-1/p,p}(\Gamma_D)$ *and* $F \in [W^{2-s,p'}(\Omega)]'$.

Proof If $u \in B_s^{p,q}(\Omega)$, then $g \in B_{s-1/p}^{p,q}(\Gamma_D)$ by Theorem 1.24.7. If $u \in W^{s,p}(\Omega)$, then $g \in W^{s-1/p,p}(\Gamma_D)$ by Theorem 1.24.8.

Choose $\varphi \in \mathcal{C}_c^\infty(R^m)$ such that $\varphi = 1$ on a neighborhood of Γ_D and $\varphi = 0$ on a neighborhood of Γ_R. Denote $u_1 = u\varphi$, $u_2 = u(1-\varphi)$. If $u \in B_s^{p,q}(\Omega)$ then $u_1, u_2 \in B_s^{p,q}(\Omega)$ by Lemma 1.23.9. If $u \in W^{s,p}(\Omega)$, then $u_1, u_2 \in W^{s,p}(\Omega)$ by

Lemma 1.21.17. Define $h = 0$ on Γ_D. Then there exists a distribution F_2 supported on $\overline{\Omega}$ such that u_2 is a very weak solution of the Robin problem

$$-\Delta u_2 = F_2|\Omega \text{ in } \Omega, \quad \frac{\partial u_2}{\partial n} + hu_2 = F_2|\partial\Omega \text{ on } \partial\Omega.$$

If $u \in B^{p,q}_s(\Omega)$, then $F_2 \in [B^{p',q'}_{2-s}(\Omega)]'$; if the function $u \in W^{s,p}(\Omega)$, then $F_2 \in [W^{2-s,p'}(\Omega)]'$. (See Proposition 7.24.3.)

If $u \in B^{p,q}_s(\Omega)$, then $-\Delta u_1 \in B^{p,q}_{s-2}(\Omega)$ by Theorem 7.10.1. According to Theorem 1.23.1 and Hahn-Banach's theorem (Theorem 1.5.2), there exists $F_1 \in [B^{p',q'}_{2-s}(\Omega)]'$ such that $F_1 = -\Delta u_1$ in Ω.

If $u \in W^{s,p}(\Omega)$, then Theorem 7.10.2 gives that $-\Delta u_1 \in W^{s-2,p}(\Omega)$. According to Hahn-Banach's theorem (Theorem 1.5.2), there exists $F_1 \in [W^{2-s,p'}(\Omega)]'$ such that $F_1 = -\Delta u_1$ in Ω.

Since $u_2 = 0$ on a neighborhood of Γ_D, u_2 is a very weak solution of the mixed problem

$$-\Delta u_2 = F_2|\Omega \text{ in } \Omega, \; u_2 = 0 \text{ on } \Gamma_D, \; \frac{\partial u_2}{\partial n} + hu_2 = F_2|\Gamma_R \text{ on } \Gamma_R.$$

Since $u_1 = 0$ on a neighborhood of Γ_R, u_1 is very weak solution of the mixed problem

$$-\Delta u_1 = F_1 \text{ in } \Omega, \; u_1 = g \text{ on } \Gamma_D, \; \frac{\partial u_1}{\partial n} + hu_1 = 0 \text{ on } \Gamma_R,$$

The rest is a consequence of the fact that $F = F_1 + F_2$.

Lemma 7.36.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $1 < p, q < \infty$, $1/p < s < 1 + 1/p$. If $p \neq 2$, suppose moreover that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let $h \in L^\infty(\Gamma_R)$, $h \geq 0$. Suppose that $u \in W^{s,p}(\Omega)$ or $u \in B^{p,q}_s(\Omega)$ is a very weak solution of the problem (7.145), (7.146), (7.147). If $g = 0$ and $F = 0$, then $u = 0$.*

Proof Since $\Delta u = 0$ in Ω in the sense of distributions, Theorem 2.18.2 gives that $u \in H(\Omega) \subset \mathcal{C}^2(\Omega)$. Choose a bounded open set G_D with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_D \subset G_D$ and $\Gamma_R \cap \overline{G_D} = \emptyset$. Define $g = u$ on $\Omega \cap \partial G_D$. Then u is a solution of the Dirichlet problem $\Delta u = 0$ in $G_D \cap \Omega$, $u = g$ on $\partial(G_D \cap \Omega)$. Since $g \in \mathcal{C}^2(\partial(G_D \cap \Omega))$, Theorems 7.10.1 and 7.10.2 give that $u \in W^{1,2}(G_D \cap \Omega)$.

Choose a bounded open set G_R with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset G_R$ and $\Gamma_D \cap \overline{G_R} = \emptyset$. Define $h = 1, f = \partial u/\partial n + u$ on $\Omega \cap \partial G_R$ and $h = 0$ outside $\Gamma_R \cup (\Omega \cap \partial G_R)$. Choose $\psi \in \mathcal{C}^\infty_c(G_R)$ such that $\psi = 1$ on a neighborhood of Γ_R. Choose an open set ω with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset \omega$ and $\psi = 1$ on a neighborhood of $\overline{\omega}$. Put $\Omega(R) = \Omega \cap G_R$. If $\varphi \in \mathcal{C}_c^\infty(R^m)$, then Theorem 2.2.1 gives

$$\begin{aligned}
&\int\limits_{\Omega(R)} u(-\Delta\varphi)\,\mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega(R)} u\left(\frac{\partial\varphi}{\partial n} + h\varphi\right)\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega(R)} u[-\Delta(\psi\varphi)]\,\mathrm{d}\mathcal{H}_m \\
&\quad + \int\limits_{\partial\Omega(R)} u\left[\frac{\partial(\psi\varphi)}{\partial n} + h\varphi\psi\right]\,\mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\Omega(R)} u\{-\Delta[(1-\psi)\varphi]\}\,\mathrm{d}\mathcal{H}_m \\
&\quad + \int\limits_{\partial\Omega(R)} u\left\{\frac{\partial[(1-\psi)\varphi]}{\partial n} + h\varphi(1-\psi)\right\}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\Omega} u[-\Delta(\psi\varphi)]\,\mathrm{d}\mathcal{H}_m \\
&\quad + \int\limits_{\partial\Omega} u\left[\frac{\partial(\psi\varphi)}{\partial n} + h\varphi\psi\right]\,\mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\Omega(R)\setminus\overline{\omega}} u\{-\Delta[(1-\psi)\varphi]\}\,\mathrm{d}\mathcal{H}_m \\
&\quad + \int\limits_{\partial(\Omega(R)\setminus\overline{\omega})} u\left\{\frac{\partial[(1-\psi)\varphi]}{\partial n} + h\varphi(1-\psi)\right\}\,\mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial G(R)\setminus\Gamma_R} f\varphi\,\mathrm{d}\mathcal{H}_{m-1}.
\end{aligned}$$

So, u is a very weak solution of the Robin problem $-\Delta u = 0$ in $\Omega(R)$, $\partial u/\partial n + hu = \tilde{F}$ on $\partial\Omega(R)$, where $\tilde{F} = f\mathcal{H}_{m-1}|[\partial\Omega(R) \setminus \Gamma_R]$. Since $f \in L^2(\partial\Omega(R) \setminus \Gamma_R)$, we infer that $\tilde{F} \in [W^{1,2}(\Omega(R))]'$. (See Theorems 1.24.1 and 1.14.3.) Theorem 7.24.6 forces that $u \in W^{1,2}(\Omega(R))$.

$u \in \mathcal{C}_c^\infty(\Omega)$, $u \in W^{1,2}(\Omega \cap G_D)$, $u \in W^{1,2}(\Omega \cap G_R)$ force $u \in W^{1,2}(\Omega)$. Clearly, there exists $\mathcal{F} \in [W^{1,2}(\Omega)]'$ such that u is a weak solution of the problem $-\Delta u = \mathcal{F}|\Omega$ in Ω, $u = g$ on Γ_D, $\partial u/\partial n + hu = \mathcal{F}|\partial\Omega$ on Γ_R. Proposition 7.36.1 gives that u is a very weak solution of this problem. Thus, $\mathcal{F} = F = 0$. Theorem 7.35.1 gives that $u \equiv 0$.

Theorem 7.36.4 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. If* $m = 2$*, suppose moreover that* $\overline{\Omega} \subset B(0;1)$*. Let* Γ_D *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_D \cup \Gamma_R$*. Let* $1 < p, q < \infty$*,* $1/p < s < 1 + 1/p$*. If* $p \neq 2$*, suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*. Let* $h \in L^\infty(\Gamma_R)$*,* $h \geq 0$*. Define*

$$T\psi = \begin{cases} \mathcal{S}^\Omega\psi, & \text{on } \Gamma_D, \\ \left(\frac{1}{2}I - K'_\Omega\right)\psi + h\mathcal{S}^\Omega\psi, & \text{on } \Gamma_R. \end{cases}$$

Then

$$T : B^{p,q}_{s-1/p-1}(\partial\Omega) \to B^{p,q}_{s-1/p}(\Gamma_D) \times B^{p,q}_{s-1/p-1}(\Gamma_R),$$

$$T : W^{s-1/p-1,p}(\partial\Omega) \to W^{s-1/p,p}(\Gamma_D) \times W^{s-1/p-1,p}(\Gamma_R)$$

are isomorphisms.

- *If* $g \in W^{s-1/p,p}(\Gamma_D)$ *and* $F \in W^{s-1/p-1,p}(\Gamma_R)$*, then* $u = \mathcal{S}^\Omega T^{-1}(g, F)$ *is a unique very weak solution of the problem (7.145), (7.146), (7.147) in* $W^{s,p}(\Omega)$*.*
- *If* $g \in B^{p,q}_{s-1/p}(\Gamma_D)$ *and* $F \in B^{p,q}_{s-1/p-1}(\Gamma_R)$*, then* $u = \mathcal{S}^\Omega T^{-1}(g, F)$ *is a unique very weak solution of the problem (7.145), (7.146), (7.147) in* $B^{p,q}_s(\Omega)$*.*

Proof Uniqueness follows form Lemma 7.36.3.

We shall look for a solution in the form $\mathcal{S}^\Omega\psi$ where $\psi \in B^{p,q}_{s-1/p-1}(\partial\Omega)$ or $\psi \in W^{s-1/p-1,p}(\partial\Omega)$. If $\psi \in B^{p,q}_{s-1/p-1}(\partial\Omega)$, then $\mathcal{S}^\Omega\psi \in B^{p,q}_s(\Omega)$. If $\psi \in W^{s-1/p-1,p}(\partial\Omega)$, then $\mathcal{S}^\Omega\psi \in W^{s,p}(\Omega)$. (See Proposition 7.6.1.) Propositions 7.6.2 and 7.24.1 give that $\mathcal{S}^\Omega\psi$ is a very weak solution of the problem (7.145), (7.146), (7.147) if and only if $T\psi = (g, F)$.

$$T : B^{p,q}_{s-1/p-1}(\partial\Omega) \to B^{p,q}_{s-1/p}(\Gamma_D) \times B^{p,q}_{s-1/p-1}(\Gamma_R),$$
$$T : W^{s-1/p-1,p}(\partial\Omega) \to W^{s-1/p,p}(\Gamma_D) \times W^{s-1/p-1,p}(\Gamma_R)$$

are Fredholm operators with index 0 according to Proposition 7.22.3, Theorems 7.6.3, 1.23.1, 1.23.3, 1.8.3 and 1.13.7 and 1.23.14. Suppose that $T\psi = 0$. Since $\mathcal{S}^\Omega\psi$ is a very weak solution of the problem (7.145), (7.146), (7.147) with $g = 0$, $F = 0$, we have $\mathcal{S}^\Omega\psi = 0$ in Ω. Proposition 7.6.2 gives that $\mathcal{S}^\Omega\psi = 0$ on $\partial\Omega$. Thus, $\psi = 0$ by Theorem 7.6.3. Since T is a Fredholm operator with index 0 and $\operatorname{Ker} T = \{0\}$, we infer that T is onto. So, T is an isomorphism by Theorem 1.6.3.

Theorem 7.36.5 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary. Let* Γ_D *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_D \cup \Gamma_R$*. Let* $1 < p, q < \infty$, $1/p < s < 1+1/p$, $p' = p/(p-1)$, $q' = q/(q-1)$*. If* $p \neq 2$*, suppose moreover that* $\partial\Omega$ *is of class* $\mathcal{C}^{1,\alpha}$ *with* $0 < \alpha < 1$*. Let* $h \in L^\infty(\Gamma_R)$, $h \geq 0$*.*

- *There exists a very weak solution* $u \in B^{p,q}_s(\Omega)$ *of the problem (7.145), (7.146), (7.147) if and only if* $g \in B^{p,q}_{s-1/p}(\Gamma_D)$, $F \in [B^{p',q'}_{2-s}(\Omega)]'$*. This solution is unique and*
$$\|u\|_{B^{p,q}_s(\Omega)} \leq C\left[\|g\|_{B^{p,q}_{s-1/p}(\Gamma_D)} + \|F\|_{[B^{p',q'}_{2-s}(\Omega)]'}\right] \tag{7.149}$$
where a constant C depends only on Ω, Γ_D, Γ_R, p, q*, and* s*.*
- *There exists a weak solution* $u \in W^{1,p}(\Omega)$ *of the problem (7.145), (7.146), (7.147) if and only if* $g \in W^{1-1/p,p}(\Gamma_D)$, $F \in [W^{1,p'}(\Omega)]'$*. This solution is unique and*
$$\|u\|_{W^{1,p}(\Omega)} \leq C\left[\|g\|_{W^{1-1/p,p}(\Gamma_D)} + \|F\|_{[W^{1,p'}(\Omega)]'}\right] \tag{7.150}$$
where a constant C depends only on Ω, Γ_D, Γ_R*, and* p*.*

Proof Uniqueness follows form Lemma 7.36.3.

Without loss of generality, we can suppose that $\overline{\Omega} \subset B(0; 1)$. If there exists a very weak solution $u \in B^{p,q}_s(\Omega)$ $(u \in W^{s,p}(\Omega))$ of the problem (7.145), (7.146), (7.147), then $g \in B^{p,q}_{s-1/p}(\Gamma_D)$ ($g \in W^{s-1/p,p}(\Omega)$), $F \in [B^{p',q'}_{2-s}(\Omega)]'$ ($F \in [W^{s-2,p'}(\Omega)]'$), respectively. (See Lemma 7.36.2)

Suppose now $g \in B^{p,q}_{s-1/p}(\Gamma_D)$, $F \in [B^{p',q'}_{2-s}(\Omega)]'$ or $g \in W^{1-1/p,p}(\Gamma_D)$, $F \in [W^{1,p'}(\Omega)]'$ be given. Since F is a distribution with compact support, we can define $v = F * h_m$. Then, $-\Delta v = F$ in the sense of distributions by Theorem 3.1.5. If $F \in [W^{1,p'}(\Omega)]'$, then $F \in [W^{1,p'}(R^m)]' = W^{-1,p}(R^m)$ and $v \in W^{1,p}(\Omega)$ and

$$\|v\|_{W^{1,p}(\Omega)} \leq C_1 \| F\|_{[W^{1,p'}(\Omega)]'} \tag{7.151}$$

where C_1 depends only on Ω and p (see Proposition 3.18.5). If $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then $F \in [B^{p',q'}_{2-s}(R^m)]' = B^{p,q}_{s-2}(R^m)$ (see Theorem 1.23.1) and $v \in B^{p,q}_s(\Omega)$ with

$$\|v\|_{B^{p,q}_s(\Omega)} \leq C_1 \| F\|_{[B^{p',q'}_{2-s}(\Omega)]'} \tag{7.152}$$

where C_1 depends only on Ω, p, q, and s (see Proposition 3.18.5). Denote by v_0 the trace of v. If $F \in [W^{1,p'}(\Omega)]'$, then $v_0 \in W^{1-1/p,p}(\Gamma_D)$ by Theorem 1.24.2; if $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then $v_0 \in B^{p,q}_{s-1/p}(\Gamma_D)$ by Theorem 1.24.7. If $F \in [W^{1,p'}(\Omega)]'$, then there exists $\tilde{F} \in [W^{1,p'}(\Omega)]'$ such that v is a weak solution of the problem

$$-\Delta v = \tilde{F}|\Omega \text{ in } \Omega, \quad v = v_0 \text{ on } \Gamma_D, \quad \frac{\partial v}{\partial n} + hv = \tilde{F}|\Gamma_R \text{ on } \Gamma_R. \tag{7.153}$$

(It is also a very weak solution of the problem by Proposition 7.36.1.) If $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then there exists $\tilde{F} \in [B^{p',q'}_{2-s}(\Omega)]'$ such that v is a very weak solution of the problem (7.153) (see Lemma 7.36.2). Since $F = -\Delta v = \tilde{F}$ in Ω, we deduce that $F - \tilde{F}$ is supported on $\partial\Omega$. If $F \in [W^{1,p'}(\Omega)]'$, then $F - \tilde{F} \in W^{-1/p,p}(\partial\Omega)$ by Theorems 1.24.2, 1.23.3, and 1.23.1. If $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then $F - \tilde{F} \in B^{p,q}_{s-1/p-1}(\partial\Omega)$ by Theorems 1.24.7 and 1.23.1. Let T be the operator from Theorem 7.36.4. Then $w = \mathcal{S}^{\Omega} T^{-1}(g - v_0, F - \tilde{F})$ is a very weak solution of the problem

$$-\Delta w = (F - \tilde{F})|\Omega \text{ in } \Omega, \quad w = g - v_0 \text{ on } \Gamma_D, \ \frac{\partial w}{\partial n} + hw = (F - \tilde{F})|\Gamma_R \text{ on } \Gamma_R.$$

If $g \in B^{p,q}_{s-1/p}(\Gamma_D)$, $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then $w \in B^{p,q}_s(\Omega)$. If now $g \in W^{1-1/p,p}(\Gamma_D)$, $F \in [W^{1,p'}(\Omega)]'$, then $w \in W^{1,p}(\Omega)$. (See Theorem 7.36.4.) Put $u = v + w$. Then u is a very weak solution of the problem (7.145), (7.146), (7.147). If $g \in B^{p,q}_{s-1/p}(\Gamma_D)$, $F \in [B^{p',q'}_{2-s}(\Omega)]'$, then $u \in B^{p,q}_s(\Omega)$. The estimate (7.149) follows from (7.152), Theorems 1.24.7 and 7.36.4, and Proposition 7.6.1. Let now $g \in W^{1-1/p,p}(\Gamma_D)$, $F \in [W^{1,p'}(\Omega)]'$. Then $u \in W^{1,p}(\Omega)$. Clearly, there exists $G \in [W^{1,p'}(\Omega)]'$ such that u is a weak solution of the problem

$$-\Delta u = G|\Omega \text{ in } \Omega, \quad u = g \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = G|\Gamma_R \text{ on } \Gamma_R.$$

Proposition 7.36.1 gives that u is a very weak solution of this problem. Thus, $F - G$ is supported on Γ_D. Therefore, u is a weak solution of the problem (7.145), (7.146), (7.147). The estimate (7.150) follows from (7.151), Theorems 1.24.2 and 7.36.4, and Proposition 7.6.1.

Theorem 7.36.6 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^2$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in \mathcal{C}^1(\Gamma_R)$, $h \geq 0$, $1 < p < \infty$, $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\Gamma_R)$, $u_0 \in W^{2-1/p,p}(\Gamma_D)$. Then there exists a unique solution $u \in W^{2,p}(\Omega)$ of the mixed problem*

$$-\Delta u = f \text{ in } \Omega, \quad u = u_0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \Gamma_R. \tag{7.154}$$

Moreover,

$$\|u\|_{W^{2,p}(\Omega)} \leq C\big(\|f\|_{L^p(\Omega)} + \|u_0\|_{W^{2-1/p,p}(\Gamma_D)} + \|g\|_{W^{1-1/p,p}(\Gamma_R)}\big)$$

where a constant C does not depend on f, g, and u_0.

Proof According to Theorem 7.36.5, there exist a unique solution $u \in W^{1,p}(\Omega)$ of the problem. Corollary 7.14.5 and Theorem 7.26.3 give that $u \in D^{2,p}(\Omega)$.

Since the mapping $u \mapsto [u, \partial u/\partial n + hu, -\Delta u]$ is a one-to-one continuous mapping from $W^{2,p}(\Omega)$ onto $W^{2-1/p,p}(\Gamma_D) \times W^{1-1/p,p}(\Gamma_R) \times L^p(\Omega)$, it is an isomorphism by Theorem 1.6.3.

Theorem 7.36.7 *Let $\Omega \subset R^2$ be a bounded domain with boundary of class $\mathcal{C}^{k+2}$ where $k \in N_0$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in \mathcal{C}^{k+1}(\Gamma_R)$, $h \geq 0$, $1 < p < \infty$, $f \in W^{k,p}(\Omega)$, $g \in W^{k+1-1/p,p}(\Gamma_R)$, $u_0 \in W^{k+2-1/p,p}(\Gamma_D)$. Then there exists a unique solution $u \in W^{k+2,p}(\Omega)$ of the mixed problem (7.154). Moreover,*

$$\|u\|_{W^{k+2,p}(\Omega)} \leq C\big(\|f\|_{W^{k,p}(\Omega)} + \|u_0\|_{W^{k+2-1/p,p}(\Gamma_D)} + \|g\|_{W^{k+1-1/p,p}(\Gamma_R)}\big)$$

where a constant C does not depend on f, g, and u_0.

Proof According to Theorem 7.36.6, there exists a unique solution $u \in W^{2,p}(\Omega)$ of the mixed problem (7.154). Corollary 7.14.5 gives that $u \in W^{2+k,p}_{loc}(\overline{\Omega} \setminus \Gamma_R)$. Choose a bounded domain ω with boundary of class $\mathcal{C}^\infty$ such that $\Gamma_R \subset \omega$ and $\Gamma_D \cap \overline{\omega} = \emptyset$. Define $h = 1$, $u_0 = u$, $g = \nabla u \cdot n^\omega + u$ on $\Omega \cap \partial\omega$. Then $g \in W^{k+1-1/p,p}(\partial(\Omega \cap \omega))$, $u_0 \in W^{k+2-1/p,p}(\partial(\Omega \setminus \overline{\omega}))$ by Theorem 1.24.6. Since u is a solution of the problem $-\Delta u = f$ in $\Omega \cap \omega$, $\partial u/\partial n + hu = g$ on $\partial(\Omega \cap \omega)$, Theorem 7.30.2 yields that $u \in W^{k+2,p}(\Omega \cap \omega)$. Thus, $u \in W^{k+2,p}(\Omega)$.

Since the mapping $u \mapsto [u, \partial u/\partial n + hu, -\Delta u]$ is a one-to-one continuous mapping from $W^{k+2,p}(\Omega)$ onto $W^{k+2-1/p,p}(\Gamma_D) \times W^{k+1-1/p,p}(\Gamma_R) \times W^{k,p}(\Omega)$, it is an isomorphism by Theorem 1.6.3.

7.37 The Mixed Dirichlet-Robin Problem in $D^{1,p}(\Omega)$ for Unbounded Domains

In this section, we prove the existence of a solution of the Robin-Dirichlet problem in $D^{1,p}(\Omega)$ for an unbounded domain with compact Ljapunov boundary and discuss the uniqueness of a solution.

Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in L^\infty(\partial\Omega)$, $1 < p < \infty$, $p' = p/(p-1)$. Let $u_0 \in W^{1-1/p}(\partial\Omega)$, $F \in [\tilde{W}^{1,p'}(\Omega)]'$. We say that $u \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem

$$-\Delta u = F|\Omega \text{ in } \Omega, \quad u = u_0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = F|\Gamma_R \text{ on } \Gamma_R \tag{7.155}$$

if $u = u_0$ on Γ_D in the sense of traces and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} hu\varphi \, \mathrm{d}\mathcal{H}_{m-1} = \langle F, \varphi \rangle \ \forall \varphi \in \tilde{W}^{1,p'}(\Omega), \varphi = 0 \text{ on } \Gamma_D.$$

Lemma 7.37.1 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in L^\infty(\partial\Omega)$, $h \geq 0$. If $1 < p \leq m/(m-1)$ and $u \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.155) with $u_0 \equiv 0$ and $F \equiv 0$, then $u \equiv 0$.*

Proof We show that $u \in D^{1,p'}(\Omega)$, where $p' = p/(p-1)$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Define $\omega = \Omega \cap B(0; r)$, $u_0 = u$ on $\partial B(0; r)$. Then u is a weak solution of the mixed problem

$$\Delta u = 0 \text{ in } \omega, \quad \frac{\partial u}{\partial n} + hu = 0 \text{ on } \Gamma_R, \quad u = u_0 \text{ on } \partial\omega \setminus \Gamma_R.$$

Thus, $u \in W^{1,p'}(\omega)$ by Theorem 7.36.5. Since $|\nabla u| \in L^p(\Omega)$, Proposition 2.17.7 forces $u \in D^{1,p'}(\Omega)$.

Since $p' \geq m$, we have $D^{1,p'}(\Omega) = \tilde{W}^{1,p'}(\Omega)$ by Lemma 3.38.11. Since $u \in \tilde{W}^{1,p'}(\Omega)$ the definition of a weak solution of the mixed problem gives

$$\int_\Omega \nabla u \cdot \nabla u \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} huu \, \mathrm{d}\mathcal{H}_{m-1} = 0.$$

Thus, $\nabla u = 0$ in Ω. Therefore, there exists a constant c such that $u = c$ in Ω (see Lemma 1.20.1). Since $u = 0$ on Γ_D, we infer that $c = 0$.

Lemma 7.37.2 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in L^\infty(\partial\Omega)$, $h \geq 0$. If $1 < p < m$ and $u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.155) with $u_0 \equiv 0$ and $F \equiv 0$, then $u \equiv 0$.*

Proof If $p \leq m/(m-1)$, then the proposition follows from Lemma 7.37.1.

Let now $m/(m-1) < p < m$. Then $m > 2$. We show that $u \in \tilde{W}^{1,2}(\Omega)$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Define $\omega = \Omega \cap B(0;r)$, $u_0 = u$ on $\partial B(0;r)$. Then u is a weak solution of the mixed problem

$$\Delta u = 0 \text{ in } \omega, \quad \frac{\partial u}{\partial n} + hu = 0 \text{ on } \Gamma_R, \quad u = u_0 \text{ on } \partial\omega \setminus \Gamma_D.$$

Thus, $u \in W^{1,2}(\omega)$ by Theorem 7.36.5. Since $u \in \tilde{W}^{1,p}(\Omega)$, we have $u \in L^{mp/(m-p)}(R^m \setminus B(0;r))$ by Proposition 3.38.5. Proposition 2.17.7 gives that $u \in D^{1,2}(\Omega)$. Since $u \in L^{mp/(m-p)}(R^m \setminus B(0;r))$, Proposition 3.38.5 forces $u \in \tilde{W}^{1,2}(\Omega)$.

Since $u \in \tilde{W}^{1,2}(\Omega)$, the definition of a weak solution of the mixed problem gives

$$\int_\Omega \nabla u \cdot \nabla u \, d\mathcal{H}_m + \int_{\partial\Omega} huu \, d\mathcal{H}_{m-1} = 0.$$

Thus, $\nabla u = 0$ in Ω. Therefore, there exists a constant c such that $u = c$ in Ω (see Lemma 1.20.1). Since $u = 0$ on Γ_D, we infer that $c = 0$.

Proposition 7.37.3 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $1 < p < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. For $f \in W^{-1/p,p}(\partial\Omega)$ define*

$$Pf = \frac{\langle f, 1\rangle}{\mathcal{H}_{m-1}(\partial\Omega)} \mathcal{H}_{m-1}|\partial\Omega,$$

$$Tf = \begin{cases} [(1/2)I - K'_\Omega](I-P)f + h[\mathcal{S}^\Omega(I-P)f + Pf] & \text{on } \Gamma_R, \\ \mathcal{S}^\Omega(I-P)f + Pf & \text{on } \Gamma_D. \end{cases}$$

Then $T : W^{-1/p,p}(\partial\Omega) \mapsto W^{-1/p,p}(\Gamma_R) \times W^{1-1/p,p}(\Gamma_D)$ is an isomorphism. If $g \in W^{-1/p,p}(\Gamma_R)$, $u_0 \in W^{1-1/p,p}(\Gamma_D)$ then

$$u = \mathcal{S}^\Omega(I-P)T^{-1}[g, u_0] + PT^{-1}[g, u_0] \tag{7.156}$$

is a weak solution of the mixed problem for the Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \Gamma_R \tag{7.157}$$

in $D^{1,p}(\Omega)$.

Proof T is a Fredholm operator with index 0 by Proposition 7.22.3 and Theorems 7.6.3, 1.23.3, and 1.8.3. Let $f \in W^{-1/p,p}(\partial\Omega)$. Then $u = \mathcal{S}^{\Omega}(I-P)f + Pf \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.157) with $[g, u_0] = Tf$. (See Lemma 7.27.1 and Proposition 7.6.2.) Let now $Tf = 0$. Fix $q \in (1, m/(m-1))$. Then $f \in W^{-1/q,q}(\partial\Omega)$ by Lemma 1.8.4. Since $u \in D^{1,q}(\Omega)$ is a weak solution of the mixed problem (7.157) with $g = 0$ and $u_0 = 0$, Lemma 7.37.1 forces that $u \equiv 0$. Thus, $\mathcal{S}^{\Omega}(I-P)f = -Pf$ is constant. So, $\mathcal{S}^{\Omega}(I-P)f \in D^{1,p}(\Omega)$ is a weak solution of the homogeneous Neumann problem $\Delta v = 0$ in Ω, $\partial v/\partial n = 0$ on $\partial\Omega$. Thus, $[(1/2)I - K'_{\Omega}](I-P)f = 0$ by Lemma 7.27.1. So, $(I-P)f = 0$ by Proposition 7.22.3. Hence, $-Pf = \mathcal{S}^{\Omega}(I-P)f = 0$. Thus, $f = (I-P)f + Pf = 0$. Since T is a Fredholm operator with index 0 on $W^{-1/p,p}(\partial\Omega)$ with trivial kernel, it is onto. Theorem 1.6.3 gives that T is an isomorphism.

Proposition 7.37.4 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ and $m > 2$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $1 < p < \infty$, $h \in L^{\infty}(\partial\Omega)$, $h \geq 0$. For $f \in W^{-1/p,p}(\partial\Omega)$, define*

$$\tilde{T}f = \begin{cases} [(1/2)I - K'_{\Omega}]f + h\mathcal{S}^{\Omega}f & \text{on } \Gamma_R, \\ \mathcal{S}^{\Omega}f & \text{on } \Gamma_D. \end{cases}$$

Then $\tilde{T} : W^{-1/p,p}(\partial\Omega) \mapsto W^{-1/p,p}(\Gamma_R) \times W^{1-1/p,p}(\Gamma_D)$ is an isomorphism. If $p > m/(m-1)$, $g \in W^{-1/p,p}(\Gamma_R)$, and $u_0 \in W^{1-1/p,p}(\Gamma_D)$, then $u = \mathcal{S}^{\Omega}\tilde{T}^{-1}[g, u_0] \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the mixed problem for the Laplace equation (7.157).

Proof $\tilde{T}$ is a Fredholm operator with index 0 by Proposition 7.37.3 and Theorem 1.8.3. If $f \in W^{-1/p,p}(\partial\Omega)$ and $p > m/(m-1)$, then $u = \mathcal{S}^{\Omega}f \in \tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.157) with $[g, u_0] = \tilde{T}f$. (See Lemma 7.27.1 and Propositions 7.6.2 and 7.6.4.) Let now $\tilde{T}f = 0$, $f \in W^{-1/p,p}(\partial\Omega)$ and $m/(m-1) < p < m$. Since $u \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.157) with $g = 0$ and $u_0 = 0$, Lemma 7.37.2 forces that $u \equiv 0$. By virtue of Proposition 7.6.2 and Theorem 7.6.3, we obtain that $f \equiv 0$. If $1 < p < \infty$ is arbitrary, then $\operatorname{Ker}\tilde{T}$ is trivial by Lemma 1.8.4. Since $\tilde{T}$ is a Fredholm operator with index 0 on $W^{-1/p,p}(\partial\Omega)$ with trivial kernel, it is onto. Theorem 1.6.3 gives that $\tilde{T}$ is an isomorphism.

Theorem 7.37.5 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $1 < p < \infty$, $p' = p/(p-1)$, $h \in L^{\infty}(\partial\Omega)$, $h \geq 0$, $u_0 \in W^{1-1/p,p}(\partial\Omega)$, $F \in [\tilde{W}^{1,p'}(\Omega)]'$.*

- *If $p \leq m/(m-1)$, then there exists a unique weak solution $u \in D^{1,p}(\Omega)$ of the mixed problem for the Poisson equation (7.155).*
- *If $p > m/(m-1)$, then there exists a weak solution $u \in \tilde{W}^{1,p}(\Omega) \subset D^{1,p}(\Omega)$ of the mixed problem for the Poisson equation (7.155). If moreover $p < m$ then this solution is unique.*

- *If $p > m/(m-1)$ and $F \equiv 0$ and $u_0 \equiv 0$, then the set of all weak solutions $u \in D^{1,p}(\Omega)$ of the mixed problem for the Poisson equation (7.155) is a one-dimensional space.*

Proof Suppose first that F is supported on $\partial\Omega$. By virtue of Theorem 1.24.2, we deduce that $F \in [W^{1-1/p',p'}(\partial\Omega)]' = W^{-1/p,p}(\partial\Omega)$. If T and P are the operators from Proposition 7.37.3, then

$$T : W^{-1/p,p}(\partial\Omega) \to W^{-1/p,p}(\Gamma_R) \times W^{1-1/p,p}(\Gamma_D)$$

is an isomorphism. If $m > 2$ and $\tilde{T}$ is the operator from Proposition 7.37.4, then

$$\tilde{T} : W^{-1/p,p}(\partial\Omega) \to W^{-1/p,p}(\Gamma_R) \times W^{1-1/p,p}(\Gamma_D)$$

is an isomorphism. The function $v = \mathcal{S}^\Omega(I-P)T^{-1}[F,u_0]+PT^{-1}[F,u_0] \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem (7.155). If $p \le m/(m-1)$, then this solution is unique by Lemma 7.37.1. If $p \ge m$, then $v \in \tilde{W}^{1,p}(\Omega)$ by Proposition 3.38.4. If $m/(m-1) < p < m$, then $m > 2$, and $\tilde{v} = \mathcal{S}^\Omega\tilde{T}^{-1}[F,u_0] \in \tilde{W}^{1,p}(\Omega)$ is a weak solution of (7.155) by Proposition 7.37.4.

Let now $F \equiv 0$, $u_0 \equiv 0$, $p > m/(m-1)$ and $m = 2$. Fix $z \in R^2 \setminus \overline{\Omega}$. Denote $\rho(x) = \ln|x-z|$ in Ω, $\tilde{\rho}(x) = \ln|x-z|$ on $\partial\Omega$, $\Phi = \partial\rho/\partial n + h\rho$ on $\partial\Omega$. Then $\rho \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem

$$\Delta\rho = 0 \text{ in } \Omega, \quad \rho = \tilde{\rho} \text{ on } \Gamma_D, \quad \frac{\partial\rho}{\partial n} + h\rho = \Phi \text{ on } \Gamma_R.$$

But $\Psi = \mathcal{S}^\Omega(I-P)T^{-1}[\Phi,\tilde{\rho}]+PT^{-1}[\Phi,\tilde{\rho}]$ is another weak solution of this problem. According to Proposition 3.1.6, we have $\Psi(x) = O(1)$ as $|x| \to \infty$. Since $\rho(x) \to \infty$ as $|x| \to \infty$, we infer that $\psi = \rho - \Psi \in D^{1,p}(\Omega)$ is a nontrivial weak solution of the mixed problem (7.155). Let now $u \in D^{1,p}(\Omega)$ be a weak solution of the mixed problem (7.155). According to Proposition 2.17.6, there exists a constant c such that $u(x) - c\ln|x-z| = O(1)$ as $|x| \to \infty$. So, $u(x) - c\psi(x) = O(1)$ as $|x| \to \infty$. Proposition 2.17.3 gives that $|\nabla[u(x) - c\psi(x)]| = O(|x|^{-2})$ as $|x| \to \infty$. Since $p > 2$, we deduce that $u - c\psi \in D^{1,2}(\Omega)$. Lemma 7.37.1 gives that $u - c\psi \equiv 0$.

Let now $F \equiv 0$, $u_0 \equiv 0$, $p > m/(m-1)$ and $m > 2$. The function $\varphi \equiv 1 \in D^{1,p}(\Omega)$ is a weak solution of the mixed problem

$$\Delta\varphi = 0 \text{ in } \Omega, \quad \varphi = 1 \text{ on } \Gamma_D, \quad \frac{\partial\varphi}{\partial n} + h\rho = h \text{ on } \Gamma_R.$$

But the function $\tilde{\varphi} = \mathcal{S}^\Omega\tilde{T}^{-1}[h,1] \in D^{1,p}(\Omega)$ is a weak solution of this problem, too. Since $\tilde{\varphi}(x) \to 0$ as $|x| \to \infty$ by Proposition 3.1.6, the function $\Psi = \varphi - \tilde{\varphi} \in D^{1,p}(\Omega)$ is a nontrivial weak solution of (7.155). Let now $u \in D^{1,p}(\Omega)$ be a weak solution of the mixed problem (7.155). According to Proposition 2.17.5, there exists a constant c such that $u(x) - c = O(|x|^{2-m})$ as $|x| \to \infty$. So, $u(x) -$

$c\Psi(x) = O(|x|^{2-m})$ as $|x| \to \infty$. Proposition 2.12.1 gives that $|\nabla[u(x) - c\Psi(x)]| = O(|x|^{1-m})$ as $|x| \to \infty$. So, there exists $q \in (1, m)$ such that $u - c\Psi \in D^{1,q}(\Omega)$. Since $u(x) - c\Psi(x) \to 0$ as $|x| \to \infty$, $u - c\Psi \in \tilde{W}^{1,q}(\Omega)$ by Lemma 3.38.11 and Proposition 3.38.5. Hence, $u - c\psi \equiv 0$ by Lemma 7.37.2.

Let now F be general. According to Theorem 7.13.3, there exists $w \in D^{1,p}(\Omega)$ such that $-\Delta w = F$ in Ω. If $p > m/(m-1)$, we can suppose that $w \in \tilde{W}^{1,p}(\Omega)$. According to Theorem 1.24.2, the trace $w \in W^{1-1/p,p}(\partial\Omega)$. Define g by

$$\langle g, \varphi\rangle = \langle F, \varphi\rangle - \int_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{d}\mathcal{H}_m - \int_{\partial\Omega} hw\varphi \,\mathrm{d}\mathcal{H}_{m-1}.$$

Theorem 1.24.2 forces that $g \in [\tilde{W}^{1,p'}(\Omega)]'$. If $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then $\langle g, \varphi\rangle = 0$ by Theorem 3.19.3. So, g is supported on $\partial\Omega$. We have shown that there exists a weak solution $v \in D^{1,p}(\Omega)$ of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad v = u_0 - w \text{ on } \Gamma_D, \quad \frac{\partial v}{\partial n} + hv = g \text{ on } \Gamma_R.$$

If $p > m/(m-1)$, then we can suppose that $v \in \tilde{W}^{1,p}(\Omega)$. Clearly, $u = w + v$ is a solution of the original problem (7.155).

7.38 The Mixed Dirichlet-Robin Problem in $D^{2,p}(\Omega)$ on Exterior Domains

Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in \mathcal{C}^1(\partial\Omega)$, $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$, $u_0 \in W^{2-1/p,p}(\partial\Omega)$, $1 < p < \infty$. We say that $u \in D^{2,p}(\Omega)$ is a strong solution of the mixed problem

$$-\Delta u = f \text{ in } \Omega, \quad u = u_0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \Gamma_R \tag{7.158}$$

if $-\Delta u = f$ in Ω in the sense of distributions and the boundary conditions are fulfilled in the sense of traces.

Theorem 7.38.1 *Let $\Omega \subset R^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^2$. Let Γ_D and Γ_R be disjoint nonempty closed subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \Gamma_R$. Let $h \in \mathcal{C}^1(\partial\Omega)$, $h \geq 0$, $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$, $u_0 \in W^{2-1/p,p}(\partial\Omega)$, $1 < p < \infty$. Then there exists a strong solutions $u \in D^{2,p}(\Omega)$ of the mixed problem (7.158). If $f \equiv 0$, $g \equiv 0$, and $u_0 \equiv 0$, then the space of all solutions $u \in D^{2,p}(\Omega)$ of the mixed problem (7.158) has dimension $m + 1$.*

Proof Suppose first that $f \equiv 0$. According to Theorem 7.37.5, there exists a weak solution $u \in D^{1,p}(\Omega)$ of the problem (7.158). Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Then $u \in D^{2,p}(R^m \setminus \overline{B(0;r)})$ by Proposition 2.17.7. Define $u_0 = u$ on $\partial B(0;r)$, $\omega = \Omega \cap B(0;r)$. Then u is a weak solution of the mixed problem

$$\Delta u = 0 \text{ in } \omega, \quad u = u_0 \text{ on } \partial\omega \setminus \Gamma_R, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \Gamma_R.$$

Since $g \in W^{1-1/p,p}(\Gamma_R)$ and $u_0 \in W^{2-1/p,p}(\partial\omega)$, we infer that $u \in D^{2,p}(\omega)$ is a strong solution of the problem by Theorems 7.36.6 and 7.36.5. So, $u \in D^{2,p}(\Omega)$.

Let now $f \equiv 0$, $g \equiv 0$, and $u_0 \equiv 0$. For $j \in \{1,\dots,m\}$, define $\tau_j(x) = x_j$. Then, $\tau_j \in D^{2,p}(\Omega) \setminus D^{1,p}(\Omega)$. Fix $z \in R^m \setminus \overline{\Omega}$ and define $\tau_{m+1}(x) = h_m(x-z)$. Define $g_j = \partial\tau_j/\partial n + h\tau_j$ on Γ_R. We have proved that there exists a strong solution $\tilde{\tau}_j \in D^{2,p}(\Omega) \cap D^{1,p}(\Omega)$ of the problem

$$\Delta\tau = 0 \text{ in } \Omega, \ \tau = \tau_j \text{ on } \Gamma_D, \ \frac{\partial\tau(x)}{\partial n} + h(x)\tau(x) = g_j \text{ on } \Gamma_R.$$

Define $v_j = \tau_j - \tilde{\tau}_j$ for $j \in \{1,\dots,m\}$. Then $v_j \in D^{2,p}(\Omega) \setminus D^{1,p}(\Omega)$ is a solution of the problem (7.158). If $p \le m/(m-1)$, then $v_{m+1} = \tau_{m+1} - \tilde{\tau}_{m+1} \in D^{2,p}(\Omega) \setminus D^{1,p}(\Omega)$ is a solution of the problem (7.158). If $p > m/(m-1)$, then there exists a nontrivial solution $v_{m+1} \in D^{1,p}(\Omega)$ of the problem (7.158). (See Theorem 7.37.5.) We have proved that $v_{m+1} \in D^{2,p}(\Omega)$. Since $v_1,\dots,v_{m+1}$ are linearly independent, the space of all solutions $u \in D^{2,p}(\Omega)$ of the mixed problem (7.158) has dimension at least $m+1$.

Let now $f \equiv 0$, $g \equiv 0$, $u_0 \equiv 0$, and $u \in D^{2,p}(\Omega)$ be a solution of the mixed problem (7.158). According to Proposition 2.17.5, there exist constants $c_1,\dots,c_{m+1}$ such that $v(x) := u(x) - c_1x_1 - \cdots - c_mx_m - c_{m+1}h_m(x-z) = O(|x|^{1-m})$ as $|x| \to \infty$. Since $|\nabla v(x)| = O(|x|^{-m})$ as $|x| \to \infty$ by Proposition 2.12.1, we infer that $v \in D^{1,p}(\Omega)$. This forces $w := u - c_1v_1 - \cdots - c_{m+1}v_{m+1} \in D^{1,p}(\Omega)$. Remember that w is a solution of (7.158). If $p \le m/(m-1)$, then $w \equiv 0$ by Theorem 7.37.5. If $p > m/(m-1)$, then w is a multiple of v_{m+1} by Theorem 7.37.5. Since u is a linear combination of $v_1,\dots,v_{m+1}$, we deduce that the space of all solutions $u \in D^{2,p}(\Omega)$ of the mixed problem (7.158) has dimension $m+1$.

Let now f be arbitrary. Define $f = 0$ on $R^m \setminus \Omega$. According to Proposition 7.4.2, there exists $v \in D^{2,p}(R^m)$ such that $-\Delta v = f$ in R^m. Define $\tilde{g} = g - \partial v/\partial n - hv$, $\tilde{u}_0 := u_0 - v$ on $\partial\Omega$. Then $\tilde{u}_0 \in W^{2-1/p,p}(\partial\Omega)$, $\tilde{g} \in W^{1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. We have proved that there exists a strong solution $w \in D^{2,p}(\Omega)$ of the mixed problem

$$-\Delta w = 0 \text{ in } \Omega, \quad w = \tilde{u}_0 \text{ on } \Gamma_D, \quad \frac{\partial w}{\partial n} + hw = \tilde{g} \text{ on } \Gamma_R.$$

Clearly, $u = v + w \in D^{2,p}(\Omega)$ is a strong solution of the problem (7.158).

7.39 The Mixed Dirichlet-Robin Problem in $D^{k+2,p}(\Omega)$ on Exterior Planar Domains

Theorem 7.39.1 *Let* $\Omega \subset R^2$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{k+2}$*, where* $k \in N_0$*. Let* Γ_D *and* Γ_R *be disjoint nonempty closed subsets of* $\partial\Omega$ *such that* $\partial\Omega = \Gamma_D \cup \Gamma_R$*. Let* $h \in \mathcal{C}^{k+1}(\partial\Omega)$*,* $h \geq 0$*,* $f \in D^{k,p}(\Omega)$*,* $g \in W^{k+1-1/p,p}(\partial\Omega)$*,* $u_0 \in W^{k+2-1/p,p}(\partial\Omega)$*,* $1 < p < \infty$*. Then, there exists a strong solution* $u \in D^{k+2,p}(\Omega)$ *of the mixed problem (7.158). If* $f \equiv 0$*,* $g \equiv 0$*, and* $u_0 \equiv 0$*, then the space of all solutions* $u \in D^{k+2,p}(\Omega)$ *of the mixed problem (7.158) has dimension equal to* $\dim[H(R^m) \cap \tilde{P}_{k+1}(R^m)]$*, where* $H(R^m)$ *is the space of all harmonic function in* R^m *and* $\tilde{P}_{k+1}(R^m)$ *is the space of all polynomials of degree at most* $k+1$.

Proof For $k = 0$, see Theorem 7.38.1. So, we can suppose that $k \in N$.

Suppose first that $f \equiv 0$. According to Theorem 7.38.1, there exists a solution $u \in D^{2,p}(\Omega)$ of the problem (7.158). Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Then $u \in D^{k+2,p}(R^m \setminus \overline{B(0;r)})$ by Proposition 2.17.7. Define $u_0 = u$ on $\partial B(0;r)$, $\omega = \Omega \cap B(0;r)$. Then u is a solution of the mixed problem

$$\Delta u = 0 \text{ in } \omega, \quad u = u_0 \text{ on } \partial\omega \setminus \Gamma_R, \quad \frac{\partial u}{\partial n} + hu = g \text{ on } \Gamma_R.$$

Since $g \in W^{k+1-1/p,p}(\Gamma_R)$ and $u_0 \in W^{k+2-1/p,p}(\partial\omega)$, we infer that $u \in D^{k+2,p}(\omega)$ by Theorem 7.36.7. So, $u \in D^{k+2,p}(\Omega)$.

Let now $f \equiv 0$, $u_0 \equiv 0$, and $g \equiv 0$. Let $u \in D^{k+2,p}(\Omega)$ be a solution of the problem (7.158). Proposition 7.9.2 gives

$$u = \mathcal{D}^{\Omega} u + \mathcal{S}^{\Omega}(\partial u/\partial n) + P,$$

where $P \in \tilde{P}_{k+1}(R^m)$. Denote $X = \{Q \in \tilde{P}_{k+1}(R^m); Q(0) = 0, \nabla Q(0) = 0\}$. There exist $Q_1 \in \tilde{P}_1(R^m)$ and $Q_2 \in X$ such that $P = Q_1 + Q_2$. Put $v = \mathcal{D}^{\Omega} u + \mathcal{S}^{\Omega}(\partial u/\partial n) + Q_1$. Since $u, v \in H(\Omega)$, we infer that $Q_2 = u - v \in H(\Omega)$. Moreover, $v = u - Q_2 \in D^{k+2,p}(\Omega)$ is a solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad v = -Q_2 \text{ on } \Gamma_D, \quad \partial v/\partial n + hv = -\partial Q_2/\partial n - hQ_2 \text{ on } \Gamma_R.$$

Let now $Q_2 \in X \cap H(\Omega)$ be given. According to Theorem 7.38.1, there exists a solution $v \in D^{2,p}(\Omega)$ of this problem. We have proved that $v \in D^{k+2,p}(\Omega)$. So, $u = v + Q_2 \in D^{k+2,p}(\Omega)$ is a solution of the problem $\Delta u = 0$ in Ω, $u = 0$ on $\partial\Omega$. By virtue of Theorem 7.38.1, we deduce that the space of all solutions $u \in D^{k+2,p}(\Omega)$ of the mixed problem (7.158) has dimension equal to $m + 1 + \dim(X \cap H(\Omega)) = \dim(\tilde{P}_{k+1}(R^m) \cap H(\Omega))$.

Let now f be arbitrary. According to Theorem 7.15.3, there exists $v \in D^{k+2,p}(\Omega)$ such that $-\Delta v = f$ in Ω. Define $\tilde{g} = g - \partial v/\partial n - hv$, $\tilde{u}_0 := u_0 - v$ on $\partial\Omega$. Then $\tilde{u}_0 \in W^{k+2-1/p,p}(\partial\Omega)$, $\tilde{g} \in W^{k+1-1/p,p}(\partial\Omega)$ by Theorem 1.24.5. We have proved that there exists a strong solution $w \in D^{k+2,p}(\Omega)$ of the mixed problem

$$-\Delta w = 0 \text{ in } \Omega, \quad w = \tilde{u}_0 \text{ on } \Gamma_D, \quad \frac{\partial w}{\partial n} + hw = \tilde{g} \text{ on } \Gamma_R.$$

Clearly, $u = v + w \in D^{k+2,p}(\Omega)$ is a strong solution of the problem (7.158).

7.40 The Mixed Transmission Problem

In this section, we study the mixed transmission problem in $\tilde{W}^{1,2}(\Omega_\pm)$. We find a necessary and sufficient condition for the existence of a solution and discuss the uniqueness.

Theorem 7.40.1 *Let $G \subset R^m$ be a domain with nonempty compact Lipschitz boundary. Let $\Gamma_R \subset \partial G$ be closed. Denote $\Gamma_D = \partial G \setminus \Gamma_R$. Let Ω_+ be a bounded open set with Lipschitz boundary such that $\overline{\Omega}_+ \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}_+$. Let b_+, b_- be positive constants. Let $h \in L^\infty(\Gamma_R)$, $h_+, h_- \in L^\infty(\partial\Omega_+)$, $h, h_+, h_- \geq 0$. Suppose that*

$$\int_{\Gamma_D} 1 \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\Gamma_R} h \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega_+} (h_+ + h_-) \, \mathrm{d}\mathcal{H}_{m-1} > 0.$$

Denote by n the unit exterior normal with respect to Ω_-. Let $u_0 \in H^{1/2}(\partial G)$, $f \in H^{-1/2}(\partial G)$, $u_c \in H^{1/2}(\partial\Omega_+)$, $f_c \in H^{-1/2}(\partial\Omega_+)$ be given. Then the mixed transmission problem

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \tag{7.159}$$

$$u_- = u_0 \quad \text{on } \Gamma_D, \quad \frac{\partial u_-}{\partial n} + hu_- = f \text{ on } \Gamma_R, \tag{7.160}$$

$$u_- - u_+ = u_c, \quad b_- \frac{\partial u_-}{\partial n} - b_+ \frac{\partial u_+}{\partial n} + h_+ u_+ + h_- u_- = f_c \quad \text{on } \partial\Omega_+ \tag{7.161}$$

has a unique solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$.

Proof Define $b = b_+$ on Ω_+, $b = b_-$ on Ω_-, $h = 1$ on Γ_D and

$$(u, v) = \int_G b \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} b_- huv \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega_+} (h_+ + h_-) uv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then (,) is a scalar product on $\tilde{W}^{1,2}(G)$, and the corresponding norm is equivalent to the original norm (compare Corollary 7.21.4).

Let $f \in H^{-1/2}(\partial G), f_c \in H^{-1/2}(\partial\Omega_+)$, $u_0 \equiv 0$ $u_c \equiv 0$. Define

$$X = \{u \in \tilde{W}^{1,2}(G); u = 0 \text{ on } \Gamma_D\}.$$

According to Riesz representation theorem (Theorem 1.9.10), there exists a unique $u \in X$ such that

$$(u, v) = (b_- f, v) + (f_c, v) \quad \forall v \in X.$$

Put $u_\pm = u$ in $\Omega_\pm$. Then $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ is a solution of the mixed problem (7.159), (7.160), (7.161).

Suppose $u_0 \in H^{1/2}(\partial G), f \in H^{-1/2}(\partial G)$, $u_c \in H^{1/2}(\partial\Omega_+), f_c \in H^{-1/2}(\partial\Omega_+)$. According to Theorem 7.1.2, there exists $v_- \in \tilde{W}^{1,2}(\Omega_-)$ such that

$$\Delta v_- = 0 \text{ in } \Omega_-, \quad v_- = u_0 \text{ on } \Gamma_D, \quad v_- = u_c \text{ on } \partial\Omega_+.$$

Put $v_+ \equiv 0$. Define $\tilde{f} = f - \partial v_-/\partial n - hv_-$ on $\partial G, \tilde{f}_c = f_c - b_-\partial v_-/\partial n - h_- v_-$ on $\partial\Omega_+$. Then $\tilde{f} \in H^{-1/2}(\partial G), \tilde{f}_c \in H^{-1/2}(\partial\Omega_+)$. We have proved that there exist $w_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ such that

$$\Delta w_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$w_- = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w_-}{\partial n} + hw_- = \tilde{f} \text{ on } \Gamma_R,$$

$$w_- - w_+ = 0, \quad b_- \frac{\partial w_-}{\partial n} - b_+ \frac{\partial w_+}{\partial n} + h_+ w_+ + h_- w_- = \tilde{f}_c \quad \text{on } \partial\Omega_+.$$

Clearly, $u_\pm = v_\pm + w_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ is a solution of the mixed problem (7.159), (7.160), (7.161).

Let now $u_0 \equiv 0$, $u_c \equiv 0, f \equiv 0, f_c \equiv 0$, and $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ be a solution of the mixed problem (7.159), (7.160), (7.161). Define $u = u_\pm$ on $\Omega_\pm$. Then $u \in \tilde{W}^{1,2}(\Omega)$ by Theorem 1.21.1 and Lemma 1.21.12. So, $u \in X$ and $(u, v) = 0$ for all $v \in X$. We have proved that $u \equiv 0$.

Theorem 7.40.2 *Let $G \subset R^m$ be a domain with nonempty compact Lipschitz boundary. Suppose that Ω is bounded or $m = 2$. Let Ω_+ be a bounded open set with Lipschitz boundary such that $\overline{\Omega_+} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega_+}$. Denote by n the unit exterior normal with respect to Ω_-. Let b_+, b_- be positive constants. If $f \in H^{-1/2}(\partial G)$, $u_c \in H^{1/2}(\partial\Omega_+), f_c \in H^{-1/2}(\partial\Omega_+)$, then there exists a solution $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ of the mixed transmission problem*

$$\Delta u_\pm = 0 \quad \text{in } \Omega_\pm, \qquad \frac{\partial u_-}{\partial n} = f \text{ on } \partial G, \tag{7.162}$$

$$u_- - u_+ = u_c, \quad b_- \frac{\partial u_-}{\partial n} - b_+ \frac{\partial u_+}{\partial n} = f_c \quad \text{on } \partial\Omega_+ \tag{7.163}$$

if and only if

$$\langle f_c, 1\rangle + b_-\langle f, 1\rangle = 0. \tag{7.164}$$

If $v_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ *is a particular solution of the problem (7.162), (7.163), then general form of a solution is* $u_\pm = v_\pm + c$, *where c is a constant.*

Proof $H_\pm := H(\Omega_\pm) \cap \tilde{W}^{1,2}(\Omega_\pm)$ is a closed subspace of $\tilde{W}^{1,2}(\Omega_\pm)$ by Proposition 2.11.4. Define

$$T_h[u_+, u_-] = \left[\frac{\partial u_-}{\partial n} + hu, u_- - u_+, b_-\frac{\partial u_-}{\partial n} - b_+\frac{\partial u_+}{\partial n}\right].$$

Then T_1 is a one-to-one bounded linear operator from $H_+ \times H_-$ onto $H^{-1/2}(\partial G) \times H^{1/2}(\partial\Omega_+) \times H^{-1/2}(\partial\Omega_+)$ by Theorem 7.40.1. Theorem 1.6.3 yields that T_1 is an isomorphism. Since $T_1 - T_0$ is a compact operator by Theorem 1.24.1, T_0 is a Fredholm operator with index 0.

Define $\varphi_\pm \equiv 1$. Then $\varphi_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ by Proposition 3.38.4. Clearly $T_0[\varphi_+, \varphi_-] = 0$. Let now $T_0[u_+, u_-] = 0$. Define $u = u_\pm$ on $\Omega_\pm$. Then $u \in \tilde{W}^{1,2}(G)$ by Theorems 1.21.1 and 1.24.2. Since

$$0 = \int\limits_{\partial G} b_- u\frac{\partial u}{\partial n} + \int\limits_{\partial\Omega_+} u\left[b_-\frac{\partial u}{\partial n} - b_-\frac{\partial u}{\partial n}\right] = \int\limits_{\Omega_-} b_-|\nabla u|^2 + \int\limits_{\Omega_+} b_+|\nabla u|^2,$$

we infer that $\nabla u = 0$ in G. According to Lemma 1.20.1, there exists a constant c such that $u_\pm = c$.

Denote by X the set of all $[f, u_c, f_c] \in H^{-1/2}(\partial G) \times H^{1/2}(\partial\Omega_+) \times H^{-1/2}(\partial\Omega_+)$ satisfying (7.164). Let now $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ be a solution of (7.162), (7.163). Define $\varphi \equiv 1$. Then $\varphi \in \tilde{W}^{1,2}(G)$ by Proposition 3.38.4. Thus,

$$\langle f_c, \varphi\rangle + b_-\langle f, \varphi\rangle = \int\limits_{\Omega_-} b_-\nabla u_- \cdot \nabla\varphi \, d\mathcal{H}_m + \int\limits_{\Omega_+} b_+\nabla u_+ \cdot \nabla\varphi \, d\mathcal{H}_m = 0.$$

Thus, $T_0(H_+ \times H_-) \subset X$. We have proved that $\dim \operatorname{Ker} T_0 = 1$. Since T_0 is a Fredholm operator with index 0, the codimension of $T_0(H_+ \times H_-)$ is equal to 1. Thus, $T_0(H_+ \times H_-) = X$.

Theorem 7.40.3 *Let* $G \subset R^m$ *be an unbounded domain with nonempty compact Lipschitz boundary,* $m > 2$. *Let* Ω_+ *be a bounded open set with Lipschitz boundary such that* $\overline{\Omega}_+ \subset G$. *Denote* $\Omega_- = G \setminus \overline{\Omega}_+$. *Denote by n the unit exterior normal with respect to* Ω_-. *Let* b_+, b_- *be positive constants. If* $f \in H^{-1/2}(\partial G)$, $u_c \in H^{1/2}(\partial\Omega_+)$, *and* $f_c \in H^{-1/2}(\partial\Omega_+)$, *then there exists a unique solution* $u_\pm \in \tilde{W}^{1,2}(\Omega_\pm)$ *of the mixed transmission problem (7.162), (7.163). Moreover,*

$$\|u_\pm\|_{\tilde{W}^{1,2}(\Omega_\pm)} \le C[\|f\|_{H^{-1/2}(\partial G)} + \|f_c\|_{H^{-1/2}(\partial\Omega_+)} + \|u_c\|_{H^{1/2}(\partial\Omega_+)}]$$

where C does not depend on f, f_c, *and* u_c.

Proof $H_\pm := H(\Omega_\pm) \cap \tilde{W}^{1,2}(\Omega_\pm)$ is a closed subspace of $\tilde{W}^{1,2}(\Omega_\pm)$ by Proposition 2.11.4. Define

$$T_h[u_+, u_-] = \left[\frac{\partial u_-}{\partial n} + hu, u_- - u_+, b_-\frac{\partial u_-}{\partial n} - b_+\frac{\partial u_+}{\partial n}\right].$$

Then T_1 is a one-to-one bounded linear operator from $H_+ \times H_-$ onto $H^{-1/2}(\partial G) \times H^{1/2}(\partial\Omega_+) \times H^{-1/2}(\partial\Omega_+)$ by Theorem 7.40.1. Theorem 1.6.3 yields that T_1 is an isomorphism. Since $T_1 - T_0$ is a compact operator by Theorem 1.24.1, T_0 is a Fredholm operator with index 0.

Let $T_0[u_+, u_-] = 0$. Define $u = u_\pm$ on $\Omega_\pm$. Then $u \in \tilde{W}^{1,2}(G)$ by Theorems 1.21.1 and 1.24.2. Since

$$0 = \int\limits_{\partial G} b_- u\frac{\partial u}{\partial n} + \int\limits_{\partial\Omega_+} u\left[b_-\frac{\partial u}{\partial n} - b_-\frac{\partial u}{\partial n}\right] = \int\limits_{\Omega_-} b_-|\nabla u|^2 + \int\limits_{\Omega_+} b_+|\nabla u|^2,$$

we infer that $\nabla u = 0$ in G. So u is constant by Lemma 1.20.1. Proposition 3.38.5 gives that $u \equiv 0$. Since T_0 is a Fredholm operator with index 0 and trivial kernel, it is onto. Theorem 1.6.3 gives that T_0 is an isomorphism.

7.41 The Dirichlet Eigenvalue Problem

In this section, we study the Dirichlet eigenvalue problem for a bounded domain. We show that eigenvalues form a sequence of positive numbers converging to infinity and the set of corresponding eigenfunctions is an orthogonal basis in $\overset{\circ}{W}{}^{1,2}(\Omega)$ and in $L^2(\Omega)$. We also study how the regularity of $\partial\Omega$ forces smoothness of eigenfunctions.

Definition 7.41.1 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. A complex number λ is an eigenvalue of the Dirichlet problem for the Laplace equation in Ω if there is a nontrivial solution $u \in \tilde{W}^{1,2}(\Omega, C)$ of the problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{7.165}$$

A solution u is a corresponding eigenfunction.

Lemma 7.41.2 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. Let λ be an eigenvalue of the Dirichlet problem for the Laplace equation in $\tilde{W}^{1,2}(\Omega)$ and u be a corresponding eigenfunction. Then $u \in \mathcal{C}^\infty(\Omega)$ and $u = \lambda G_\Omega u$ where G_Ω is the Green function of Ω.*

Proof Since $\Delta^j u = (-\lambda)^j u$ for each $j \in N$, Proposition 3.18.2 and Theorem 1.21.13 give that $u \in \mathcal{C}^\infty(\Omega)$. Choose a sequence of bounded open sets Ω_k with boundary of class $\mathcal{C}^\infty$ such that $\overline{\Omega_k} \subset \Omega_{k+1}$, $\cup\Omega_k = \Omega$. According to Corollary 3.9.6 and

Proposition 3.9.8, there exist the Green function G_Ω of Ω and Green functions G_k of Ω_k. According to Theorem 6.4.2, there exists a unique classical solution of the Dirichlet problem

$$-\Delta u_k = \lambda u \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } \partial\Omega_k. \tag{7.166}$$

We have $u_k \in \mathcal{C}^{1,0}(\overline{\Omega_k})$ by Theorem 6.11.3. Hence, u_k is a solution of (7.166) in $\tilde{W}^{1,2}(\Omega_k)$. Extend u_k by 0 on $\Omega \setminus \Omega_k$. Then $u_k \to u$ in $\tilde{W}^{1,2}(\Omega)$ by Theorem 7.2.1.

$u_k = \lambda G_k u$ by Proposition 6.13.1. Proposition 3.9.5 gives $0 \le G_k \le G_\Omega$. According to Propositions 3.9.14 and 3.9.15, there exists a constant c such that $G_\Omega(x,y) \le h_m(x-y) + c$. Since $u \in \mathcal{C}^\infty(\Omega) \cap L^2(\Omega)$, we infer that $G|u| < \infty$ in Ω. Since $G_k \uparrow G_\Omega$ by Theorem 3.9.13, Lebesgue lemma (Lemma 1.13.2) forces $G_k u \to G_\Omega u$. Since $\lambda G_k u = u_k \to u$ in $\tilde{W}^{1,2}(\Omega)$, we deduce that $u = \lambda G_\Omega u$.

Lemma 7.41.3 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$. Let λ be an eigenvalue of the Dirichlet eigenvalue problem (7.165) with an eigenfunction u. Then $\lambda > 0$ and*

$$\lambda = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m}{\int_\Omega |u|^2 \, \mathrm{d}\mathcal{H}_m}. \tag{7.167}$$

Proof According to the definition,

$$\int\limits_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \lambda \int\limits_\Omega uv \, \mathrm{d}\mathcal{H}_m$$

for all $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$. For $v = \overline{u}$, we obtain the equality (7.167). Since $\|\nabla v\|_{L^2(\Omega)}$ is an equivalent norm on $\overset{\circ}{W}{}^{1,2}(\Omega, C)$ (see Lemma 1.21.10), we get $\lambda > 0$.

Theorem 7.41.4 *Let $\Omega \subset R^m$ be a bounded open set, $m \ge 2$. Then there exist a sequence of eigenvalues $\{\lambda_j\}$ of the Dirichlet eigenvalue problem (7.165) and a sequence of corresponding eigenfunctions u_j such that $0 < \lambda_j \le \lambda_{j+1}$, $\lambda_j \to \infty$ as $j \to \infty$, $\{u_j\}$ is a complete orthonormal sequence in $L^2(\Omega)$. Moreover, $\{u_j/\sqrt{\lambda_j + 1}\}$ is a complete orthonormal sequence in $\overset{\circ}{W}{}^{1,2}(\Omega)$. Fix an eigenvalue λ. Then the space $V(\lambda) = \{u \in \overset{\circ}{W}{}^{1,2}(\Omega); -\Delta u = \lambda u\}$ has a finite dimension $k(\lambda)$. Moreover, there is $l \in N$ such that $\lambda_j = \lambda$ if and only if $l \le j \le l + k(\lambda) - 1$ and $\{u_j; l \le j \le l + k(\lambda) - 1\}$ is a basis of $V(\lambda)$.*

Proof For each $f \in L^2(\Omega)$, there is a unique $u_f \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that

$$\int\limits_\Omega \nabla u_f \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \int\limits_\Omega vf \, \mathrm{d}\mathcal{H}_m, \quad \forall v \in \overset{\circ}{W}{}^{1,2}(\Omega)$$

(see Theorem 7.1.2). Define $Lf := u_f$. Then L is a bounded linear operator from $L^2(\Omega)$ to $\overset{\circ}{W}{}^{1,2}(\Omega)$ (see Theorem 7.1.2). Since the injection of $\overset{\circ}{W}{}^{1,2}(\Omega)$ to $L^2(\Omega)$ is compact by Lemma 1.21.7, the operator L is a compact linear operator on $L^2(\Omega)$ (see Theorem 1.7.4). If $u = Lf$ and $v = Lg$, then

$$\int\limits_\Omega f(Lg)\, d\mathcal{H}_m = \int\limits_\Omega \nabla u \cdot \nabla v\, d\mathcal{H}_m = \int\limits_\Omega (Lf)g\, d\mathcal{H}_m.$$

So, the operator L is self-adjoint. According to Corollary 1.9.19, there exists a complete orthonormal sequence of eigenfunctions u_j in $L^2(\Omega)$ with eigenvalues ν_j such that $|\nu_j| \downarrow 0$ as $j \to \infty$, $\{\nu_j\}$ are all eigenvalues of L and $\{u_j; \nu_j = \nu_k\}$ is a basis of $\operatorname{Ker}(L - \nu_k I)$. Clearly $\nu_j \neq 0$. Put $\lambda_j = 1/\nu_j$. Then, λ_j is an eigenvalue of the Dirichlet problem (7.165) and u_j is a corresponding eigenfunction. According to Lemma 7.41.3, each eigenvalue λ_j is positive and thus $\lambda_j \uparrow \infty$.

If $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$, then

$$\int\limits_\Omega \nabla u_j \cdot \nabla v\, d\mathcal{H}_m + \int\limits_\Omega u_j v\, d\mathcal{H}_m = (\lambda_j + 1) \int\limits_\Omega u_j v\, d\mathcal{H}_m.$$

Thus, $\{u_j/\sqrt{\lambda_j + 1}\}$ is an orthonormal sequence in $\overset{\circ}{W}{}^{1,2}(\Omega)$. Moreover, the orthogonal complement of $\{u_j\}$ in $\overset{\circ}{W}{}^{1,2}(\Omega)$ is trivial. Corollary 1.9.7 gives that $\{u_j/\sqrt{\lambda_j + 1}\}$ is a complete sequence in $\overset{\circ}{W}{}^{1,2}(\Omega)$.

Theorem 7.41.5 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. Put*

$$\lambda_1(\Omega) = \inf_{\substack{v \in \overset{\circ}{W}{}^{1,2}(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2\, d\mathcal{H}_m}{\int_\Omega |v|^2\, d\mathcal{H}_m}. \tag{7.168}$$

Then $\lambda_1(\Omega)$ is the smallest eigenvalue of the Dirichlet eigenvalue problem (7.165). If $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$ is nontrivial and $\lambda_1(\Omega) = \|\nabla v\|^2_{L^2(\Omega)}/\|v\|^2_{L^2(\Omega)}$, then v is an eigenfunction corresponding to $\lambda_1(\Omega)$.

Proof Choose a sequence $v_k \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that

$$\|\nabla v_k\|^2_{L^2(\Omega)}/\|v_k\|^2_{L^2(\Omega)} \to \lambda_1(\Omega).$$

Put $w_k = v_k/\|v_k\|_{L^2(\Omega)}$. Then $\|w_k\|_{L^2(\Omega)} = 1$ and $\|\nabla w_k\|^2_{L^2(\Omega)} \to \lambda_1(\Omega)$. Since w_k is a bounded sequence in $\overset{\circ}{W}{}^{1,2}(\Omega)$, there is a subsequence u_k of w_k and $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that u_k converges weakly to u in $\overset{\circ}{W}{}^{1,2}(\Omega)$ (see Theorem 1.5.10). Since the injection of $\overset{\circ}{W}{}^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact (see Lemma 1.21.7), Proposition 1.7.8

gives that $u_k \to u$ in $L^2(\Omega)$. Therefore, $\|u\|_{L^2(\Omega)} = 1$. Since $\|\nabla v\|_{L^2(\Omega)}$ is an equivalent norm on $\overset{\circ}{W}{}^{1,2}(\Omega)$ (see Lemma 1.21.10) and u_k converges weakly to u in $\overset{\circ}{W}{}^{1,2}(\Omega)$, Lemma 1.5.8 gives that $\|\nabla u\|^2_{L^2(\Omega)} \leq \liminf \|\nabla u_k\|^2_{L^2(\Omega)} = \lambda_1(\Omega)$. Since $\|u\|_{L^2(\Omega)} = 1$, we have

$$\frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m}{\int_\Omega |u|^2 \, \mathrm{d}\mathcal{H}_m} \leq \lambda_1(\Omega) = \inf_{\substack{v \in \overset{\circ}{W}{}^{1,2}(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, \mathrm{d}\mathcal{H}_m}{\int_\Omega |v|^2 \, \mathrm{d}\mathcal{H}_m}.$$

Fix $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$. Put $V(t) = \|\nabla(u+tv)\|^2_{L^2(\Omega)}/\|(u+tv)\|^2_{L^2(\Omega)}$. Then its derivative is equal to

$$V'(t) = \|(u+tv)\|^{-4}_{L^2(\Omega)}\Bigg[\|(u+tv)\|^2_{L^2(\Omega)} \int\limits_\Omega (2\nabla u \cdot \nabla v + 2t|\nabla v|^2) \, \mathrm{d}\mathcal{H}_m$$

$$-\|\nabla(u+tv)\|^2_{L^2(\Omega)} \int\limits_\Omega (2uv + 2t|v|^2) \, \mathrm{d}\mathcal{H}_m\Bigg].$$

Since $V(t)$ attains its minimum in $t = 0$, we have

$$0 = V'(0) = \int\limits_\Omega 2\nabla u \cdot \nabla v \mathrm{d}\mathcal{H}_m - \lambda_1(\Omega) \int\limits_\Omega 2uv \mathrm{d}\mathcal{H}_m.$$

So, $\lambda_1(\Omega)$ is an eigenvalue of the Dirichlet problem for the Laplace equation in Ω, and u is a corresponding eigenfunction. Lemma 7.41.3 gives that $\lambda_1(\Omega)$ is the smallest eigenvalue of the Dirichlet eigenvalue problem (7.165).

Theorem 7.41.6 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$, $\lambda_1(\Omega)$ be the smallest eigenvalue of the Dirichlet eigenvalue problem, and u be a corresponding eigenfunction. Then, u does not change sign in Ω. If K is a compact subset of Ω, then* $\inf\{|u(x)|; x \in K\} > 0$. *If v is another eigenfunction corresponding to $\lambda_1(\Omega)$, then there is a real constant c such that $v = cu$.*

Proof According to Lemma 7.41.3,

$$\lambda_1(\Omega) = \|\nabla w\|^2_{L^2(\Omega)}/\|w\|^2_{L^2(\Omega)} \tag{7.169}$$

for $w = u$. Since $|\nabla|u|| = |\nabla u|$ by Theorem 1.21.5, the relation (7.169) holds true also for $w = |u|$. Since $|u| \in \overset{\circ}{D}{}^{1,2}(\Omega)$ (see Lemma 1.25.10), $|u|$ is an eigenfunction corresponding to $\lambda_1(\Omega)$ by Theorem 7.41.5. Since u is nontrivial, u^+ or u^- is nontrivial. Without loss of generality, we can assume that u^+ is nontrivial. Since u and $|u|$ are eigenfunctions corresponding to $\lambda_1(\Omega)$, $u^+ = (u + |u|)/2$ is an eigenfunction corresponding to $\lambda_1(\Omega)$, too. Since $\lambda_1(\Omega) > 0$ by Lemma 7.41.3, we have for $\varphi \in \mathcal{C}_c^\infty(\Omega)$, $\varphi \geq 0$

$$\int_\Omega \nabla u^+ \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \int_\Omega \lambda_1(\Omega) u^+ \varphi \, \mathrm{d}\mathcal{H}_m \geq 0.$$

So, u^+ is a superharmonic function by Theorem 3.19.4. If K is a compact subset of Ω, then there exists $x_0 \in K$ such that $u^+(x) \geq u^+(x_0)$ for each x in K (see Lemma 3.2.2). If $u^+(x_0) = 0$, then $u^+ \equiv 0$ (see Proposition 3.3.5), what is a contradiction. Since $u^+ > 0$ in Ω, we obtain $u = u^+$.

Let now v be another eigenfunction corresponding to $\lambda_1(\Omega)$. For a fixed $x \in \Omega$, put $c = v(x)/u(x)$. Since u and v are eigenfunctions corresponding to $\lambda_1(\Omega)$, the function $v - cu$ is an eigenfunction corresponding to $\lambda_1(\Omega)$ or $v - cu \equiv 0$. Since $v(x) - cu(x) = 0$, the function $v - cu$ is not an eigenfunction corresponding to $\lambda_1(\Omega)$. Thus, $v = cu$.

Theorem 7.41.7 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$. Let $\{\lambda_j\}$ be the sequence of all eigenvalues of the Dirichlet problem for the Laplace equation in Ω, and $\lambda \in C$.*

- *If $\lambda \in C \setminus \{\lambda_j\}$, then there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Dirichlet problem*

  $$\Delta u + \lambda u = f \quad in\, \Omega, \qquad u = g \quad on\, \partial\Omega \tag{7.170}$$

  for all $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$. Moreover,

  $$\|u\|_{\tilde{W}^{1,2}(\Omega)} \leq C\left[\|g\|_{\tilde{W}^{1,2}(\Omega)} + \|f\|_{D^{-1,2}(\Omega)}\right]$$

  where C depends only on Ω and λ.
- *Let λ be an eigenvalue of the Dirichlet problem for the Laplace equation in Ω, $f \in D^{-1,2}(\Omega)$, $g \in \tilde{W}^{1,2}(\Omega)$. Then there exists a solution $u \in \tilde{W}^{1,2}(\Omega)$ of the Dirichlet problem (7.170) if and only if*

  $$\langle f, w\rangle + \int_\Omega \nabla g \cdot \nabla w \, \mathrm{d}\mathcal{H}_m - \lambda \int_\Omega g w \, \mathrm{d}\mathcal{H}_m = 0 \tag{7.171}$$

  for all eigenfunctions w corresponding to λ.

Proof Put $v = u - g$. Then u is a solution of the problem (7.170) if and only if $v \in \overset{\circ}{D}{}^{1,2}(\Omega)$, $\Delta v + \lambda v = F = f - \Delta g - \lambda g$.

Δ is an isomorphism from $\overset{\circ}{D}{}^{1,2}(\Omega)$ onto $D^{-1,2}(\Omega)$ by Theorems 7.1.2 and 3.19.3. Since the injection of the $\overset{\circ}{D}{}^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact by Lemma 1.21.7, and $L^2(\Omega)$ is continuously imbedded into $D^{-1,2}(\Omega)$ by Hölder's inequality (Proposition 1.14.1), we infer that the identity is a compact operator from $\overset{\circ}{D}{}^{1,2}(\Omega)$ to

$D^{-1,2}(\Omega)$ (see Theorem 1.7.4). Therefore, $\Delta + \lambda I$ is a Fredholm operator with index 0 (see Theorem 1.8.3). If λ is not an eigenvalue, then $\Delta + \lambda I$ is injective and therefore surjective. Theorem 1.6.3 gives that $\Delta + \lambda I$ is an isomorphism from $\overset{\circ}{D}{}^{1,2}(\Omega)$ onto $D^{-1,2}(\Omega)$.

Let now λ be an eigenvalue. If $\varphi, \psi \in \overset{\circ}{D}{}^{1,2}(\Omega)$, then

$$\langle (\Delta + \lambda I)\varphi, \psi \rangle = \int_{\Omega} [-\nabla\varphi \cdot \nabla\psi + \lambda\varphi\psi] \, \mathrm{d}\mathcal{H}_m$$

by Theorem 3.19.3. So, the adjoint operator of the operator $\Delta + \lambda I$ is the same operator $\Delta + \lambda I$. Proposition 1.5.13 gives that $F = f - \Delta g - \lambda g$ is in the range of $\Delta + \lambda I$ if and only if $\langle F, w \rangle = 0$ for all $w \in \operatorname{Ker}(\Delta + \lambda I)$. But $\operatorname{Ker}(\Delta + \lambda I)$ is the space of all eigenfunctions corresponding to λ. If w is an eigenfunction, then $w \in \overset{\circ}{D}{}^{1,2}(\Omega)$ and

$$\langle F, w \rangle = \langle f, w \rangle + \int_{\Omega} \nabla g \cdot \nabla w \, \mathrm{d}\mathcal{H}_m - \lambda \int_{\Omega} g w \, \mathrm{d}\mathcal{H}_m$$

by Theorem 3.19.3.

Corollary 7.41.8 *Let $\Omega \subset R^m$ be a bounded domain, $m \geq 2$. Let $\{\lambda_j; j \in N\}$ be the sequence of all eigenvalues of the Dirichlet problem for the Laplace equation in Ω. Denote $\mathcal{D}(T) = \{u \in \tilde{W}^{1,2}(\Omega, C); -\Delta u \in L^2(\Omega, C)\}$, $Tu = -\Delta u$ for $u \in \mathcal{D}(T)$. Then T is a closed linear operator in $L^2(\Omega, C)$ defined on the dense subset $\mathcal{D}(T)$ and its spectrum $\sigma(T) = \{\lambda_j; j \in N\}$.*

Proof Since $\mathcal{C}_c^\infty(\Omega, C) \subset \mathcal{D}(T) \subset L^2(\Omega, C)$, Proposition 1.14.7 gives that $\mathcal{D}(T)$ is a dense subset of $L^2(\Omega, C)$.

Let $\lambda \in C \setminus \{\lambda_j; j \in N\}$. Then $-\Delta - \lambda I : \overset{\circ}{W}{}^{1,2}(\Omega, C) \to D^{-1,2}(\Omega, C)$ is an isomorphism by Theorem 7.41.7. Since $L^2(\Omega, C)$ is continuously imbedded into $D^{-1,2}(\Omega, C)$ by Hölder's inequality (Proposition 1.14.1) and $\overset{\circ}{W}{}^{1,2}(\Omega, C)$ is continuously imbedded into $L^2(\Omega, C)$, we infer that $(T - \lambda I)^{-1}$ is a continuous linear operator on $L^2(\Omega, C)$. Thus, $T - \lambda I$ and T are closed operators and $\lambda \notin \sigma(T)$.

If λ_j is an eigenvalue and u is a corresponding eigenfunction, then $u \neq 0$ and $Tu - \lambda_j u = 0$. Thus, $\lambda_j \in \sigma(T)$.

Proposition 7.41.9 *Let $\Omega \subset R^m$ be a bounded regular domain (e.g., a domain with Lipschitz boundary), $\lambda \in C$. Then, u is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ in $\tilde{W}^{1,2}(\Omega)$ if and only if u is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ in $\mathcal{C}^0(\overline{\Omega})$.*

Proof Let $u \in \tilde{W}^{1,2}(\Omega)$ be a solution of (7.165). Lemma 7.41.2 gives that $u \in \mathcal{C}^\infty(\Omega)$ and $u = \lambda G_\Omega u$, where G_Ω is the Green function of Ω. So, u is an eigenfunction of the Green potential. So, $u \in \mathcal{C}^{0,0}(\Omega)$ by Theorem 3.20.1.

Theorem 4.6.3 gives that $u \in \mathcal{C}^0(\overline{\Omega})$. Thus, u is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ in $\mathcal{C}^0(\overline{\Omega})$.

According to Theorem 7.41.4, there exists a sequence of eigenfunctions u_j from $\tilde{W}^{1,2}(\Omega)$ that is a complete orthonormal sequence in $L^2(\Omega)$. Moreover, an eigenfunction u corresponding to λ is a linear combination of a finite set u_j corresponding to λ. The same is true for eigenfunctions from $\mathcal{C}^0(\overline{\Omega})$. (See Theorem 6.22.3.) Since each eigenfunction in $\tilde{W}^{1,2}(\Omega)$ is an eigenfunction in $\mathcal{C}^0(\overline{\Omega})$, we deduce that u is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ in $\tilde{W}^{1,2}(\Omega)$ if and only if u is an eigenfunction of the Dirichlet problem for the Poisson equation corresponding to the eigenvalue λ in $\mathcal{C}^0(\overline{\Omega})$

Corollary 7.41.10 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. If u is an eigenfunction of the Dirichlet problem for the Poisson equation in $\tilde{W}^{1,2}(\Omega)$, then $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ for all $\lambda \in (0, \alpha)$.*

Proof u is an eigenfunction of the Dirichlet problem for the Poisson equation in $\mathcal{C}^0(\overline{\Omega})$ by Proposition 7.41.9. Theorem 6.22.3 gives $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ for all $\lambda \in (0, \alpha)$.

Proposition 7.41.11 *Let $\Omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^{k+2}$ with $k \in N_0$. If u is an eigenfunction of the Dirichlet problem for the Poisson equation in $\tilde{W}^{1,2}(\Omega)$, then*

$$u \in \bigcap_{0<\lambda<1} \mathcal{C}^{k+1,\lambda}(\overline{\Omega}) \cap \bigcap_{1<p<\infty} W^{k+2,p}(\Omega).$$

Proof Suppose first that $k = 0$. Corollary 7.41.10 gives $u \in \mathcal{C}^{1,\lambda}(\overline{\Omega})$ for all $\lambda \in (0, 1)$. Fix $1 < p < \infty$. Since $u \in \tilde{W}^{1,2}(\Omega)$ is a solution of (7.165), Theorems 7.14.3 and 7.1.2 give that $u \in W^{2,p}(\Omega)$.

Let now $k \in N$ and the proposition hold true for $k - 1$. Fix $1 < p < \infty$. Since $u \in W^{k+1,p}(\Omega)$ is a solution of (7.165), Theorem 7.14.3 gives $u \in W^{k+2,p}(\Omega)$. Since p was arbitrary, Theorem 1.21.13 forces $u \in \mathcal{C}^{k+1,\lambda}(\overline{\Omega})$ for each $\lambda \in (0, 1)$.

7.42 Neumann Eigenvalue Problem in $\tilde{W}^{1,2}(\Omega)$

In this section, we study the Neumann eigenvalue problem in $W^{1,2}(\Omega)$ for a bounded domain with Lipschitz boundary. We show that eigenvalues form a sequence of nonnegative numbers converging to infinity and the set of corresponding eigenfunctions is an orthogonal basis in $W^{1,2}(\Omega)$ and in $L^2(\Omega)$.

Definition 7.42.1 Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. We say that $\lambda \in C$ is an eigenvalue of the Neumann problem for the Laplace equation in $\tilde{W}^{1,2}(\Omega, C)$ if there exists a nontrivial $u \in \tilde{W}^{1,2}(\Omega, C)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\mathcal{H}_m = \int_\Omega \lambda u v \, d\mathcal{H}_m \qquad \forall v \in \tilde{W}^{1,2}(\Omega, C),$$

i.e., u is a solution of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

The function u is called an eigenfunction corresponding to λ.

Lemma 7.42.2 *Let $\Omega \subset R^m$ be a bounded open set, $m \geq 2$. Let λ be an eigenvalue of the Neumann eigenvalue problem for the Laplace equation with an eigenfunction u. Then $\lambda \geq 0$ and*

$$\lambda = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m}{\int_\Omega |u|^2 \, \mathrm{d}\mathcal{H}_m}. \tag{7.172}$$

If μ is an another eigenvalue with an eigenfunction w, then

$$\int\limits_\Omega uw \, \mathrm{d}\mathcal{H}_m = \int\limits_\Omega \nabla u \cdot \nabla w \, \mathrm{d}\mathcal{H}_m = 0. \tag{7.173}$$

Proof According to the definition,

$$\int\limits_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \lambda \int\limits_\Omega uv \, \mathrm{d}\mathcal{H}_m$$

for all $v \in \tilde{W}^{1,2}(\Omega)$. For $v = \overline{u}$, we obtain the equality (7.172). Since

$$\mu \int\limits_\Omega uw \, \mathrm{d}\mathcal{H}_m = \int\limits_\Omega \nabla u \cdot \nabla w \, \mathrm{d}\mathcal{H}_m = \lambda \int\limits_\Omega uw \, \mathrm{d}\mathcal{H}_m$$

we obtain the equalities (7.173).

Theorem 7.42.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$. Then there exist a sequence of eigenvalues $\{\lambda_j\}$ of the Neumann eigenvalue problem in $\tilde{W}^{1,2}(\Omega, C) = W^{1,2}(\Omega, C)$ and a sequence of corresponding eigenfunctions u_j such that $0 \leq \lambda_j \leq \lambda_{j+1}$, $\lambda_j \to \infty$ as $j \to \infty$, u_j is a complete orthonormal sequence in $L^2(\Omega)$. Moreover, $u_j/\sqrt{\lambda_j + 1}$ is a complete orthonormal sequence in $W^{1,2}(\Omega)$. Fix an eigenvalue λ. Then the space $V(\lambda) = \{0\} \cup \{u$ an eigenfunction corresponding to $\lambda\}$ has a finite dimension $k(\lambda)$. Moreover, there is $l \in N$ such that $\lambda_j = \lambda$ if and only if $l \leq j \leq l + k(\lambda) - 1$ and $\{u_j; l \leq j \leq l + k(\lambda) - 1\}$ is a basis of $V(\lambda)$. We have $\lambda_1 = 0 < \lambda_2$, and u_1 is constant.*

Proof Every eigenvalue is nonnegative by Lemma 7.42.2. Clearly, $u_1 \equiv 1/\mathcal{H}_m(\Omega)$ is an eigenfunction corresponding to 0. If u is an eigenfunction corresponding to $\lambda_1 = 0$, then $\nabla u = 0$ by Lemma 7.42.2 and thus u is constant.

Define the operator T from $\tilde{W}^{1,2}(\Omega, C)$ to $[\tilde{W}^{1,2}(\Omega, C)]'$ by

$$\langle Tu, v\rangle = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m.$$

Then T is a bounded linear operator. Clearly, $\langle Tu, v\rangle = \langle u, Tv\rangle$. In particular, if $Tu, Tv \in L^2(\Omega)$, then

$$\int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m.$$

Denote by $\tilde{T}$ the restriction of T onto $W_0^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega); \int u \, \mathrm{d}\mathcal{H}_m = 0\}$. Then $\tilde{T}$ is an isomorphism $W_0^{1,2}(\Omega)$ onto $\{f \in [\tilde{W}^{1,2}(\Omega)]'; \langle f, 1\rangle = 0\}$. (See Theorem 7.20.2.) Denote by S the restriction of $\tilde{T}^{-1}$ onto $L_0^2(\Omega) := \{f \in L^2(\Omega); \langle f, 1\rangle = 1\}$. Since $L^2(\Omega) \hookrightarrow [W^{1,2}(\Omega)]'$ by Hölder's inequality (Proposition 1.14.1) and $W^{1,2}(\Omega)$ is compactly imbedded into $L^2(\Omega)$ (see Lemma 1.21.8), the operator S is a compact linear operator on $L_0^2(\Omega)$ (see Theorem 1.7.4). Moreover,

$$\int\limits_{\Omega} fSg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (TSf)Sg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)(TSg) \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)g \, \mathrm{d}\mathcal{H}_m$$

and the operator S is self-adjoint on $L_0^2(\Omega)$. According to Corollary 1.9.19, there exists a complete orthonormal sequence of eigenfunctions $u_2, u_3, \dots$ of S in $L_0^2(\Omega)$ and a sequence of corresponding eigenvalues μ_j such that $|\mu_j| \downarrow 0$ as $j \to \infty$, $\{\mu_j\}$ is the set of all eigenvalues of S and $\{u_j; \mu_k = \mu_j\}$ is a basis of $\mathrm{Ker}(S - \mu_k I)$. Since S is one-to-one, we have $\mu_j \neq 0$ for all $j \geq 2$. If $\lambda \in R \setminus \{0\}$, then λ is an eigenvalue of T with an eigenfunction u if and only if $\mu = 1/\lambda$ is an eigenvalue of S with an eigenfunction u. Put $\lambda_j = 1/\mu_j$ for $j \geq 2$. If $\lambda \neq 0$ is an eigenvalue of the Neumann problem with an eigenfunction u, then $u \in L_0^2(\Omega)$ by Lemma 7.42.2. Thus, $\{\lambda_j\}$ is a sequence of all eigenvalues of T with eigenfunctions u_j. Moreover, $\{u_j\}$ is a complete orthonormal sequence in $L^2(\Omega)$ and $0 = \lambda_1 < \lambda_2 \leq \dots \lambda_j \leq \lambda_{j+1} \to \infty$.

If $v \in W^{1,2}(\Omega)$, then

$$\int\limits_{\Omega} \nabla u_j \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\Omega} u_j v \, \mathrm{d}\mathcal{H}_m = (\lambda_j + 1) \int\limits_{\Omega} u_j v \, \mathrm{d}\mathcal{H}_m.$$

Thus, $\{u_j/\sqrt{\lambda_j + 1}\}$ is an orthonormal sequence in $W^{1,2}(\Omega)$. Moreover, the orthogonal complement of $\{u_j\}$ in $W^{1,2}(\Omega)$ is trivial. Corollary 1.9.7 gives that $\{u_j/\sqrt{\lambda_j + 1}\}$ is complete sequence in $W^{1,2}(\Omega)$.

Theorem 7.42.4 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $m \geq 2$.

- *If* $\lambda \in C$ *is not an eigenvalue of the Neumann problem for the Laplace equation in* $\tilde{W}^{1,2}(\Omega, C)$*, then for each* $F \in [\tilde{W}^{1,2}(\Omega, C)]'$*, there exists a unique solution* $u \in \tilde{W}^{1,2}(\Omega, C)$ *of Neumann the problem*

$$\int_\Omega [\nabla u \cdot \nabla v - \lambda uv] \, \mathrm{d}\mathcal{H}_m = \langle F, v\rangle \qquad \forall v \in \tilde{W}^{1,2}(\Omega, C). \tag{7.174}$$

  Moreover, there exists a constant C *depending only on* Ω *and* λ *such that*

$$\|u\|_{W^{1,2}(\Omega)} \leq C\| F\|_{[W^{1,2}(\Omega)]'}.$$

- *Let* λ *be an eigenvalue of the Neumann problem for the Laplace equation in* $\tilde{W}^{1,2}(\Omega, C)$*,* $F \in [\tilde{W}^{1,2}(\Omega, C)]'$*. Then there exists a solution* $u \in \tilde{W}^{1,2}(\Omega, C)$ *of the Neumann problem (7.174) if and only if* $\langle F, v\rangle = 0$ *for each eigenfunction* v *corresponding to* λ.

Proof Define the bounded linear operator T

$$\langle Tu, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m.$$

from $\tilde{W}^{1,2}(\Omega, C)$ to $[\tilde{W}^{1,2}(\Omega, C)]'$. Theorem 7.21.2 gives that $T + I$ is an isomorphism. The identity operator is a compact operator from $\tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega)$ to $L^2(\Omega)$ (see Lemma 1.21.8). Since the identity operator is a bounded linear operator from $L^2(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ by the Hölder inequality (Proposition 1.14.1), we infer that the identity operator is a compact linear operator from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ (see Theorem 1.7.4). Hence, $T - \lambda I$ is a Fredholm operator from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ with index 0 by Theorem 1.8.3. If λ is not an eigenvalue, then the kernel of $T - \lambda I$ is trivial. Since the index of $T - \lambda I$ is 0, the operator $T - \lambda I$ is onto. Theorem 1.6.3 gives that $T - \lambda I$ is continuously invertible.

Let now λ be an eigenvalue. Since $\lambda \geq 0$ by Lemma 7.42.2, we can restrict ourselves onto the space of real functions $\tilde{W}^{1,2}(\Omega)$. Since $\langle (T - \lambda I)u, v\rangle = \langle u, (T - \lambda I)v\rangle$, the adjoint operator of $T - \lambda I$ is the operator $T - \lambda I$. Proposition 1.5.13 gives that the Neumann problem (7.174) is solvable if and only if $\langle F, v\rangle = 0$ for each eigenfunction v corresponding to λ.

Corollary 7.42.5 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $m \geq 2$*. Let* $\{\lambda_j; j \in N\}$ *be the sequence of all eigenvalues of the Neumann problem for the Laplace equation in* $\tilde{W}^{1,2}(\Omega)$*. Define the bounded linear operator* T *from* $\tilde{W}^{1,2}(\Omega, C)$ *to* $[\tilde{W}^{1,2}(\Omega, C)]'$ *by*

$$\langle Tu, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m.$$

Denote $\mathcal{D}(\tilde{T}) = \{u \in W^{1,2}(\Omega, C); Tu \in L^2(\Omega, C)\}$, *i.e.,* $\mathcal{D}(\tilde{T}) = \{u \in W^{1,2}(\Omega, C); \Delta u \in L^2(\Omega, C), \partial u/\partial n = 0\}$, *and* $\tilde{T}u = Tu$ *for* $u \in \mathcal{D}(\tilde{T})$, *i.e.,* $\tilde{T}u = -\Delta u$. *Then,* $\tilde{T}$ *is a closed linear operator in* $L^2(\Omega, C)$ *defined on the dense subset* $\mathcal{D}(\tilde{T})$ *and its spectrum* $\sigma(\tilde{T}) = \{\lambda_j; j \in N\}$.

Proof Since $\mathcal{C}_c^\infty(\Omega, C) \subset \mathcal{D}(\tilde{T}) \subset L^2(\Omega, C)$, Proposition 1.14.7 gives that $\mathcal{D}(\tilde{T})$ is a dense subset of $L^2(\Omega, C)$.

Let $\lambda \in C \setminus \{\lambda_j; j \in N\}$. Then $T - \lambda I$ is an isomorphism from $W^{1,2}(\Omega, C)$ onto $[W^{1,2}(\Omega, C)]'$ by Theorem 7.42.4. Since $L^2(\Omega, C)$ is continuously imbedded into $[W^{1,2}(\Omega, C)]'$ by Hölder's inequality (Proposition 1.14.1) and $W^{1,2}(\Omega, C)$ is continuously imbedded into $L^2(\Omega, C)$, we infer that $(\tilde{T} - \lambda I)^{-1}$ is a continuous linear operator on $L^2(\Omega, C)$. Thus, $\tilde{T} - \lambda I$ and $\tilde{T}$ are closed operators and $\lambda \notin \sigma(\tilde{T})$.

If λ_j is an eigenvalue and u is a corresponding eigenfunction, then $u \neq 0$ and $\tilde{T}u - \lambda_j u = 0$. Thus, $\lambda_j \in \sigma(\tilde{T})$.

7.43 The Robin Eigenvalue Problem

Suppose that $\Omega \subset R^m$ is a bounded domain with Lipschitz boundary. In this section, we study the Robin eigenvalue problem for the Poisson equation

$$-\Delta u = f \qquad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + bu = g \qquad \text{on } \partial\Omega, \tag{7.175}$$

with $b \in L^\infty(\partial\Omega)$, $b \geq 0$,

$$\int_{\partial\Omega} b \, \mathrm{d}\mathcal{H}_{m-1} > 0. \tag{7.176}$$

We say that $\lambda \in C$ is an eigenvalue of the Robin problem (7.175) in $\tilde{W}^{1,2}(\Omega, C) = W^{1,2}(\Omega, C)$ if there exists a nontrivial $u \in W^{1,2}(\Omega, C)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial\Omega} buv \, \mathcal{H}_{m-1} = \lambda \int_\Omega uv \, d\mathcal{H}_m \quad \forall v \in W^{1,2}(\Omega, C), \tag{7.177}$$

i.e., u is a solution of the Robin problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + bu = 0 \quad \text{on } \partial\Omega. \tag{7.178}$$

This u is called an eigenfunction corresponding to λ.

Theorem 7.43.1 *Let* $\Omega \subset R^m$ *be a bounded domain with Lipschitz boundary,* $m \geq 2$, $b \in L^\infty(\Omega)$, $b \geq 0$ *and (7.176) hold. Then, there exist a sequence of eigenvalues*

$\{\lambda_j\}$ of the Robin eigenvalue problem in $\tilde{W}^{1,2}(\Omega, C) = W^{1,2}(\Omega, C)$ and a sequence of corresponding eigenfunctions $u_j \in W^{1,2}(\Omega)$ such that $0 < \lambda_j \le \lambda_{j+1}$, $\lambda_j \to \infty$ as $j \to \infty$, u_j is a complete orthonormal sequence in $L^2(\Omega)$. Denote on $W^{1,2}(\Omega, C)$ the inner product

$$(u, v) = \int\limits_{\Omega} \nabla u \cdot \nabla \overline{v} \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} bu\overline{v} \, \mathrm{d}\mathcal{H}_{m-1}$$

where $\overline{v}$ is the complex conjugate of v. Then $\|u\| = \sqrt{(u, u)}$ is an equivalent norm on $W^{1,2}(\Omega, C)$. The sequence $\{u_j/\sqrt{\lambda_j}\}$ is a complete orthonormal system in $W^{1,2}(\Omega, C)$ and in $W^{1,2}(\Omega)$ equipped with the inner product $(\, , \,)$. Fix an eigenvalue λ. Then the space $V(\lambda) = \{0\} \cup \{u$ an eigenfunction corresponding to $\lambda\}$ has a finite dimension $k(\lambda)$. Moreover, there is $l \in N$ such that $\lambda_j = \lambda$ if and only if $l \le j \le l + k(\lambda) - 1$ and $\{u_j; l \le j \le l + k(\lambda) - 1\}$ is a basis of $V(\lambda)$.

Proof Proposition 1.21.9 gives that $\| \, \|$ is an equivalent norm on $W^{1,2}(\Omega, C)$. If λ is an eigenvalue and u is a corresponding eigenfunction, then we get from (7.177) for $v = \overline{u}$ that $\lambda = \|u\|^2/\|u\|^2_{L^2(\Omega)} > 0$.

Define the operator T from $W^{1,2}(\Omega)$ to $[W^{1,2}(\Omega)]'$:

$$\langle Tu, v\rangle = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} buv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T is an isomorphism by Theorem 7.21.3. Clearly, $\langle Tu, v\rangle = \langle u, Tv\rangle$. In particular, if $Tu, Tv \in L^2(\Omega)$, then

$$\int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m.$$

Denote by S the restriction of T^{-1} onto $L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded into $[W^{1,2}(\Omega)]'$ by Hölder's inequality (Proposition 1.14.1) and $W^{1,2}(\Omega)$ is compactly imbedded into $L^2(\Omega)$ (see Lemma 1.21.8), the operator S is a compact linear operator on $L^2(\Omega)$ (see Theorem 1.7.4). Moreover,

$$\int\limits_{\Omega} fSg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (TSf)Sg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)(TSg) \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)g \, \mathrm{d}\mathcal{H}_m$$

and the operator S is self-adjoint on $L^2(\Omega)$. According to Corollary 1.9.19, there exist a complete orthonormal sequence $\{u_j\}$ of eigenfunctions of S and a sequence of corresponding eigenfunctions μ_j such that $|\mu_j| \ge |\mu_{j+1}| \to 0$, $\{\mu_j\}$ is the set of all eigenfunctions and $\{u_j; \mu_j = \mu_k\}$ is a basis of $\mathrm{Ker}(S - \mu_k I)$. Since S is one-to-one, we have $\mu_j \neq 0$ for all j. If $\lambda \in R \setminus \{0\}$, then λ is an eigenvalue of T with an eigenfunction u if and only if $\mu = 1/\lambda$ is an eigenvalue of S with an eigenfunction u. Now we put $\lambda_j = 1/\mu_j$.

If $v \in W^{1,2}(\Omega)$, then

$$(u_j, v) = \lambda_j \int\limits_{\Omega} u_j \overline{v} \, \mathrm{d}\mathcal{H}_m.$$

Thus, $\{u_j/\sqrt{\lambda_j}\}$ is an orthonormal sequence in $W^{1,2}(\Omega)$. Moreover, the orthogonal complement of $\{u_j\}$ in $W^{1,2}(\Omega)$ is trivial. Corollary 1.9.7 gives that $\{u_j/\sqrt{\lambda_j}\}$ is complete sequence in $W^{1,2}(\Omega)$.

Theorem 7.43.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$, $b \in L^\infty(\Omega)$, $b \geq 0$ and (7.176) hold.*

- *If $\lambda \in C$ is not an eigenvalue of the Robin problem for the Laplace equation (7.178) in $\tilde{W}^{1,2}(\Omega, C)$, then for each $F \in [\tilde{W}^{1,2}(\Omega, C)]'$, there exists a unique solution $u \in \tilde{W}^{1,2}(\Omega, C)$ of the Robin problem*

  $$\int\limits_{\Omega} [\nabla u \cdot \nabla v - \lambda u v] \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} b u v \, \mathrm{d}\mathcal{H}_{m-1} = \langle F, v \rangle \tag{7.179}$$

  for all $v \in \tilde{W}^{1,2}(\Omega, C)$. Moreover, there exists a constant C depending only on Ω, b, and λ such that

  $$\|u\|_{W^{1,2}(\Omega)} \leq C \|F\|_{[W^{1,2}(\Omega)]'}.$$

- *Let λ be an eigenvalue of the Robin problem for the Laplace equation (7.178) in $\tilde{W}^{1,2}(\Omega, C)$, $F \in [\tilde{W}^{1,2}(\Omega, C)]'$. Then there exists a solution $u \in \tilde{W}^{1,2}(\Omega, C)$ of the Robin problem (7.179) if and only if $\langle F, v \rangle = 0$ for each eigenfunction v corresponding to λ.*

Proof Define the operator T from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$:

$$\langle Tu, v \rangle = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} b u v \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T is an isomorphism by Theorem 7.21.3. The identity operator is a compact operator from $\tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega)$ to $L^2(\Omega)$ (see Lemma 1.21.8). Since the identity operator is a bounded linear operator from $L^2(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ by the Hölder inequality (Proposition 1.14.1), we infer that the identity operator is a compact linear operator from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ (see Theorem 1.7.4). Hence, $T - \lambda I$ is a Fredholm operator from $\tilde{W}^{1,2}(\Omega)$ to $[\tilde{W}^{1,2}(\Omega)]'$ with index 0 by Theorem 1.8.3. If λ is not an eigenvalue, then the kernel of $T - \lambda I$ is trivial. Since the index of $T - \lambda I$ is 0, the operator $T - \lambda I$ is onto. Theorem 1.6.3 gives that $T - \lambda I$ is an isomorphism.

Let now λ be an eigenvalue. Since $\lambda > 0$ by Theorem 7.43.1, we can restrict ourselves onto the space of real functions $\tilde{W}^{1,2}(\Omega)$. Since $\langle (T-\lambda I)u, v\rangle = \langle u, (T-\lambda I)v\rangle$, the adjoint operator of $T - \lambda I$ is the operator $T - \lambda I$. Proposition 1.5.13 gives that the Robin problem (7.178) is solvable if and only if $\langle F, v\rangle = 0$ for all $v \in \operatorname{Ker}(T - \lambda I) = \{0\} \cup \{v$ eigenfunction corresponding to $\lambda\}$.

Corollary 7.43.3 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$, $b \in L^\infty(\Omega)$, $b \geq 0$ and (7.176) hold. Let $\{\lambda_j; j \in N\}$ be the sequence of all eigenvalues of the Robin problem for the Laplace equation (7.178) in $\tilde{W}^{1,2}(\Omega)$. Define the bounded linear operator T from $\tilde{W}^{1,2}(\Omega, C)$ to $[\tilde{W}^{1,2}(\Omega, C)]'$*

$$\langle Tu, v\rangle = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} buv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Denote $\mathcal{D}(\tilde{T}) = \{u \in W^{1,2}(\Omega, C); Tu \in L^2(\Omega, C)\}$, i.e., $\mathcal{D}(\tilde{T}) = \{u \in W^{1,2}(\Omega, C); \Delta u \in L^2(\Omega, C), \partial u/\partial n + bu = 0$ on $\partial\Omega\}$, and $\tilde{T}u = Tu$ for $u \in \mathcal{D}(\tilde{T})$, i.e., $\tilde{T}u = -\Delta u$. Then $\tilde{T}$ is a closed linear operator in $L^2(\Omega, C)$ defined on the dense subset $\mathcal{D}(\tilde{T})$ and its spectrum $\sigma(\tilde{T}) = \{\lambda_j; j \in N\}$.

Proof Since $\mathcal{C}_c^\infty(\Omega, C) \subset \mathcal{D}(\tilde{T}) \subset L^2(\Omega, C)$, Proposition 1.14.7 gives that $\mathcal{D}(\tilde{T})$ is a dense subset of $L^2(\Omega, C)$.

Let $\lambda \in C \setminus \{\lambda_j; j \in N\}$. Then $T - \lambda I$ is an isomorphism from $W^{1,2}(\Omega, C)$ onto $[W^{1,2}(\Omega, C)]'$ by Theorem 7.43.2. Since $L^2(\Omega, C)$ is continuously imbedded into $[W^{1,2}(\Omega, C)]'$ by Hölder's inequality (Proposition 1.14.1) and $W^{1,2}(\Omega, C)$ is continuously imbedded into $L^2(\Omega, C)$, we infer that $(\tilde{T} - \lambda I)^{-1}$ is a continuous linear operator on $L^2(\Omega, C)$. Thus, $\tilde{T} - \lambda I$ and $\tilde{T}$ are a closed operator and $\lambda \notin \sigma(\tilde{T})$.

If λ_j is an eigenvalue and u is a corresponding eigenfunction, then $u \neq 0$ and $\tilde{T}u - \lambda_j u = 0$. Thus, $\lambda_j \in \sigma(\tilde{T})$.

Proposition 7.43.4 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \geq 2$, $b \in L^\infty(\partial\Omega)$, $b \geq 0$, $\lambda \in R^1$. Let $u \in W^{1,2}(\Omega)$ be a solution of (7.178). Then, $u \in W^{s,2}(\Omega)$ for all $s \in (1, 3/2)$. If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, $\beta \in (0, \alpha)$, and $b \in \mathcal{C}^{0,\beta}(\partial\Omega)$, then $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$. If $\partial\Omega$ is of class $\mathcal{C}^2$ and $b \in \mathcal{C}^1(\partial\Omega)$, then $u \in W^{2,p}(\Omega)$ for every $p \in (1, \infty)$. If $\Omega \subset R^2$ has boundary of class $\mathcal{C}^{k+2}$ with $k \in N_0$ and $b \in \mathcal{C}^{k+1}(\partial\Omega)$, then $u \in W^{k+2,p}(\Omega)$ for every $p \in (1, \infty)$.*

Proof If $v \in W^{1,2}(\Omega)$, $-\Delta v = \lambda u$ in Ω, $\partial v/\partial n + bv = 0$ on $\partial\Omega$, then $u - v$ is constant by Theorems 7.20.2 and 7.21.3. If $1 \leq s < 3/2$, then $u \in W^{s,2}(\Omega)$ by Theorems 7.24.5 and 7.24.7.

Let $\partial\Omega$ be of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$. Then $u \in W^{1,p}(\Omega)$ for all $p \in (1, \infty)$ by Theorems 7.24.5 and 7.24.7. If $\beta \in (0, \alpha)$, then $u \in \mathcal{C}^{0,\beta}(\overline{\Omega})$ by Theorem 1.21.13. If moreover $b \in \mathcal{C}^{0,\beta}(\partial\Omega)$, then $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ by Theorems 6.10.8 and 6.10.10.

If $\partial\Omega$ is of class $\mathcal{C}^2$ and $b \in \mathcal{C}^1(\partial\Omega)$, then $u \in W^{2,p}(\Omega)$ for every $p \in (1, \infty)$ by Theorem 7.26.3.

Suppose that $\Omega \subset R^2$ has boundary of class $\mathcal{C}^{k+2}$ with $k \in N_0$ and $b \in \mathcal{C}^{k+1}(\partial\Omega)$. Fix $p \in (1, \infty)$. If $u \in W^{k,p}(\Omega)$, then $u \in W^{k+2,p}(\Omega)$ by Theorems 7.30.1 and

7.30.2. We have proved that $u \in W^{2,p}(\Omega)$. So, the induction argument gives that $u \in W^{k+2,p}(\Omega)$.

7.44 The Mixed Eigenvalue Problem

In this section, we study the mixed eigenvalue problem in $W^{1,2}(\Omega)$ for a bounded domain with Lipschitz boundary. We show that eigenvalues form a sequence of positive numbers converging to infinity and the set of corresponding eigenfunctions is an orthogonal basis in $L^2(\Omega)$.

In this section, we suppose that $\Omega \subset R^m$ is a bounded domain with Lipschitz boundary and Γ_R is a nonempty closed subset of $\partial\Omega$, $\Gamma_D = \partial\Omega \setminus \Gamma_R \neq \emptyset$, $b \in L^\infty(\partial\Omega)$, $b \geq 0$. If $\lambda \in C$, $u \in W^{1,2}(\Omega, C)$ is a nontrivial solution of the mixed problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{7.180}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{7.181}$$

$$\frac{\partial u}{\partial n} + bu = 0 \qquad \text{on } \Gamma_R, \tag{7.182}$$

then λ is called an eigenvalue of the mixed problem (7.180), (7.181), (7.182), and u is a corresponding eigenfunction.

Theorem 7.44.1 *There exist a sequence of eigenvalues $\{\lambda_j\}$ of the mixed problem (7.180), (7.181), (7.182) in $W^{1,2}(\Omega, C)$ and a sequence of corresponding eigenfunctions $u_j \in W^{1,2}(\Omega)$ such that $0 < \lambda_j \leq \lambda_{j+1}$, $\lambda_j \to \infty$ as $j \to \infty$, u_j is a complete orthonormal sequence in $L^2(\Omega)$. Denote on $W^{1,2}(\Omega, C)$ the inner product*

$$(u, v) = \int_\Omega [\nabla u \cdot \nabla \overline{v} + u\overline{v}]\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} bu\overline{v}\, \mathrm{d}\mathcal{H}_{m-1}$$

where $\overline{v}$ is the complex conjugate of v. Then $\|u\| = \sqrt{(u, u)}$ is an equivalent norm on $W^{1,2}(\Omega, C)$. The sequence $\{u_j/\sqrt{\lambda_j + 1}\}$ is a complete orthonormal system in $V = \{v \in W^{1,2}(\Omega); v = 0 \text{ on } \Gamma_D\}$ equipped with the inner product $(\ , \ )$. Fix an eigenvalue λ. Then the space $E(\lambda) = \{0\} \cup \{u$ an eigenfunction corresponding to $\lambda\}$ has a finite dimension $k(\lambda)$. Moreover, there is $l \in N$ such that $\lambda_j = \lambda$ if and only if $l \leq j \leq l + k(\lambda) - 1$ and $\{u_j; l \leq j \leq l + k(\lambda) - 1\}$ is a basis of $E(\lambda)$.

Proof If λ is an eigenvalue and u is a corresponding eigenfunction, then we get from the definition that

$$\int_\Omega \nabla u \cdot \nabla \overline{u}\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} bu\overline{u}\, \mathrm{d}\mathcal{H}_{m-1} = \lambda \int_\Omega u\overline{u}\, \mathrm{d}\mathcal{H}_m.$$

Thus, $\lambda \geq 0$. If $\lambda = 0$, then $\nabla u \equiv 0$ and u is constant. Since $u = 0$ on Γ_D, we infer that $u \equiv 0$, what is a contradiction.

Define the operator T from V to V' by

$$\langle Tu, v \rangle = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} buv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T is a bounded linear operator. If $u \in \operatorname{Ker} T$, then u is a solution of (7.180), (7.181), (7.182) with $\lambda = 0$. We have proved that $u \equiv 0$. Every $F \in V'$ can be extended as an element of $[W^{1,2}(\Omega)]'$ with the same norm (see Theorem 1.5.2). Theorem 7.35.1 gives that T is an isomorphism. Clearly, $\langle Tu, v \rangle = \langle u, Tv \rangle$. In particular, if $Tu, Tv \in L^2(\Omega)$, then

$$\int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} uTv \, \mathrm{d}\mathcal{H}_m.$$

Denote by S the restriction of T^{-1} onto $L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded into V' by Hölder's inequality (Proposition 1.14.1) and V is compactly imbedded into $L^2(\Omega)$ (see Lemma 1.21.8), the operator S is a compact linear operator on $L^2(\Omega)$ (see Theorem 1.7.4). Moreover,

$$\int\limits_{\Omega} fSg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (TSf)Sg \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)(TSg) \, \mathrm{d}\mathcal{H}_m = \int\limits_{\Omega} (Sf)g \, \mathrm{d}\mathcal{H}_m$$

and the operator S is self-adjoint on $L^2(\Omega)$. According to Corollary 1.9.19, there exist a complete orthonormal sequence $\{u_j\}$ of eigenfunctions of S and a sequence of corresponding eigenfunctions μ_j such that $|\mu_j| \geq |\mu_{j+1}| \to 0$, $\{\mu_j\}$ is the set of all eigenfunctions and $\{u_j; \mu_j = \mu_k\}$ is a basis of $\operatorname{Ker}(S - \mu_k I)$. Since S is one-to-one, we have $\mu_j \neq 0$ for all j. If $\lambda \in R \setminus \{0\}$, then λ is an eigenvalue of T with an eigenfunction u if and only if $\mu = 1/\lambda$ is an eigenvalue of S with an eigenfunction u. Now, we put $\lambda_j = 1/\mu_j$.

If $v \in V$, then

$$(u_j, v) = (\lambda_j + 1) \int\limits_{\Omega} u_j v \, \mathrm{d}\mathcal{H}_m.$$

Thus, $\{u_j/\sqrt{\lambda_j + 1}\}$ is an orthonormal sequence in V. Moreover, the orthogonal complement of $\{u_j\}$ in V is trivial. Corollary 1.9.7 gives that $\{u_j/\sqrt{\lambda_j + 1}\}$ is a complete sequence in V.

Theorem 7.44.2 *Denote* $V = \{v \in W^{1,2}(\Omega, C); v = 0$ *on* $\Gamma_D\}$. *Let* $u_0 \in W^{1,2}(\Omega, C)$, $F \in [W^{1,2}(\Omega, C)]'$, $\lambda \in C$. *Define* $G \in [W^{1,2}(\Omega, C)]'$ *by*

$$\langle G, v\rangle = \langle F, v\rangle - \int_\Omega [\nabla u_0 \cdot \nabla v - \lambda u_0 v]\, \mathrm{d}\mathcal{H}_m - \int_{\Gamma_R} b u_0 v\, \mathrm{d}\mathcal{H}_{m-1}.$$

- *If* λ *is not an eigenvalue of the mixed problem (7.180), (7.181), (7.182) in* $W^{1,2}(\Omega, C)$, *then there exists a unique solution* $u \in W^{1,2}(\Omega, C)$ *of the mixed problem for the Poisson equation*

$$u - u_0 \in V, \tag{7.183}$$

$$\int_\Omega [\nabla u \cdot \nabla v - \lambda u v]\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} b u v\, \mathrm{d}\mathcal{H}_{m-1} = \langle F, v\rangle, \quad v \in V. \tag{7.184}$$

  Moreover, there exists a constant C *depending only on* Ω, Γ_D, b, *and* λ *such that*

$$\|u\|_{W^{1,2}(\Omega)} \le C\left\{\|u_0\|_{W^{1,2}(\Omega)} + \|F\|_{[W^{1,2}(\Omega)]'}\right\}.$$

- *If* λ *is an eigenvalue of the mixed problem (7.180), (7.181), (7.182) in* $W^{1,2}(\Omega, C)$, *then there exists a solution* $u \in W^{1,2}(\Omega, C)$ *of the mixed problem for the Poisson equation (7.183), (7.184) if and only if* $\langle G, v\rangle = 0$ *for each eigenfunction* v *corresponding to* λ.

Proof If $u = u_0 + w$, then u is a solution of the problem (7.183), (7.184) if and only if $w \in V$ and

$$\int_\Omega [\nabla w \cdot \nabla v - \lambda w v]\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} b w v\, \mathrm{d}\mathcal{H}_{m-1} = \langle G, v\rangle, \quad v \in V.$$

So, we can suppose that $u_0 \equiv 0$.

Define the operator T from V to V':

$$\langle Tu, v\rangle = \int_\Omega \nabla u \cdot \nabla v\, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} b u v\, \mathrm{d}\mathcal{H}_{m-1}.$$

Then T is an isomorphism by Theorem 7.35.1. The identity operator is a compact operator from V to $L^2(\Omega)$ (see Lemma 1.21.8). Since the identity operator is a bounded linear operator from $L^2(\Omega)$ to V' by the Hölder inequality (Proposition 1.14.1), we infer that the identity operator is a compact linear operator from V to V' (see Theorem 1.7.4). Hence, $T - \lambda I$ is a Fredholm operator from V to V' with index 0 by Theorem 1.8.3. If λ is not an eigenvalue, then the kernel of $T - \lambda I$ is trivial. Since the index of $T - \lambda I$ is 0, the operator $T - \lambda I$ is onto. Theorem 1.6.3 gives that $T - \lambda I$ is an isomorphism.

Let now λ be an eigenvalue. Since $\lambda > 0$ by Theorem 7.44.1, we can restrict ourselves onto the space of real functions from V. Since $\langle (T - \lambda I)u, v\rangle = \langle u, (T - \lambda I)v\rangle$, the adjoint operator of $T - \lambda I$ is the operator $T - \lambda I$. Proposition 1.5.13 gives that the mixed problem (7.183), (7.184) is solvable if and only if $\langle G, v\rangle = 0$ for all $v \in \operatorname{Ker}(T - \lambda I) = \{0\} \cup \{v$ eigenfunction corresponding to $\lambda\}$.

Corollary 7.44.3 *Denote $V = \{u \in W^{1,2}(\Omega, C); v = 0$ on $\Gamma_D\}$. Let $\{\lambda_j; j \in N\}$ be the sequence of all eigenvalues of the mixed problem (7.180), (7.181), (7.182) in $W^{1,2}(\Omega)$. Define the bounded linear operator T from V to V'*

$$\langle Tu, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int_{\Gamma_R} buv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Denote $\mathcal{D}(\tilde T) = \{u \in V; Tu \in L^2(\Omega, C)\}$, i.e., $\mathcal{D}(\tilde T) = \{u \in W^{1,2}(\Omega, C); \Delta u \in L^2(\Omega, C), u = 0$ on $\Gamma_D, \partial u/\partial n + bu = 0$ on $\Gamma_R\}$, and $\tilde T u = Tu$ for $u \in \mathcal{D}(\tilde T)$, i.e., $\tilde T u = -\Delta u$. Then, $\tilde T$ is a closed linear operator in $L^2(\Omega, C)$ defined on the dense subset $\mathcal{D}(\tilde T)$, and its spectrum is $\sigma(\tilde T) = \{\lambda_j; j \in N\}$.

Proof Since $\mathcal{C}_c^\infty(\Omega, C) \subset \mathcal{D}(\tilde T) \subset L^2(\Omega, C)$, Proposition 1.14.7 gives that $\mathcal{D}(\tilde T)$ is a dense subset of $L^2(\Omega, C)$.

Let $\lambda \in C \setminus \{\lambda_j; j \in N\}$. Then $T - \lambda I$ is an isomorphism from V onto V' by Theorems 1.6.3 and 7.44.2. Since $L^2(\Omega, C)$ is continuously imbedded into V' by Hölder's inequality (Proposition 1.14.1) and V is continuously imbedded into $L^2(\Omega, C)$, we infer that $(\tilde T - \lambda I)^{-1}$ is a continuous linear operator on $L^2(\Omega, C)$. Thus, $\tilde T - \lambda I$ and $\tilde T$ are a closed operator and $\lambda \notin \sigma(\tilde T)$.

If λ_j is an eigenvalue and u is a corresponding eigenfunction, then $u \neq 0$ and $\tilde T u - \lambda_j u = 0$. Thus, $\lambda_j \in \sigma(\tilde T)$.

7.45 Relations Between Eigenvalues

In this section, we compare the eigenvalues corresponding to the Dirichlet problem and the eigenvalues corresponding to the Robin problem. We compare also the Dirichlet eigenvalues corresponding to different domains.

Proposition 7.45.1 *Let $\Omega \subset R^m$ be a bounded domain. Denote*

$$(u, v) = \int_\Omega uv \, \mathrm{d}\mathcal{H}_m, \qquad [u, v]_0 = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m.$$

- *Let u_j^D be a complete orthogonal system of eigenfunction corresponding to the Dirichlet problem and $\lambda_1^D \le \lambda_2^D \le \ldots$ be the corresponding eigenvalues. Denote* *by N_k the system of k-dimensional subspaces of* $\overset{\circ}{W}{}^{1,2}(\Omega)$*. Then*

$$\lambda_k^D = \min_{Y \in N_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_0}{(u,u)}.$$

- *Let Ω have Lipschitz boundary, $c \in L^\infty(\partial\Omega)$, $c \geq 0$. Denote*

$$[u,v]_c = \int\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + \int\limits_{\partial\Omega} cuv \, \mathrm{d}\mathcal{H}_{m-1}.$$

Let $u_j^{R,c}$ be a complete orthogonal system of eigenfunction corresponding to the Robin problem $-\Delta u = \lambda u$ in Ω, $\partial u/\partial n + cu = 0$ on $\partial\Omega$. Let $\lambda_1^{R,c} \leq \lambda_2^{R,c} \leq \dots$ be the corresponding eigenvalues. Denote by M_k the system of k-dimensional subspaces of $W^{1,2}(\Omega)$. Then

$$\lambda_k^{R,c} = \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_c}{(u,u)}.$$

Proof For the Dirichlet problem, denote $X = \overset{\circ}{W}{}^{1,2}(\Omega)$, $\lambda_j = \lambda_j^D$, $u_j = u_j^D$, $[u,v] = [u,v]_0$, and $L_k = N_k$. For the Robin problem, denote $X = W^{1,2}(\Omega)$, $\lambda_j = \lambda_j^{R,c}$, $u_j = u_j^{R,c}$, $[u,v] = [u,v]_c$, and $L_k = M_k$. We can suppose that $(u_j, u_j) = 1$ and $(u_j, u_k) = 0$, $[u_j, u_k] = 0$ for $j \neq k$ (see Theorems 7.41.4, 7.42.3, and 7.43.1).

Suppose first that Y is the space spanned by $u_1, \dots, u_k$. If

$$u = \sum_{j=1}^{k} \alpha_j u_j$$

then

$$\frac{[u,u]}{(u,u)} = \frac{\sum \alpha_j^2 [u_j, u_j]}{\sum \alpha_j^2 (u_j, u_j)} = \frac{\sum \alpha_j^2 \lambda_j (u_j, u_j)}{\sum \alpha_j^2 (u_j, u_j)} \leq \lambda_k.$$

If $u = u_k$, then

$$\frac{[u,u]}{(u,u)} = \frac{\lambda_k (u_k, u_k)}{(u_k, u_k)} = \lambda_k.$$

Thus,

$$\sup_{u \in Y \setminus \{0\}} \frac{[u,u]}{(u,u)} = \lambda_k.$$

Let now Y be another space from L_k. Then there exists $u \in L_k$ such that $(u, u_j) = 0$ for $j = 1, \dots, k-1$. According to Theorems 7.41.4, 7.42.3, and 7.43.1,

$$u = \sum_{j=1}^{\infty} \alpha_j u_j$$

with $\alpha_j = 0$ for $j < k$. Thus,

$$\frac{[u,u]}{(u,u)} = \frac{\sum \alpha_j^2 [u_j, u_j]}{\sum \alpha_j^2 (u_j, u_j)} = \frac{\sum \alpha_j^2 \lambda_j (u_j, u_j)}{\sum \alpha_j^2 (u_j, u_j)} \geq \lambda_k.$$

Theorem 7.45.2 *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $c_1, c_2 \in L^\infty(\Omega)$, $0 \le c_1 \le c_2$. Denote by $\lambda_1^D \le \lambda_2^D \le \dots$ the eigenvalues of the Dirichlet problem and by $\lambda_1^{R,c} \le \lambda_2^{R,c} \le \dots$ the eigenvalues of the Robin problem $-\Delta u = \lambda u$, $\partial u/\partial n + cu = 0$. Then, $\lambda_j^{R,c_1} \le \lambda_j^{R,c_2} \le \lambda_j^D$.*

Proof According to Proposition 7.45.1,

$$\lambda_k^{R,c_1} = \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_{c_1}}{(u,u)} \le \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_{c_2}}{(u,u)} = \lambda_k^{R,c_2},$$

$$\lambda_k^{R,c_2} = \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_{c_2}}{(u,u)} \le \min_{Y \in N_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_{c_2}}{(u,u)}$$

$$= \min_{Y \in N_k} \sup_{u \in Y \setminus \{0\}} \frac{[u,u]_0}{(u,u)} = \lambda_k^D.$$

Theorem 7.45.3 *Let $\Omega_1 \subset \Omega_2 \subset R^m$ be bounded domains. Denote $\lambda_1(\Omega_j) \le \lambda_2(\Omega_j) \le \dots$ the eigenvalues of the Dirichlet problem $-\Delta u = \lambda u$ in Ω_j, $u = 0$ on $\partial\Omega_j$. Then $\lambda_k(\Omega_2) \le \lambda_k(\Omega_1)$.*

Proof Denote by M_k the system of k-dimensional subspaces of $\overset{\circ}{W}{}^{1,2}(\Omega_1)$ and by N_k the system of k-dimensional subspaces of $\overset{\circ}{W}{}^{1,2}(\Omega_2)$. Let $Y \in M_k$. For $u \in Y$, define $\tilde{u} = u$ in Ω_1 and $\tilde{u} = 0$ elsewhere. Then $\tilde{u} \in \overset{\circ}{W}{}^{1,2}(\Omega_2)$. Clearly, $\tilde{Y} = \{\tilde{u}; u \in Y\} \in N_k$. By virtue of Proposition 7.45.1,

$$\lambda_k(\Omega_1) = \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega_1)}}{\|u\|^2_{L^2(\Omega_1)}} = \min_{Y \in M_k} \sup_{u \in Y \setminus \{0\}} \frac{\|\nabla \tilde{u}\|^2_{L^2(\Omega_2)}}{\|\tilde{u}\|^2_{L^2(\Omega_2)}}$$

$$\ge \min_{Y \in N_k} \sup_{u \in Y \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega_2)}}{\|u\|^2_{L^2(\Omega_2)}} = \lambda_k(\Omega_2).$$

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